Analyzing GPS datasets using density ranking

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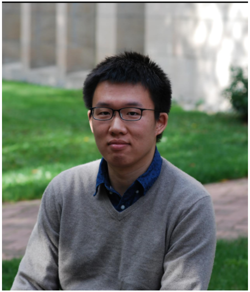
Outline

- Introduction
- Density Ranking
- Density Ranking: Multiple Datasets
- Summary

Introduction

Collaborators

Adrian Dobra

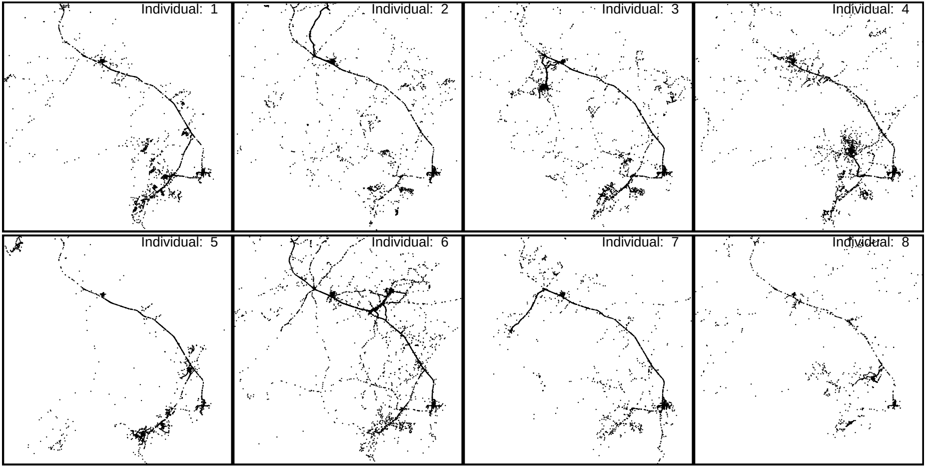


Zhihang Dong

Real Person Datasets

- This data is about 10 real person's GPS records from Chen and Dobra (2017).
- All these participants share the same work place.
- The ages of the study participants were between 34 and 48 years.
- Each person has around $3,500$ to $8,500$ GPS records.

Real Persons Datasets: Raw Data



African Animal Datasets

- This data is from the Movebank Data Repository[1] and was analyzed in Abrahms et al. (2017).
- Here we compare 4 different types of animals: elephants, jackals, vultures, and zebras.
- In this data, we have 8 elephants, 15 jackals, 10 vultures, and 9 zebras.
- Each animal has a set of GPS records with record size ranging from $1,000$ to $10,000$.

[1] https://www.datarepository.movebank.org/

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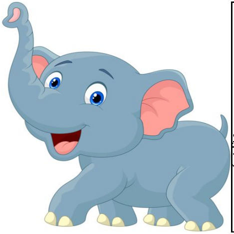

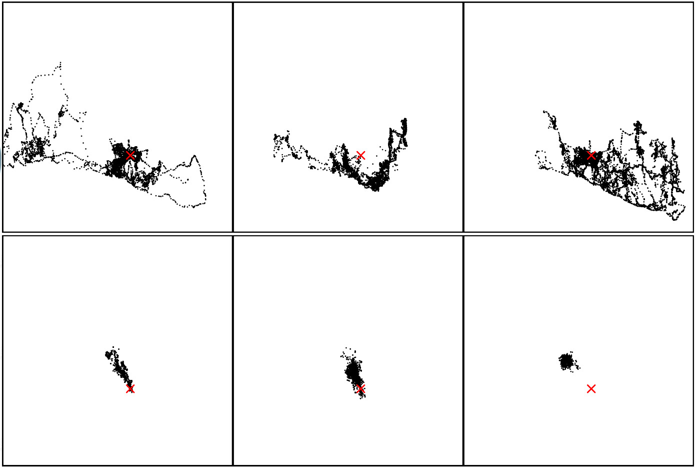

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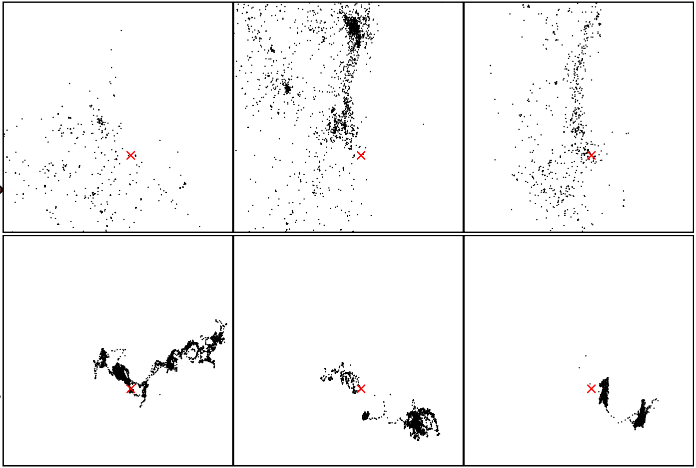

Density Ranking

A Naive Idea: KDE

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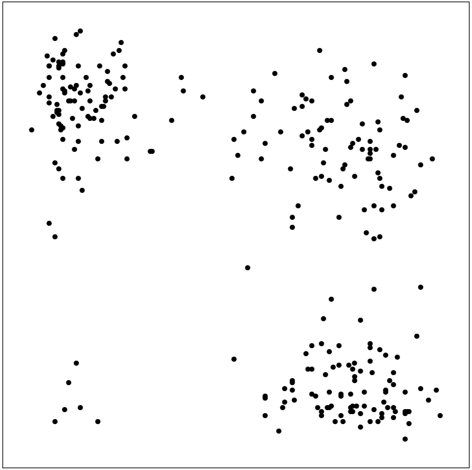
- Based on the estimated density function, we can then compare these datasets.
- Here we will use a simple approach called the kernel density estimation (KDE).
- At a given point x, KDE returns a density value

$$\widehat{p}(x) = \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right),$$

where $K(\cdot)$ is the kernel function that is often a smooth function like a Gaussian, and $h > 0$ is the smoothing bandwidth that controls the amount of smoothing.

