Appendix to the paper "Incorporating Delay Mechanism in Ordering Policies in Multi-Echelon Distribution Systems"

Kamran Moinzadeh and Yong-Pin Zhou

University of Washington Business School

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Proof of Lemma 1

When $r+Q \leq 0$, Equation (2) reduces to $\frac{\lambda}{Q}\left[-hQ+(h+\pi)Q\right]=\lambda\pi>0$. Therefore, it is never optimal to delay the orders. This is intuitive as when $r+Q<0$, all orders are backordered and any delay in the ordering will only increase additional penalty cost, with no savings in holding cost. When $r+Q>0$, the optimal delay will satisfy $FO=0$. So from (2) we get:

$$\tau_r(t)=\left[\left(\sum_{k=r+1}^{r+Q} G_k^t\right)^{-1}\left(\frac{hQ}{\pi+h}\right)-L\right]^+=\left[\left(H_{r+1,r+Q}^t\right)^{-1}\left(\frac{hQ}{\pi+h}\right)-L\right]^+.$$

□

Proof of Lemma 2

First, let X have DFR, then by Definition 1.2 in Barlow and Proschan (1976, page 55), $\overline{G}_1^t(x)=\frac{\overline{F}(x+t)}{\overline{F}(t)}$ is increasing in $t\geq 0$ for each $x\geq 0$. Thus, whenever $t_1<t_2$, $\overline{G}_1^{t_1}(x)<\overline{G}_1^{t_2}(x)$ for each $x\geq 0$. Furthermore, we have $\overline{G}_k^{t_1}(x)<\overline{G}_k^{t_2}(x)$, $\forall k, \forall x\geq 0$. Therefore,

$$H_{r+1,r+Q}^{t_1}(x)>H_{r+1,r+Q}^{t_2}(x), \forall t_1<t_2, \forall x\geq 0.$$

Since the CDFs are increasing, we must also have:

$$\left(H_{r+1,r+Q}^{t_1}\right)^{-1}\left(\frac{hQ}{\pi+h}\right)<\left(H_{r+1,r+Q}^{t_2}\right)^{-1}\left(\frac{hQ}{\pi+h}\right).$$

Therefore,

$$\tau_r(t_1) = \left[\left(H^{t_1}_{r+1,r+Q}\right)^{-1}\left(\frac{hQ}{\pi+h}\right) - L\right]^+ \le \left[\left(H^{t_2}_{r+1,r+Q}\right)^{-1}\left(\frac{hQ}{\pi+h}\right) - L\right]^+ = \tau_r(t_2).$$

Proofs of the other two cases are similar. □

Proof of Theorem 1

From Lemma 1, we know that for constant and decreasing failure rate functions, $\tau_r(t)$ is non-decreasing in *t*. So $\tau_r(0) > 0 \Rightarrow \tau_r(t) > 0, \forall t > 0, \forall r$. This means that if it's sub-optimal to order at a demand epoch, then it remains sub-optimal to order until the next order epoch. □

Proof of Corollary 1

This follows from Theorem 1 and the fact that the inter-demand time of a Poisson process is exponentially distributed, which has CFR. □

Proof of Lemma 3

For *r*+1, the first order derivative in equation (2) can be written as (using a generic variable *x*):

$$FO(r+1,t) = \frac{\lambda}{Q}\left[-hQ + (h+\pi)H^t_{r+2,r+Q+1}\left(L+x\right)\right]. \tag{A1}$$

Now fix *x*. Since $\gamma(t)$ is increasing, $\lim_{t\to\infty} G_1^t(x) = 1 - \lim_{t\to\infty} e^{-\int_t^{t+x}\gamma(s)ds}$ is increasing in *t*. Therefore,

- if $k < 0$, $G^t_{k+1}(x) \equiv 1, \forall t$;
- if $k \ge 0$, $G^t_{k+1}(x) = G^t_1 \otimes F_k(x)$ is also increasing in *t*.

