Eigen Values and Eigenvectors (Chapter 8)

Definition

Let A be an $n \times n$ matrix. A scalar λ is called an <u>eigenvalue</u> of A if there is a non-zero vector $\vec{v} \neq 0$, called an <u>eigenvector</u>, such that

$$A\vec{v} = \lambda\vec{v}. \qquad (*)$$

See section 8.1 for a motivation for eigenvalues.

How do we compute eigenvalues and eigenvectors?

Rewrite $(*)$ as

$$(A - \lambda I)\vec{v} = 0.$$

We want a non-zero vector $\vec{v}$ that is a solution of this.

What Fundamental matrix subspace does $\vec{v}$ lie in?

$$\vec{v} \in \ker(A - \lambda I).$$

So $(A - \lambda I)$ should have a non-trivial kernel. What does that tell us about

$$\det(A - \lambda I)?$$

Solving the equation

$$\det(A - \lambda I) = 0 \quad \text{characteristic equation.} \quad \{\lambda_1, \dots, \lambda_n\}$$

produces all the eigenvalues. Once these are known, we solve

$$(A - \lambda_1 I)\vec{v}_1 = 0 \qquad \text{for a } \vec{v}_1.$$

$$(A - \lambda_2 I)\vec{v}_2 = 0 \qquad \text{for a } \vec{v}_2.$$

$$\vdots$$

$$(A - \lambda_n I)\vec{v}_n = 0 \qquad \text{for } \vec{v}_n.$$

<u>Ex</u>

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$ find all eigen values and eigenvectors.

<u>Step 1</u> Characteristic equation:

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = (3-\lambda)^2 - 1$$

$$= \lambda^2 - 6\lambda + 8$$

$$= (\lambda - 2)(\lambda - 4)$$

So. $\lambda_1 = 2$
$\lambda_2 = 4$

<u>Step 2</u> Eigenvectors:

We look for solutions of:

$\lambda_1 = 2$

$$\begin{pmatrix} 3-2 & 1 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We obtain $x + y = 0$

set $y = \alpha$

then $\vec{V}_1 = \begin{pmatrix} -\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Every eigenvector for $\lambda_1 = 2$ will be a scalar multiple of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, so we just need a non-zero vector.

Choose $\alpha = 1$,

$\vec{V}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 4$: Short cut:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad -x + y = 0$$

Need ~~a~~ solution, choose $y = 1$ & $x = 1$

$\vec{V}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let's do a more involved example:

Ex

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Step 1 Characteristic Equation:

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

$$= -\lambda \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} - (-1)\det\begin{pmatrix} 1 & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

$$(-1)\det\begin{pmatrix} 1 & 2-\lambda \\ 1 & 1 \end{pmatrix}$$

$$= -\lambda\left((2-\lambda)^2 - 1\right) + (2-\lambda) - 1 - 1 + 2 - \lambda$$

$$= -\lambda^3 + 4\lambda^2 - 3\lambda + 2 - 2\lambda$$

$$= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$

How do we solve for λ? Guess a solution: $\lambda = 1$ works!

Do long division $\rightarrow$ $-(\lambda-1)(\lambda-2)$

$$\begin{array}{r|l} & -\lambda^2 + 3\lambda - 2 \\ \lambda - 1 & -\lambda^3 + 4\lambda^2 - 5\lambda + 2 \\ & -\lambda^3 + \lambda^2 \\ \hline & 3\lambda^2 - 5\lambda \\ & 3\lambda^2 - 3\lambda \\ \hline & -2\lambda + 2 \\ & -2\lambda + 2 \\ \hline & 0 \end{array}$$

So

$$-\lambda^3+4\lambda^2-5\lambda+2 = -(\lambda-1)^2(\lambda-2) = 0$$

$\lambda_1 = 1$
$\lambda_2 = 2$ $\leftarrow$ Most 3×3 matrices have 3 distinct eigenvalues, but here we have just 2. We say that $\lambda_1 = 1$ is a repeated eigenvalue.