KDE: an Illustration



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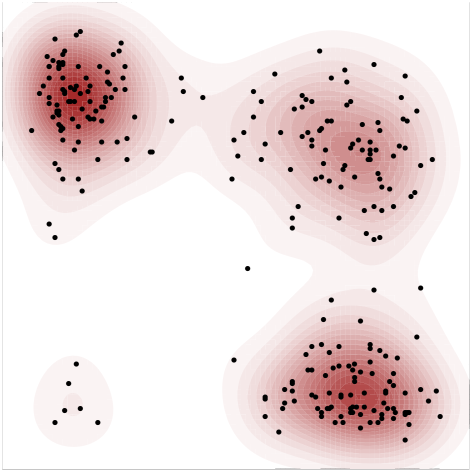
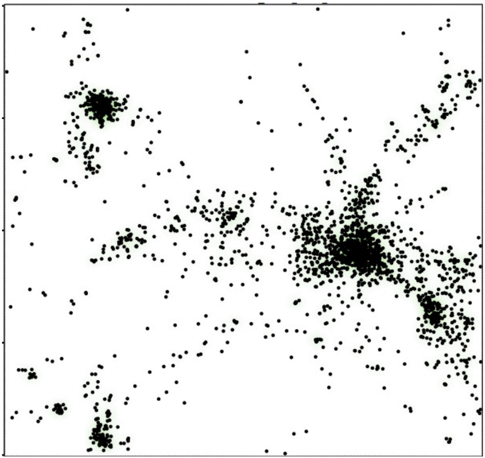


KDE: a More Complex Dataset (Astronomy)



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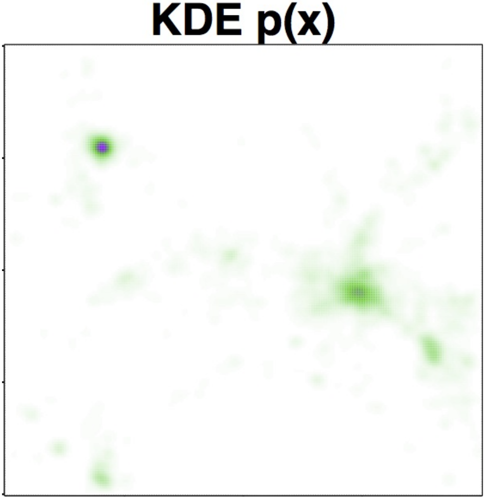

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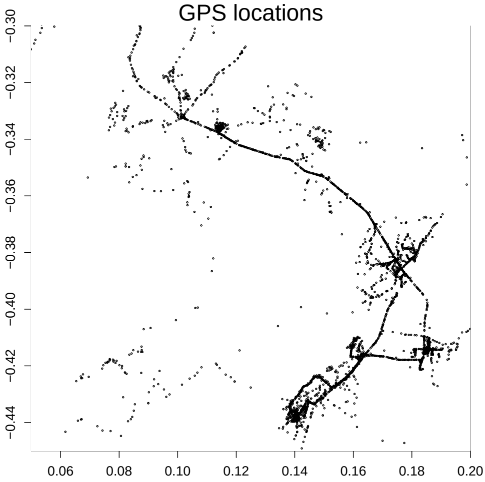

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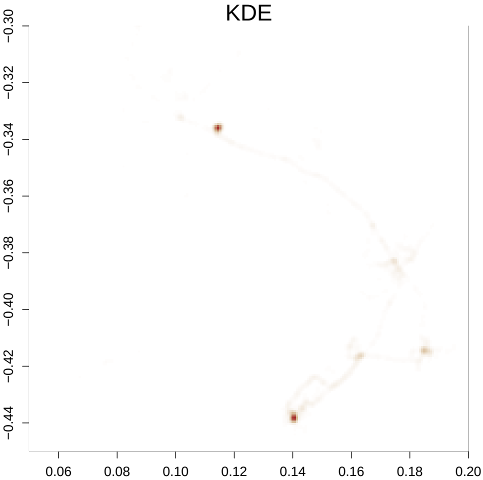

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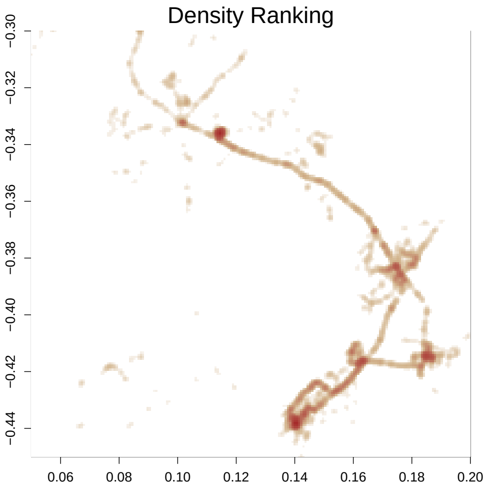