Note that $G^t_{k+1}(x)$ is increasing in *t*. This immediately implies that the *H* function, hence also the first order derivative in (A1), are increasing in *t*. Thus, $\tau_{r+1}(t)$ is decreasing in *t*, and since it's bounded by zero, the limit must exist.

Next, we study the relation between $\lim_{t\to\infty}\tau_{r+1}(t)$ and $\tau_r(0)$. Since the failure rate function $\gamma(x)$ is increasing, there can only be two cases: (Case 1) $\lim_{x\to\infty}\gamma(x)=\infty$, and (Case 2) $\lim_{x\to\infty}\gamma(x)=\gamma$ for some constant $\gamma<\infty$. We will study these two cases separately. In particular, we will first show that $\lim_{t\to\infty}\tau_{r+1}(t)$ exists and then $\lim_{t\to\infty}\tau_{r+1}(t)=\tau_r(0)$ in Case 1 and $\lim_{t\to\infty}\tau_{r+1}(t)\ge\tau_r(0)$ in the Case 2.

Case 1: $\lim_{x\to\infty}\gamma(x)=\infty$. This is satisfied by, for example, the uniform distribution and the Weibull distribution (when the parameters are such that it's IFR).

For any $x\ge 0$,

– if $k<0$, $G_{k+1}^t(x)=1=F_k(x),\forall t$;

– if $k\ge 0$, $\lim_{t\to\infty}\bar{G}_1^t(x)=\lim_{t\to\infty}e^{-\int_t^{t+x}\gamma(s)ds}=0$, and $G_{k+1}^t(x)=\int_0^x G_1^t(s)f_k(x-s)ds \xrightarrow{t\to\infty} F_k(x)$.

So overall we have $G_{k+1}^t(x)\xrightarrow{t\to\infty}F_k(x),\forall k,\forall x\ge 0$. Now let's examine the first order derivatives (as in equation 2) for r+1 and r (using x as the variable):

$$FO(r,0)=\frac{\lambda}{Q}\left[-hQ+(h+\pi)\sum_{k=r+1}^{r+Q}F_k\left(L+x\right)\right] \quad \text{(A2)}$$

It is clear then that when $t\to\infty$, the first order derivative for r+1 converges to that for r (with t=0), *i.e.*, $FO(r+1,t)\to FO(r,0)$. Therefore, we must have $\lim_{t\to\infty}\tau_{r+1}(t)=\tau_r(0)$.

Case 2: $\lim_{x\to\infty}\gamma(x)=\gamma<\infty$. This is satisfied by Erlang distributions. For example, for an Erlang(λ/Q,Q) distribution, the failure rate function converges to λ/Q.

For any $x\ge 0$,

– if $k<0$, $G_{k+1}^t(x)=1=F_k(x),\forall t$;

– if $k \geq 0$, then $\lim_{t\to\infty} \overline{G}_1^t(x) = \lim_{t\to\infty} e^{-\int_t^{t+x} \gamma(s)ds} = e^{-\gamma x}$. This means that the time until next demand arrival converges to an exponentially distributed random variable. Therefore, $G_{k+1}^t(x) = G_1^t \otimes F_k(x)$ should be stochastically larger than $F_k(x)$. More specifically,

$$\lim_{t\to\infty} G_{k+1}^t(x) = \lim_{t\to\infty} \int_0^x G_1^t(s) f_k(x-s)ds = \int_0^x e^{-\gamma s} f_k(x-s)ds \leq \int_0^x f_k(x-s)ds = F_k(x).$$

To summarize, we have: $\lim_{t\to\infty} G_{k+1}^t(x) \leq F_k(x)$. Again, if we compare the first order derivatives in (A1) and (A2), we conclude that $\lim_{t\to\infty} \tau_{r+1}(t) > \tau_r(0)$. □

Proof of Theorem 2

We observe that since $\lim_{r\to\infty} \tau_r(0) = \infty$, $\underline{r}$ exists. The conclusions of the theorem are straightforward given the definitions of *R*, *T*, and $\tau_{\underline{r}}(t)$. □