Let's find our eigenvectors:

$\lambda_1 = 1$

$$(A-I) = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-x-y-z=0$$

Let $\begin{array}{l} y = \alpha \\ z = \beta \end{array} \Rightarrow \begin{array}{l} x = -\alpha-\beta \\ y = \alpha \\ z = \beta \end{array}$

OR

$$\vec{v}_1 = \begin{pmatrix} -\alpha-\beta \\ \alpha \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

We obtain two, linearly independent eigenvectors:

$$\vec{v}_1^{(1)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_1^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Note: Finding eigenvectors is the same as finding a basis for $\ker(A-\lambda I)$ where λ is an eigenvalue.

Now, for $\lambda_2 = 2$

$$(A - 2I) = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -2 & -1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x + y &= 0 \\ -y + z &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= -y = -z \\ y &= z \end{aligned}$$

Let $z = \gamma$

$$\vec{v}_2 = \begin{pmatrix} -\gamma \\ \gamma \\ \gamma \end{pmatrix} = \gamma \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Set $\gamma = 1$, $\quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

but does this work for the second equation?

$$-2(1) + (-1-i)(-(1-i))$$

$$= -2 + (1+i)(1-i) = -2 + 1 + i - i - i^2$$

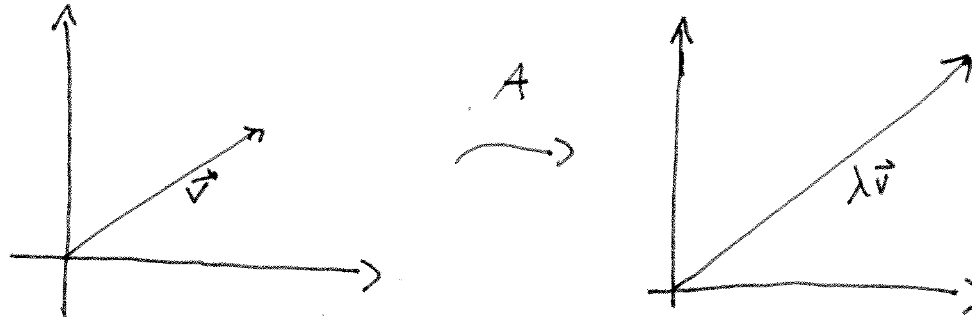
$$= -2 + 2 = 0 \checkmark$$

set $\vec{v}_1 = \begin{pmatrix} 1 \\ i-1 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} 1 \\ -i-1 \end{pmatrix} \leftarrow$ complex conjugate, replace i with $-i$.

A geometric interpretation of eigenvalues:

$$A\vec{v} = \lambda \vec{v}$$



Multiplication by A stretches an eigenvector (if $\lambda > 0$)

<u>A quick aside:</u>

Complex vector spaces: $\mathbb{C}^n$: The space of all $n \times 1$ vectors with complex entries.

$$\vec{w}, \vec{v} \in \mathbb{C}^n \quad \text{iff} \quad c\vec{w} + d\vec{v} \in \mathbb{C}^n \text{ for } c, d \in \mathbb{C}.$$

<u>Theorem</u>: If the $n \times n$ real matrix A has n distinct eigenvalues then the corresponding real eigenvectors form a basis for $\mathbb{R}^n$. If A (maybe not real) has n distinct complex eigenvalues, then the corresponding eigenvectors form a basis for $\mathbb{C}^n$.

What else can happen?

Ex

Consider

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 0 & 0 \end{pmatrix} = \lambda^2 \Rightarrow \lambda = 0 \text{ is repeated.}$$

Find eigenvectors:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x \text{ is arbitrary} \\ y = 0 \end{array}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

But no second eigenvector!

Ex

$$A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) + 2$$
$$= -1 + \lambda^2 + 2 = \lambda^2 + 1 = 0$$

$$\lambda^2 = \pm\sqrt{-1} = \pm i$$

Let's still look for eigenvectors.

$\lambda_1 = i$

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}$$

1st equation: $(1-i)x + y = 0$

choose $x = 1$, $y = -(1-i)$

Definition:

An eigenvalue λ of a matrix A is called complete if the corresponding eigenspace

$V_\lambda = \ker(A - \lambda I)$ has the same dimension as its multiplicity.

Fact: A matrix is complete if and only if the eigenvectors span $\mathbb{R}^n$ (or $\mathbb{C}^n$).

Ex

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \qquad \begin{array}{ll} \lambda_1 = 2, & \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ \lambda_2 = 4, & \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}$$

Complete

A matrix with distinct eigenvalues is complete.