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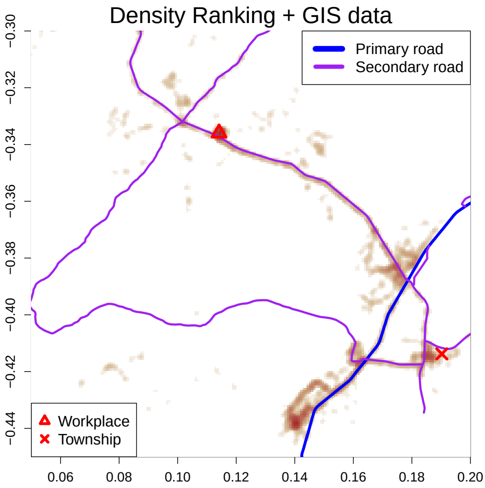

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- Namely, our probability distribution function is singular.
- However, density ranking still works!

Definition of Density Ranking

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- Namely, $\widehat{\alpha}(x) = 0.3$ implies that the (estimated) density of point x is above the (estimated) density of 30% of all observations.

Property of Density Ranking

- For an observation $X_{\max}$ with $\widehat{\alpha}(X_{\max}) = 1$, then it means

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- If an observation X_ℓ satisfies $\widehat{\alpha}(X_\ell) = 0.25$, this means that the ranking of density at X_ℓ is the 25%.
- Moreover, for any pairs of data points X_i, X_j,

$$\widehat{p}(X_i) > \widehat{p}(X_j) \Longrightarrow \widehat{\alpha}(X_i) > \widehat{\alpha}(X_j)$$
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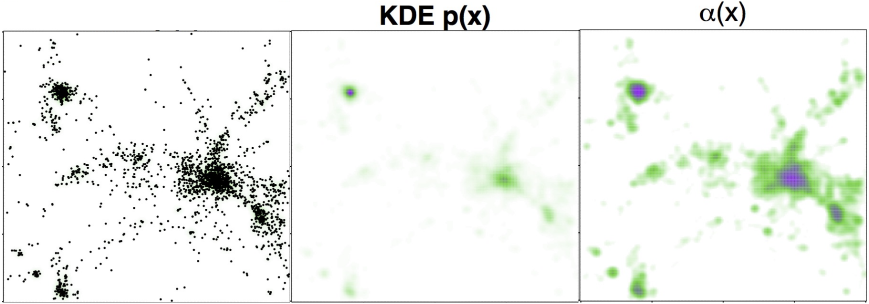
Density Ranking: Astronomy data



Density Ranking as an Estimator

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- When the distribution function has a PDF, the population version of density ranking is defined as:

$$\alpha(x) = P(p(x) \geq p(X_1)).$$

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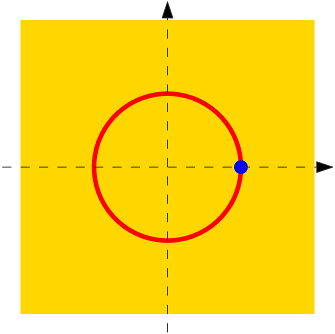
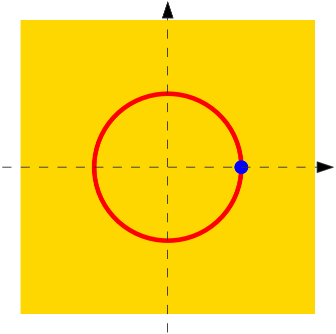
- For a point x, we then define

$$\tau(x) = \max\{s \leq d : \mathcal{H}_s(x) < \infty\}, \quad \rho(x) = \mathcal{H}_{\tau(x)}(x).$$

- Assume the distribution function P is a mixture of a $2D$ uniform distribution within $[-1, 1]^2$, a $1D$ uniform distribution over the ring $\{(x, y) : x^2 + y^2 = 0.5^2\}$, and a point mass at $(0.5, 0)$, then the support can be partitioned as follows:



Geometric Hausdorff: Example - 2



- Orange region: $\tau(x) = 2$.
- Red region: $\tau(x) = 1$.
- Blue region: $\tau(x) = 0$.

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- For two points x_1, x_2, we define an ordering such that $x_1 \succ_{\tau,\rho} x_2$ if

$$\tau(x_1) < \tau(x_2), \qquad \text{or} \quad \tau(x_1) = \tau(x_2), \quad \rho(x_1) > \rho(x_2).$$

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- Namely, we first compare the dimension of the two points, the lower dimensional structure wins. If they are on regions of the same dimension, we then compare the density of that dimension.

Constructing Density Ranking using Hausdorff Density

- Using the ordering $\succ_{\tau,\rho}$, we then define the population density ranking as

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- When the PDF exists, the ordering $\succ_{\tau,\rho}$ equals to $\succ_{d,p}$ so

$$\alpha(x) = P(x \succeq_{d,p} X_1) = P(p(x) \geq p(X_1)),$$

which recovers our original definition.

Convergence under Singular Measure: Density Ranking - 1

- When P is a singular distribution and satisfies certain regularity conditions,
$$\int \left|\widehat{\alpha}(x) - \alpha(x)\right|^2 dP(x) \xrightarrow{P} 0.$$
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- However, the speed of diverging depends on $\tau(x)$. The smaller $\tau(x)$, the faster (actually the diverging rate is $O(h^{\tau(x)-d})$).
- So eventually, we can separate different dimensional structures.

