A note on the weighted average derivative effect

Yen-Chi Chen
University of Washington
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Main reference:

> Hines, O., Diaz-Ordaz, K. and Vansteelandt, S., 2023. "Optimally weighted average derivative effects." arXiv preprint arXiv:2308.05456.

We consider a **continuous** treatment scenario in a causal inference problem. Let $Y\mathbb{R}$ be the outcome variable, $A \in \mathbb{R}$ be the treatment (continuous), $X \in \mathbb{R}^p$ be the confounders. For simplicity, we denote $Z = (A, X)$.

Let the regression function
$$\mu(A,X) = \mathbb{E}(Y|A,X).$$
In the continuous treatment problem, the so-called dose-response curve that represents the causal effect is the function
$$\eta(a) = \mathbb{E}(\mu(a,X)).$$

The difference $\eta(a+1) - \eta(a)$ is sometimes be viewed as a continuous version of the average treatment effect. Another quantity that is often considered as an average treatment effect is the **average derivative effect (ADE):**
$$\theta = \mathbb{E}(\nabla\mu(A,X)) \in \mathbb{R}^{1+p}, \tag{1}$$
where the gradient ∇ applies to both A, X.

To make it more general, we consider **weighted ADE**:
$$\theta_\omega = \mathbb{E}(\omega(A,X)\nabla\mu(A,X)), \tag{2}$$
where $\omega(a,x)$ is a known function. This allows us to place weights on a particular subpopulation of interest.

A tradition way of estimating WADE θ_ω is a plug-in estimate. We first obtain an estimator $\widehat{\mu}(a,x)$ and then obtain the estimator
$$\widehat{\theta}_\omega = \frac{1}{n}\sum_{i=1}^n \nabla\widehat{\mu}(A_i,X_i). \tag{3}$$
Or alternatively, one can estimate the derivative $\widehat{\nabla\mu}(A,X)$ and then obtain the plug-in estimate.

1 Riez Representation of WADE

The estimator in equation (3) is similar to the regression adjustment method in causal/missing data literature. One may be wondering if we can do something like the inverse probability weighting (IPW) estimator.

It turns out that the answer is yes, although the form will not be like a weighting approach. The 'IPW' approach relies on the following **Riez representation** of the WADE.

Proposition 1 *Define*

$$\ell(a,x) = -\nabla\omega(a,x) - \omega(a,x)\nabla\log p(a,x).$$

Then the WADE θ_ω *can be written as the following expression:*

$$\theta_\omega = \mathbb{E}(\ell(A,X)\mu(A,X)) = \mathbb{E}(\ell(A,X)Y).$$

Proposition 1 shows that as long as we can estimate $\ell(a,x)$, we can obtain an estimator of θ_ω. Since $\omega(a,x)$ is known, the only quantity that we need to estimate is $\nabla\log p(a,x)$. This can be done via a density estimator or a density-derivative estimator.

Proof. The proof is simply via the integration by parts.

$$\begin{aligned}
\theta_\omega &= \mathbb{E}(\omega(A,X)\nabla\mu(A,X)) \\
&= \int \omega(a,x)p(a,x)\nabla\mu(a,x)dadx \\
&= -\int [\nabla\omega(a,x)]p(a,x)\mu(a,x)dadx - \int \omega(a,x)[\nabla p(a,x)]\mu(a,x)dadx \\
&= -\int [\nabla\omega(a,x)]p(a,x)\mu(a,x)dadx - \int \omega(a,x)[\nabla\log p(a,x)]p(a,x)\mu(a,x)dadx \\
&= \mathbb{E}(\ell(A,X)\mu(A,X)).
\end{aligned}$$

□

2 WADE on the treatment

Generally, we are more interested in the derivative effect on the treatment variable only. Namely, we are interested in

$$\theta^A_\omega = \mathbb{E}\left(\omega(A,X)\frac{\partial}{\partial A}\mu(A,X)\right) \in \mathbb{R}.$$

For simplicity, we use the notation

$$f'(a,x) = \frac{\partial}{\partial a}f(a,x)$$

for any function of both a,x. With the above notation, we obtain

$$\begin{aligned}
\theta^A_\omega &= \mathbb{E}\left(\omega(A,X)\mu'(A,X)\right) \\
&= \mathbb{E}(\omega(X)\mathbb{E}(\omega(A|X)\mu'(A,X)|X)),
\end{aligned} \tag{4}$$

where

$$\omega(X) = \mathbb{E}(\omega(A,X)|X), \qquad \omega(A|X) = \frac{\omega(A,X)}{\omega(X)}.$$

Note that the quantities $\omega(X)$ and $\omega(A|X)$ both satisfies

$$\mathbb{E}(\omega(X)) = \mathbb{E}(\omega(A|X)) = 1$$

and are non-negative. So they can be interpreted as marginal weights and conditional weights.