Proof of Lemma 5

Let's examine Erlang(*λ/Q*, *Q*) for any *λ* and *Q*>1. Its PDF can be expressed as $f(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{Q-1}}{(Q-1)!}$, and its complementary CDF is $\overline{F}(t) = \int_t^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{Q-1}}{(Q-1)!}dx$. Thus, its failure (hazard) rate function satisfies:

$$\frac{1}{\gamma(t)} = \frac{\overline{F}(t)}{f(t)} = \frac{\int_t^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{Q-1}}{(Q-1)!}dx}{\frac{\lambda e^{-\lambda t}(\lambda t)^{Q-1}}{(Q-1)!}} = \int_t^\infty e^{-\lambda(x-t)} \left(\frac{x}{t}\right)^{Q-1} dx = \int_0^\infty e^{-\lambda y} \left(1+\frac{y}{t}\right)^{Q-1} dy,$$

which is clearly a decreasing function in *t*. Therefore, it has increasing failure rate function. □

Proof of Lemma 6

We will let $e_k(t)$ and $E_k(t)$ denote the PDF and the CDF of an Erlang(λ/k, k) random variable. Suppose retailer 1 has just placed an order. We let U denote the time to next order for any other retailer.

At the time retailer 1 places an order, the inventory position at any other retailer is uniformly distributed on {R+1, R+2, …, R+Q}. Therefore, U is equally likely to be an Erlang(λ/k, k) random variable for $1 \le k \le Q$.

$\Pr[U > t] = \frac{1}{Q}\sum_{k=1}^{Q}\overline{F_k}(t)$ and the hazard rate function $\gamma_U(t)$ satisfies:

$$\frac{1}{\gamma_U(t)} = -\frac{\frac{1}{Q}\sum_{k=1}^{Q}\overline{F_k}(t)}{d\left[\frac{1}{Q}\sum_{k=1}^{Q}\overline{F_k}(t)\right]\Big/dt} = \frac{\sum_{k=1}^{Q}\overline{F_k}(t)}{\sum_{k=1}^{Q}f_k(t)} = \frac{\sum_{k=1}^{Q}\int_t^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{(k-1)!}dx}{\sum_{j=1}^{Q}\frac{\lambda e^{-\lambda t}(\lambda t)^{j-1}}{(j-1)!}}$$

$$= \int_t^\infty \frac{\sum_{k=1}^{Q} e^{-\lambda(x-t)}(\lambda x)^{k-1}\big/(k-1)!}{\sum_{j=1}^{Q}(\lambda t)^{j-1}\big/(j-1)!}dx = \int_0^\infty e^{-\lambda x}\frac{\sum_{k=0}^{Q-1}(\lambda x+\lambda t)^k\big/k!}{\sum_{j=0}^{Q-1}(\lambda t)^j\big/j!}dx = \int_0^\infty e^{-\lambda x}h(x,t)dx,$$

where $h(x,t) = \frac{\sum_{k=0}^{Q-1}(\lambda x+\lambda t)^k\big/k!}{\sum_{j=0}^{Q-1}(\lambda t)^j\big/j!}$. It suffices to show that h(x, t) is decreasing in t for all x. The case of Q=1 is trivial, so we will focus on Q≥2. Differentiating h(x,t), we get:

$$\frac{\partial h(x,t)}{\partial t}=\lambda\frac{\left[\sum_{k=0}^{Q-2}\frac{(\lambda x+\lambda t)^k}{k!}\right]\left[\sum_{j=0}^{Q-1}\frac{(\lambda t)^j}{j!}\right]-\left[\sum_{k=0}^{Q-1}\frac{(\lambda x+\lambda t)^k}{k!}\right]\left[\sum_{j=0}^{Q-2}\frac{(\lambda t)^j}{j!}\right]}{\left[\sum_{j=0}^{Q-1}\frac{(\lambda t)^j}{j!}\right]^2}$$