Convergence under Singular Measure: Density Ranking - 2

- Despite the pointwise convergence and convergence in $L_2(P)$, there no guarantee for the uniform convergence $\sup_x \left|\widehat{\alpha}(x) - \alpha(x)\right|$.

Convergence under Singular Measure: Density Ranking - 2

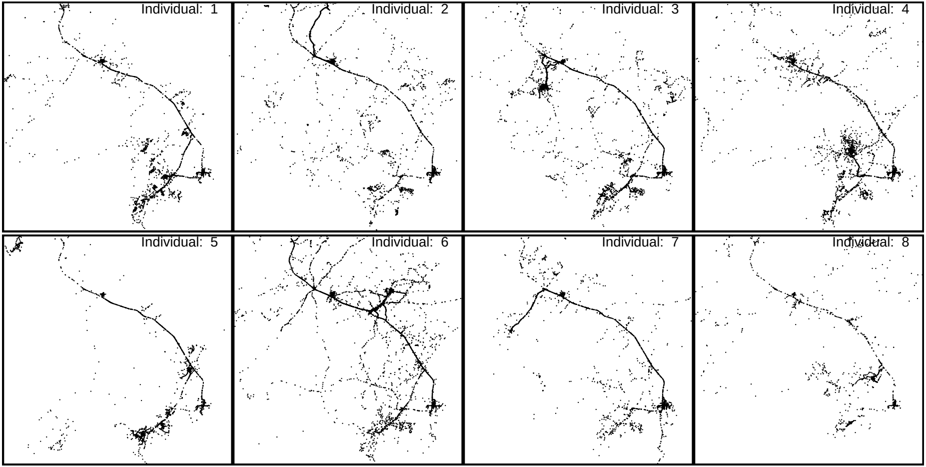
- Despite the pointwise convergence and convergence in $L_2(P)$, there no guarantee for the uniform convergence $\sup_x \left|\widehat{\alpha}(x) - \alpha(x)\right|$.
- Example of non-convergence of supreme norm: consider a sequence of points on a higher dimensional space but moving toward a lower dimensional space within distance $\frac{h}{2}$.

Convergence under Singular Measure: Density Ranking - 3

- A technical remark: $\widehat{\alpha}$ also converges to α is a 'topological' sense.
- More specifically, the cluster tree and persistent diagram of $\widehat{\alpha}$ both converge to the cluster tree and persistent diagram of α in certain metrics.

Density Ranking: Multiple Datasets

Application of Density Ranking: GPS dataset - 1



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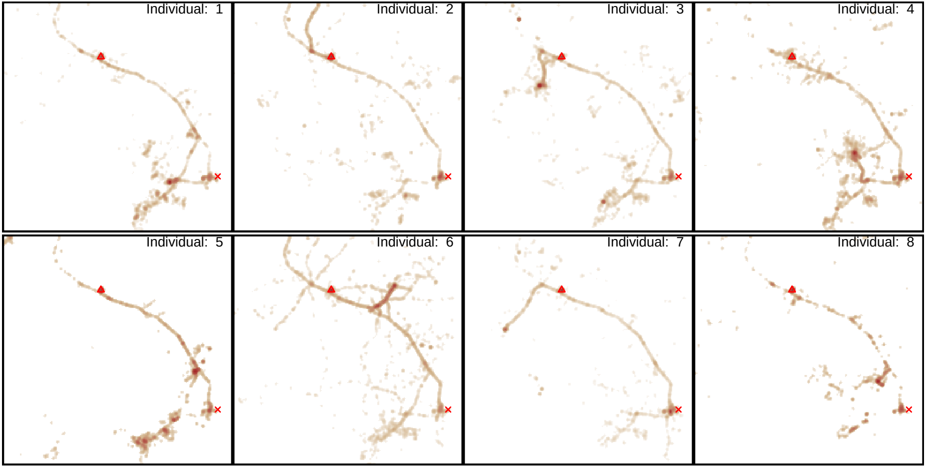

Summarizing Multiple Density Ranking: Level Plots - 1

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- To compare these density rankings, a simple approach is to overlap level plots.
- For a density ranking $\widehat{\alpha}$, let

$$\widehat{A}_\gamma = \{x : \widehat{\alpha}(x) \geq 1 - \gamma\}$$

be the (upper) level set.

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- For a density ranking $\widehat{\alpha}$, let

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  be the (upper) level set.
- We can compare the density ranking of each individual by overlapping their level sets at different levels.

Summarizing Multiple Density Ranking: Level Plots - 2

- Note that we use $1-\gamma$ as the level in the set $\widehat{A}_\gamma$.
- This is because such a set has a natural interpretation in activity space.
- Activity space: the spatial regions where an individual undertakes his/her daily life.

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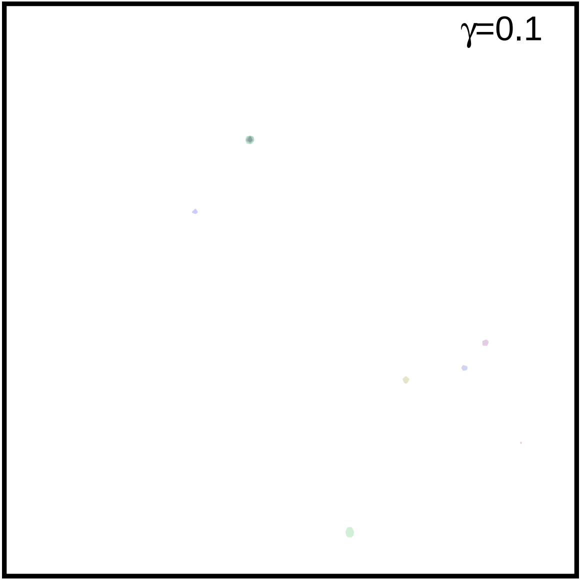
- Activity space: the spatial regions where an individual undertakes his/her daily life.
- We can interpret $\widehat{A}_\gamma$ as the (top) $\gamma \cdot 100\%$ activity space because it contains at least $\gamma \cdot 100\%$ GPS records.
- Namely, $\widehat{A}_{\gamma=0.3}$ is the (top) 30% activity space.