As one can see from equation (4), a key quantity is the conditional mean

$$Q_\omega(X) = \mathbb{E}(\omega(A|X)\mu'(A,X)|X).$$

Using a traditional regression adjustment method, we can estimate it by estimating the derivative $\mu'(a,x)$ first and then perform a plug-in procedure.

Proposition 1 shows a Riez representation of θ_ω. A similar proof can be applied to $Q_\omega(x)$, leading to the following representation:

Proposition 2 *Define*

$$Q_\omega(x) = \mathbb{E}(\omega(a|x)\mu'(a,x)|x).$$

Then Q_ω can be written as the following expression:

$$Q_\omega = \mathbb{E}(\ell(A|X)\mu(A,X)|X) = \mathbb{E}(\ell(A|X)Y|X),$$

where

$$\ell(a|x) = -\omega'(a,x) - \omega(a,x)\frac{\partial}{\partial a}\log p(a|x).$$

3 Reversed Riez representation

Propositions 1 and 2 show that if we were given $\omega(a,x)$, we are able to construction $\ell(a,x)$ or $\ell(a|x)$ to perform an 'weighted' estimator.

Now we want to study the reversed question:

> if we were given a function $\ell(a|x)$ satisfying some conditions, are we able to derive the corresponding weight $\omega(a,x)$?

The answer is yes, if $\ell(a|x)$ is a *contrast function.*

Theorem 3 *Suppose $\ell(a|x)$ is a contrast function, i.e.,*

1. $\mathbb{E}(\ell(A|X)|X) = 0$.
2. $\mathbb{E}(A\cdot\ell(A|X)|X) = 1$.

Then for any function $h(a,x)$ that is squared integrable, we have

$$\mathbb{E}(h(A,X)\ell(A|X)|X) = \mathbb{E}(\rho(A|X)h'(A,X)|X),$$

where

$$\rho(a|x) = -\frac{\mathbb{E}(\ell(A|X)|A \leq a, X = x)F(a|x)}{p(a|x)}$$

such that $F(a|x)$ is the conditional cumulative distribution function of A given X and $p(a|x)$ is the conditional PDF.

The quantity $\rho(a|x)$ is essentially the same quantity as $\omega(a|x)$. But now it is derived from $\ell(a|x)$.

Note that if we choose $h(a,x) = a$, the above result implies

$$1 = \mathbb{E}(A\ell(A|X)|X) = \mathbb{E}(\rho(A|X)|X),$$

which implies that $\rho(A|X)$ is indeed a weight function.

Proof.

Due to Proposition 2, we only need to verify that

$$\ell(a|x) = -\rho'(a|x) - \rho(a|x)\frac{\partial}{\partial a}\log p(a|x).$$

A direct computation shows

$$\begin{aligned}\rho'(a|x) = &\underbrace{-\frac{\partial}{\partial a}\left[\int_{a'=-\infty}^{a'=a} \ell(a'|x)p(a'|x)da'/F(a|x)\right]\frac{F(a|x)}{p(a|x)}}_{(A)} \underbrace{-\mathbb{E}(\ell(A|X)|A \leq a, x)}_{(B)} \\ &\underbrace{+\mathbb{E}(\ell(A|X)|A \leq a, x)\frac{F(a|x)}{p(a|x)}\frac{\partial}{\partial a}\log p(a|x)}_{(C)}.\end{aligned}$$

Term (A) can be written as

$$\begin{aligned}(A) &= -\frac{\ell(a|x)p(a|x)}{F(a|x)} \times \frac{F(a|x)}{p(a|x)} + \frac{\int_{a'=-\infty}^{a'=a} \ell(a'|x)p(a'|x)da'}{F^2(a|x)}p(a|x)\frac{F(a|x)}{p(a|x)} \\ &= -\ell(a|x) + \mathbb{E}(\ell(A|X)|A \leq a, X = x).\end{aligned}$$

Thus,

$$(A) + (B) = -\ell(a|x).$$

Now observe that

$$(C) = -\rho(a|x) \cdot \frac{\partial}{\partial a}\log p(a|x).$$

Thus, we conclude that

$$\begin{aligned}-\rho'(a|x) - \rho(a|x)\frac{\partial}{\partial a}\log p(a|x) &= (A) + (B) + (C) - \rho(a|x)\frac{\partial}{\partial a}\log p(a|x) \\ &= \ell(a|x) + \rho(a|x)\frac{\partial}{\partial a}\log p(a|x) - \rho(a|x)\frac{\partial}{\partial a}\log p(a|x) \\ &= \ell(a|x),\end{aligned}$$

which completes the proof.

□

With Theorem 3, we are able to derive the weight from a contrast function. In the causal inference problem, we choose $h(a,x) = \mu(a,x) = \mathbb{E}(Y|A = a, X = x)$. But this result is not limited to the causal problem. It is a generic approach to avoid an explicit use of a derivative, which somewhat shared a similar spirit of the Stein's lemma (for Gaussian case).