Supplemental Material

In this section the termini of the SPB core proteins are represented by points as described in the text. The goal is to prove the following two results about several points in a space. (We use capital letters to denote the points and we denote by $|AB|$ the distance between two points A and B.)

Theorem 0.1. *Let A, B, C, D, and E be points in $\mathbb{R}^3$ such that A and B lie in a plane that is parallel to the plane containing C, D, and E. Assume further that $|AB| = |DE| = |CE| > 0$, $|AD| = |AC| = |BE| = |BD| = |BC| > 0$, and $|DC| > |AB|$. Then the line through A and E is perpendicular to the two planes, $\angle CED = 120°$, and $\angle DCE = \angle EDC = 30°$.*

Theorem 0.2. *Let A, B, C, D, and E be points in $\mathbb{R}^3$ satisfying the conditions of Theorem 0.1, and let $F \in \mathbb{R}^3$ be an additional point that lies in the plane containing C, D, and E and satisfies the following relations: $|AF| = |AD|$, $|CE| = |EF| = |CF|$, and $|DF| > |DC|$. Then the points D, E and F lie on the same line. Moreover, E is the mid-point of the segment $[D, F]$, $\angle EFC = 60°$, and $\angle DCF = 90°$.*

Proof of Theorem 0.1: First we assume without loss of generality that C, D, and E lie in the xy-plane. Since the two planes are parallel, it follows that A and B have the same z-coordinate which we denote by z. Let A' and B' be the orthogonal projections of A and B respectively onto the xy-plane. Then the quadrilateral $ABB'A'$ is a rectangle, and hence $|AB| = |A'B'|$. Moreover, $|AA'| = |BB'| = z$ and (by Pythagorean Theorem) for every point P in the xy-plane, $|AP|^2 = |A'P|^2 + z^2$ and $|BP|^2 = |B'P|^2 + z^2$. Therefore we can rewrite the given equalities as

$$|A'B'| = |DE| = |CE| > 0 \text{ and } |A'D| = |A'C| = |B'E| = |B'D| = |B'C| > 0.$$

In particular, B' is the circumcenter of the triangle CDE, and $A'CB'D$ is a rhombus.

At this point it is convenient to introduce coordinates. Since we are working in the xy-plane, rather than in $\mathbb{R}^3$, we denote the coordinates of a point P by (x_P, y_P) (instead of by $(x_P, y_P, 0)$). We assume without loss of generality that $E = (0, 0)$, $D = (1, 0)$, and that $y_C \geq 0$. Then $|CE| = |DE| = 1$, and so

$$x_C^2 + y_C^2 = 1. \tag{1}$$

Since B' is the circumcenter of the triangle CDE, it is the point of intersection of perpendicular bisectors of the sides of this triangle. The perpendicular bisector of the side DE is given by equation $x = \frac{1}{2}$. Since $|CE| = |DE|$, the perpendicular bisector of the side CD passes through $E = (0, 0)$ and the mid-point, M, of the side CD, whose coordinates are given by

$$M = \left(\frac{x_C + x_D}{2}, \frac{y_C + y_D}{2}\right) = \left(\frac{x_C + 1}{2}, \frac{y_C}{2}\right). \tag{2}$$

Thus the equation of this perpendicular bisector is

$$y = \frac{y_C}{x_C + 1} \cdot x. \tag{3}$$

Substituting $x = \frac{1}{2}$ in Eq. (3), we infer that

$$B' = \left(\frac{1}{2}, \frac{y_C}{2(x_C + 1)}\right) \tag{4}$$

which together with Eq. (2) implies that

$$(5) \qquad |B'M|^2 = \frac{x_C^2}{4} + \frac{y_C^2}{4}\left(1 - \frac{1}{x_C+1}\right)^2 = \frac{x_C^2}{4}\left(1 + \frac{y_C^2}{(x_C+1)^2}\right) = \frac{x_C^2}{2(x_C+1)}$$

(where in the last step we used Eq. (1)).

Since $A'CB'D$ is a rhombus, its diagonals bisect each other. Hence their point of intersection is M and $|B'M| = \frac{1}{2}|A'B'| = \frac{1}{2}|DE| = \frac{1}{2}$. Substituting this in Eq. (5) we obtain that

$$2x_C^2 - x_C - 1 = 0,$$

and hence that either $x_C = -\frac{1}{2}$ or $x_C = 1$. Since the latter case is impossible (together with Eq. (1), it implies that $C = D$ contradicting that $|CD| > |AB| > 0$), it follows that $x_C = -\frac{1}{2}$ and $y_C = \sqrt{1 - x_C^2} = \frac{\sqrt{3}}{2}$. Thus $\cos(\angle CED) = -\frac{1}{2}$ and $\sin(\angle CED) = \frac{\sqrt{3}}{2}$, yielding that $\angle CED = 120°$. Also since $|CE| = |DE|$, we obtain that $\angle DCE = \angle EDC = \frac{1}{2}(180° - \angle CED) = 30°$.

Substituting the coordinates of point C in Eq. (4), we find that $B' = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, and hence that $|DE| = |B'D| = |B'C| = |EC| = 1$, or equivalently that $ECB'D$ is a rhombus. Since as we observed above $A'CB'D$ is a rhombus as well, it follows that A' coincides with E. Recalling that A' is the orthogonal projection of A onto the xy-plane, we conclude that AE is perpendicular to that plane, and the result follows. $\square$

Proof of Theorem 0.2: Note that since $|CE| = |EF| = |CF|$, the points C, E, and F are the vertices of an equilateral triangle, and so $\angle CFE = \angle CEF = 60°$. Combining this with Theorem 0.1 asserting that $\angle CED = 120°$, we conclude that there are only to possible cases: either $\angle DEF = 60°$ or $\angle DEF = 180°$.

We first show that the case of $\angle DEF = 60°$ is impossible. Indeed, since by our assumptions, $|DE| = |EF|$ (both of these distances are equal to $|CE|$), it follows that in such a case DEF is an equilateral triangle, and hence that $|DF| = |DE|$. Therefore in this case $|DF| = |DE| = |CE| = |EF|$, yielding that $DEFC$ is a rhombus. Hence $\angle CFD = \angle CED > 90°$, contradicting the fact that $|DF| > |DC|$.

Thus $\angle DEF = 180°$, and so the points D, E, and F lie on the same line with D and F being at the same distance (but on the opposite sides) from E. Finally, since $\angle CFE = 60°$, and since $\angle EDC = 30°$ (the latter equality follows from Theorem 0.1), we obtain that $\angle DCF = 180° - 60° - 30° = 90°$. $\square$