Level Plots: Example



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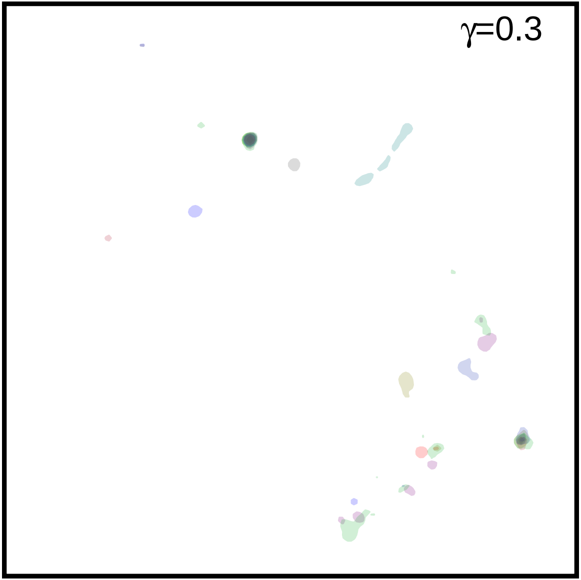

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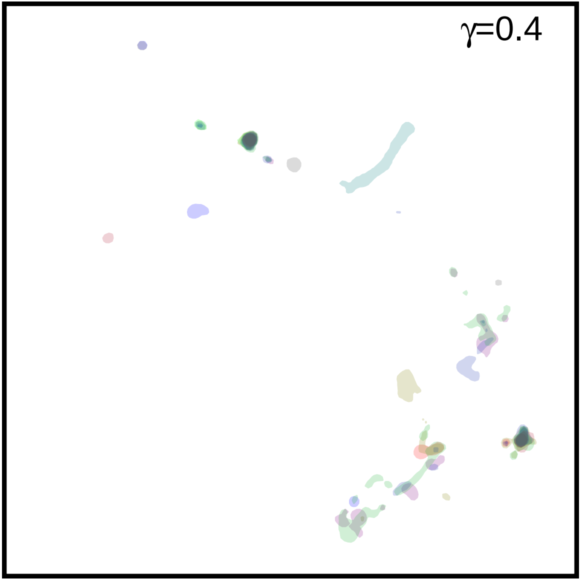

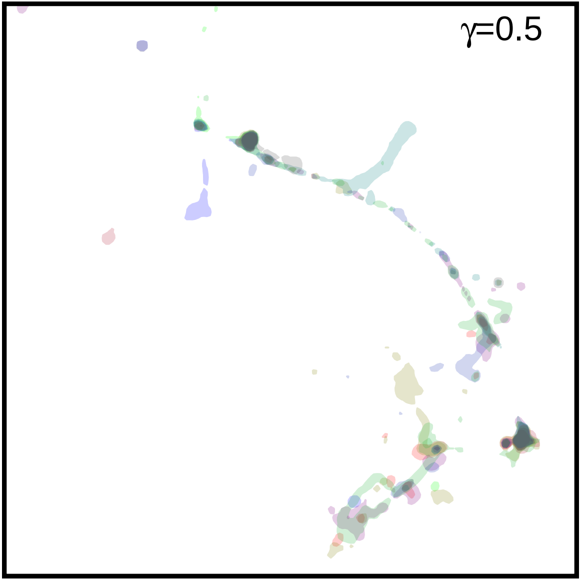
γ=0.5

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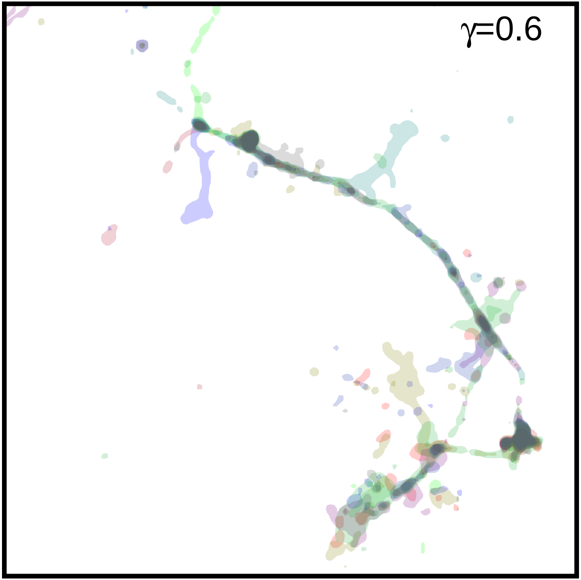