$$=\lambda\frac{\left[\sum_{k=0}^{Q-2}\frac{(\lambda x+\lambda t)^k}{k!}\right]\left[\sum_{j=0}^{Q-2}\frac{(\lambda t)^j}{j!}+\frac{(\lambda t)^{Q-1}}{(Q-1)!}\right]-\left[\sum_{k=0}^{Q-2}\frac{(\lambda x+\lambda t)^k}{k!}+\frac{(\lambda x+\lambda t)^{Q-1}}{(Q-1)!}\right]\left[\sum_{j=0}^{Q-2}\frac{(\lambda t)^j}{j!}\right]}{\left[\sum_{j=0}^{Q-1}\frac{(\lambda t)^j}{j!}\right]^2}$$

$$=\lambda\frac{\sum_{k=0}^{Q-2}\frac{(\lambda x+\lambda t)^k}{k!}\frac{(\lambda t)^{Q-1}}{(Q-1)!}-\frac{(\lambda x+\lambda t)^{Q-1}}{(Q-1)!}\sum_{j=0}^{Q-2}\frac{(\lambda t)^j}{j!}}{\left[\sum_{j=0}^{Q-1}\frac{(\lambda t)^j}{j!}\right]^2}=\lambda\frac{\sum_{k=0}^{Q-2}\left\{\frac{(\lambda x+\lambda t)^k}{k!}\frac{(\lambda t)^{Q-1}}{(Q-1)!}-\frac{(\lambda x+\lambda t)^{Q-1}}{(Q-1)!}\frac{(\lambda t)^k}{k!}\right\}}{\left[\sum_{j=0}^{Q-1}\frac{(\lambda t)^j}{j!}\right]^2}$$

$$=\lambda\frac{\sum_{k=0}^{Q-2}\left\{\frac{(\lambda x+\lambda t)^k}{k!}\frac{(\lambda t)^{Q-1}}{(Q-1)!}\left[1-\left(\frac{\lambda x+\lambda t}{\lambda t}\right)^{Q-1-k}\right]\right\}}{\left[\sum_{j=0}^{Q-1}\frac{(\lambda t)^j}{j!}\right]^2}\leq 0,$$

with equality holding true only when x=0. Therefore $h(x, t)$ is decreasing in t for all $x\geq 0$, and strictly so for x>0. We conclude that U has strictly increasing failure rate function. □

Proof of Lemma 7

Let $U_1, U_2, ..., U_m$ be independent IFR random variables, with $f_1(F_1), f_2(F_2)..., f_m(F_m)$ being their corresponding PDF(CDF). Clearly, their failure rate functions $\frac{f_i}{\overline{F}_i}$ are all increasing. Define: $\hat{U}=\min\{U_1,U_2,...,U_m\}$, and its corresponding PDF(CDF) be $f(F)$. Then:

$$\overline{F}(x)=\prod_{i=1}^{m}\overline{F}_i(x),\ \text{and}\ f(x)=-\frac{d\overline{F}(x)}{dx}=\sum_{j=1}^{m}\left[f_j(x)\prod_{\substack{i=1\\ i\neq j}}^{m}\overline{F}_i(x)\right].$$

Hence, the failure rate function of $\widehat{U}$ is:

$$\frac{f(x)}{\overline{F}(x)} = \frac{\sum_{j=1}^{m}\left[f_j(x)\prod_{\substack{i=1 \\ i\neq j}}^{m}\overline{F}_i(x)\right]}{\prod_{i=1}^{m}\overline{F}_i(x)} = \sum_{j=1}^{m}\frac{f_j(x)\prod_{\substack{i=1 \\ i\neq j}}^{m}\overline{F}_i(x)}{\prod_{i=1}^{m}\overline{F}_i(x)} = \sum_{j=1}^{m}\left[\frac{f_j(x)}{\overline{F}_j(x)}\right],$$

which is increasing. □

Proof of Proposition 1

Suppose that retailer 1 has just placed an order and its time till next order is Erlang(λ/Q, Q), which has IFR. The time to next order for all the other retailers is U, which also has IFR. The time to next order at the supplier is the minimum of all these random variables, which, by Lemma 7, also has IFR. □