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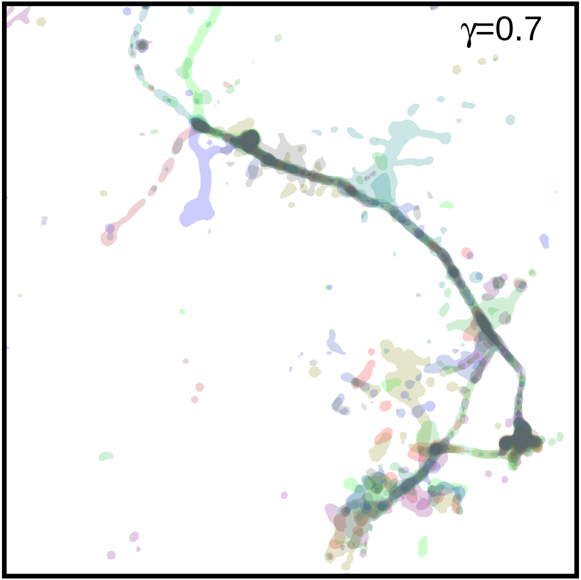

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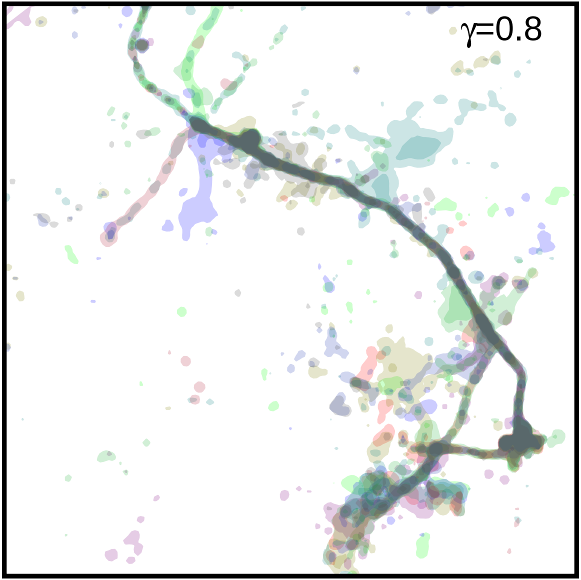

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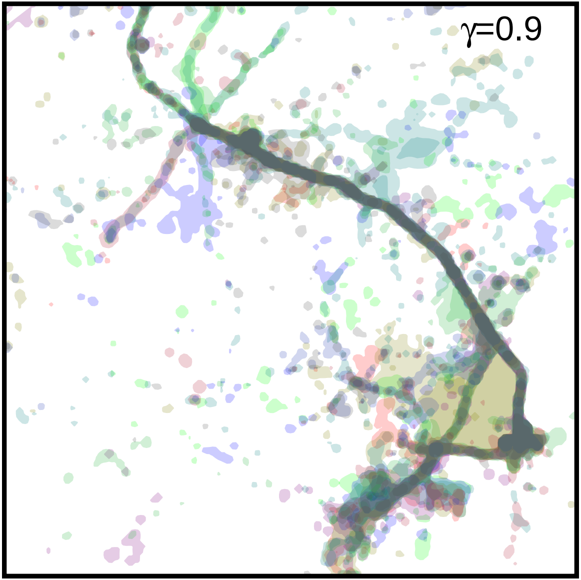

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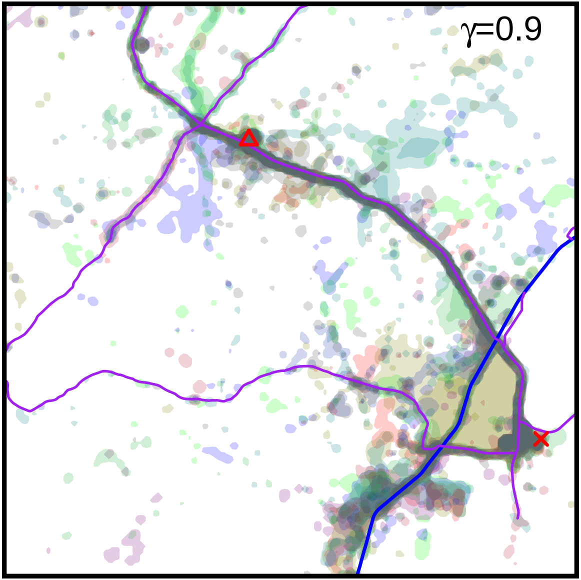

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  - When we have many individuals, this approach might not work (too many contours).
  - We often need to choose a level γ to show the plot but which level to be chosen is unclear.
- Here we introduce a few curves to summarize geometric and topological features of density ranking.

Mass-Volume Curve

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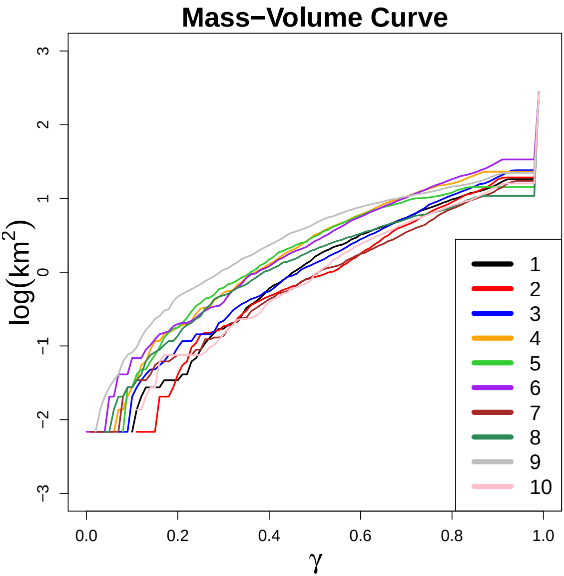
- In practice, we often plot γ versus $\log \mathsf{Vol}(\widehat{A}_\gamma)$.

Mass-Volume Curve: Example



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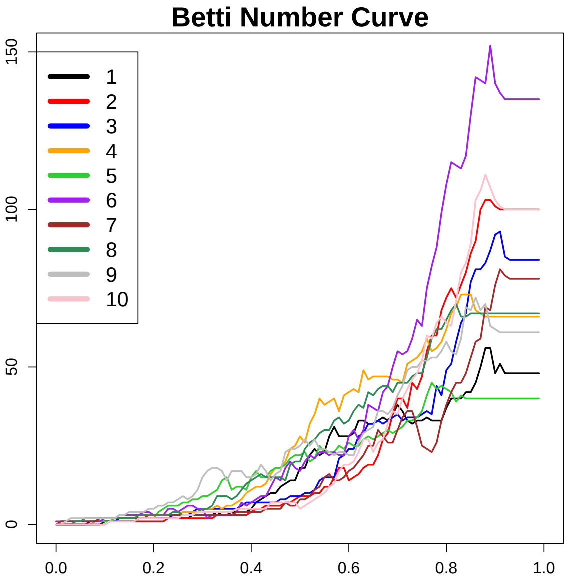
- Note that the number of connected component is called the 0th order Betti number (0th order topological structure); one can generalize this idea to higher order topological structures.

Betti Number Curve: Example



Applying to African Animal Datasets

- Now we come back to our African Animal Datasets.
- We compare their level plots and summary curves.

Level Plots: Animal Example

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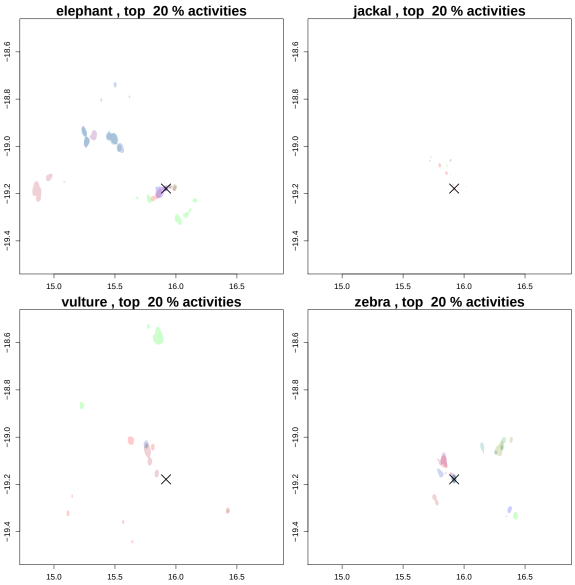
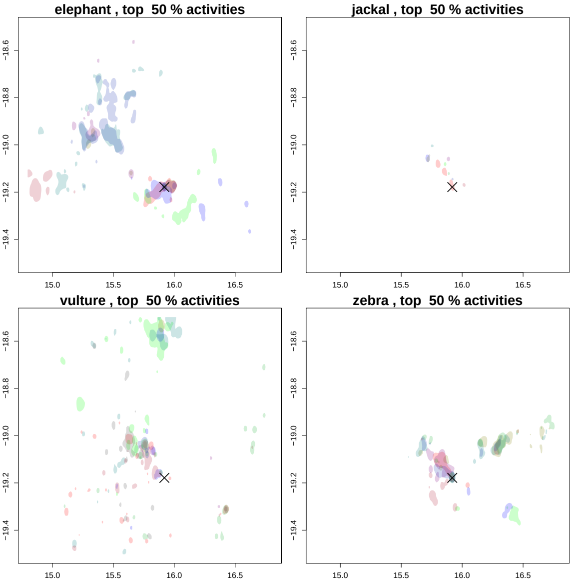
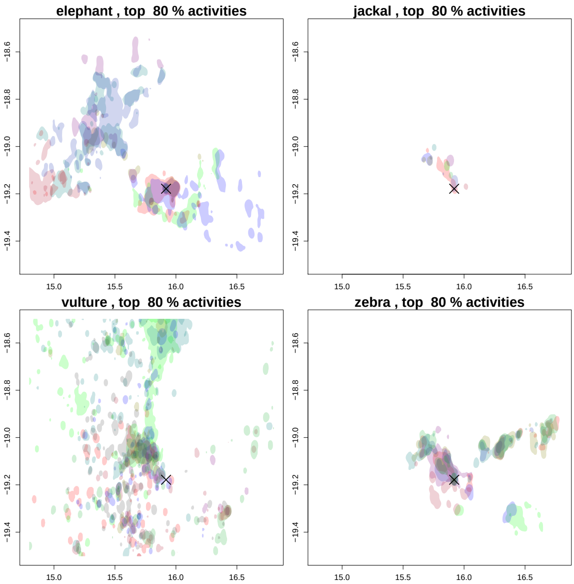


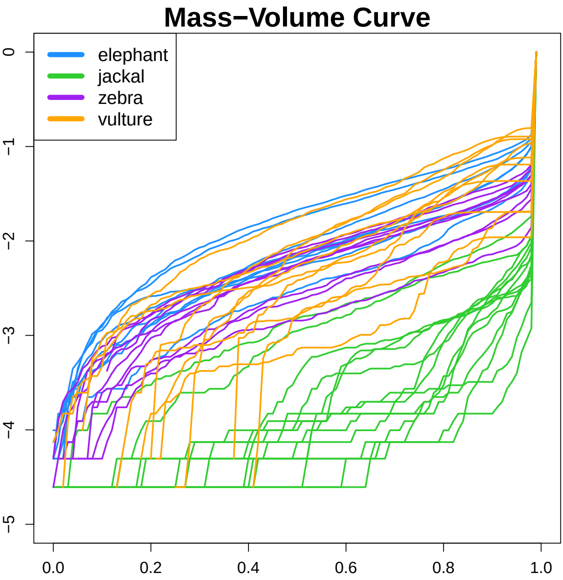
Level Plots: Animal Example



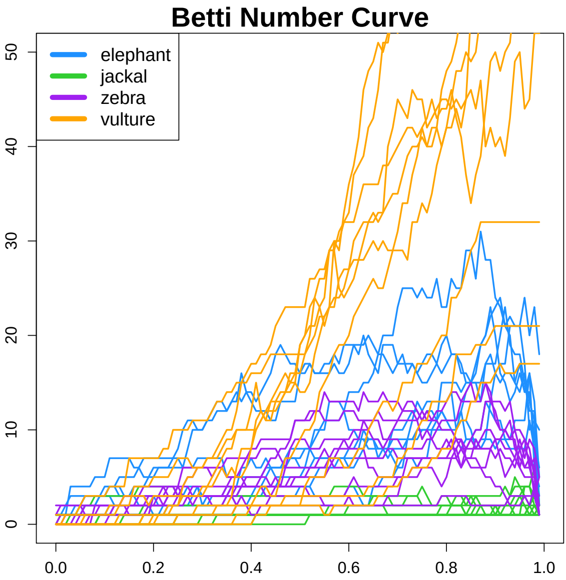
Level Plots: Animal Example



Mass-Volume Curve: Animal Example



Betti Number Curve: Animal Example



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- But we can use density ranking to analyze data.
- This quantity is stable and its estimator is statistically consistent.
- Moreover, the contours of density ranking is related to activity spaces.
- When multiple GPS datasets are available, we can summarize them by functional summaries of density ranking.
- Comparing these summaries gives us more insights about the data.

Future Work

- Sample size needed for GPS data analysis.
- Analysis of multiple summary curves.
- Analyzing activity spaces and covariates using density ranking.
- Measuring environmental exposure using density ranking.

Thank You!

An R script for density ranking:

`https://github.com/yenchic/density_ranking`

More details can be found in

`http://faculty.washington.edu/yenchic/`

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Assumptions for Regular Distributions

- **(R1)** The density function p has a compact support $\mathbb{K}$.
- **(R2)** The density function is a Morse function and is in $\mathbf{BC}^3$.
- **(K1)** The kernel function K is in $\mathbf{BC}^2$ and integrable.
- **(K2)** K satisfies the VC-type class condition.

(K2) Let

$$\mathcal{K}_r = \left\{ y \mapsto K^{(\alpha)}\left(\frac{x-y}{h}\right) : x \in \mathbb{R}^d, |\alpha| = r \right\},$$

where $K^{(\alpha)}$ is the α-th derivative and let $\mathcal{K}_l^* = \bigcup_{r=0}^{l} \mathcal{K}_r$. We assume that $\mathcal{K}_2^*$ is a VC-type class. i.e. there exists constants A, v and a constant envelope b_0 such that

$$\sup_Q N(\mathcal{K}_2^*, \mathcal{L}^2(Q), b_0\epsilon) \leq \left(\frac{A}{\epsilon}\right)^v, \tag{1}$$

where $N(T, d_T, \epsilon)$ is the ϵ-covering number for an semi-metric set T with metric d_T and $\mathcal{L}^2(Q)$ is the L_2 norm with respect to the probability measure Q.

Assumptions for Singular Distributions

(S1) The support can be partitioned into

$$K = K_0 \bigcup K_1 \bigcup \cdots \bigcup K_d,$$

where $K_\ell = \{x \in \mathbb{K} : \tau(x) = \ell\}$.

(S2) There exist $\rho_{\min}, \rho_{\max}$ such that $0 < \rho_{\min} \leq \rho(x) \leq \rho_{\max} < \infty$ for every $x \in \mathbb{K}$.

(S3) Restricted to each $\mathbb{K}_\ell$ where $\ell > 0$, $\rho(x)$ is a Morse function.

(K1') The kernel function K is in $\mathbf{BC}^2$, integrable, and supported in $[-1, 1]$.

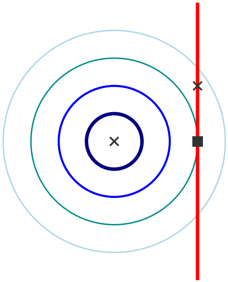
(K2) K satisfies the VC-type class condition.

Dimensional Critical Points - 1

- In singular measure, there is a new type of critical points. We call them the *dimensional critical points*.
- These critical points contribute to the change of topology of level sets as the usual critical points but they cannot be defined by setting gradient to be 0.

Dimensional Critical Points - 2

- The box in the following figure is a dimensional critical point.
- Note: this is a mixture of 2D distribution and a 1D distribution on the black line (maximum value occurs at the cross).



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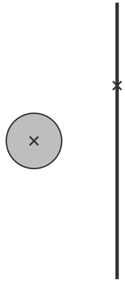

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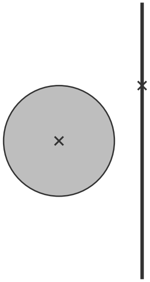

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