Ex

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad \lambda_{1,2} = 1. \quad \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 2 \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Complete

Ex $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\lambda_{1,2} = 0$ $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Not complete.

Diagonalization

Definition: A square matrix is called diagonalizable if there exists a nonsingular matrix S and a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ such that

$$S^{-1} A S = \Lambda \quad \text{or} \quad A = S \Lambda S^{-1}$$

Theorem: A matrix is complex diagonalizable if and only if it is complete.

A matrix is real diagonalizable if and only if it is complete with real eigenvalues.

Proof

Write $$AS = S\Lambda$$

Let $S = (\vec{v}_1, \dots, \vec{v}_k)$, $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

Then each column of this equation says

$$A\vec{v}_k = \vec{v}_k \lambda_k$$

So $\vec{v}_k$ is an eigen vector and λ_k is an eigenvalue

- If we have the eigenvalues / vectors we can construct S and Λ.

- If we have S and Λ we can pull out columns to find the eigenvectors.

Ex

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Eigenvalues of Symmetric Matrices

Let $K = K^T$ be a real symmetric $n \times n$ matrix. Then

a) All the eigenvalues of K are real.

b) The eigenvectors corresponding to distinct eigenvalues are orthogonal.

c) There is an orthonormal basis consisting of n eigenvectors of K.

Conclusion: Symmetric matrices are complete.

Ex

$$K = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \qquad \lambda_1 = 2 \quad \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \lambda_2 = 4 \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{v}_1 \cdot \vec{v}_2 = 0.$$

Theorem: A symmetric matrix $K = K^T$ is positive definite if and only if all of its eigenvalues are strictly positive.

Proof: 1) Pos def $\Rightarrow$ positive eigenvalues.

$\vec{x}^T K \vec{x} > 0$ for all non-zero x.

Pick an eigenvector $\vec{x}_k$:

$$0 < \vec{x}_k^T K \vec{x}_k = \vec{x}_k^T \lambda \vec{x}_k = \lambda \| x_k \|^2$$

$$\Rightarrow \lambda > 0.$$

2) Positive eigenvalues $\Rightarrow$ pos def.

Assume $\{\lambda_1, .., \lambda_n\}$ are the positive eigenvalues and $\{\vec{u}_1, \dots, \vec{u}_n\}$ are the corresponding orthonormal eigenvectors. (They exist since K is symmetric)

Consider

$$\vec{x}^T K \vec{x} = (c_1\vec{u}_1 + \dots + c_n\vec{u}_n)^T K(c_1\vec{u}_1 + \dots + c_n\vec{u}_n)$$

$$\vec{x} = c_1\vec{u}_1 + \dots + c_n\vec{u}_n$$

$$= (c_1\vec{u}_1 + \dots + c_n\vec{u}_n)^T (c_1K\vec{u}_1 + \dots + c_nK\vec{u}_n)$$

$$= (c_1\vec{u}_1 + \dots + c_n\vec{u}_n)^T (c_1\lambda_1\vec{u}_1 + \dots + c_n\lambda_n\vec{u}_n)$$

$$\vec{u}_i^T\vec{u}_j = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{otherwise} \end{cases}$$

$$= c_1^2\lambda_1 + c_2^2\lambda_2 + \dots + c_n^2\lambda_n > 0$$

since λ_i is positive for all i.

Back to the first 3 claims:

a) All the eigenvalues of K are real:

$$\vec{x}^T K\vec{x} = \vec{x}^T(K\vec{x}) = (K\vec{x})^T\vec{x}$$

This works fine for the case of $\mathbb{R}^n$

For $\mathbb{C}^n$ (which is unavoidable for eigenvectors)

we use

$$\vec{x}^T \overline{\vec{y}} = \vec{x} \cdot \vec{y} \quad \text{for our inner product.}$$

Consider

$$\vec{x}^T \overline{(K\vec{x})} = \vec{x}^T K \overline{\vec{x}} = \vec{x}^T K^T \overline{\vec{x}}$$
$$= (K\vec{x})^T \overline{\vec{x}}$$

Now let $\vec{x}$ be an eigenvector for the eigenvalue λ. $(K\vec{x} = \lambda\vec{x})$

$$\vec{x}^T \overline{(\lambda\vec{x})} = \overline{\lambda}\, \vec{x}^T \overline{\vec{x}} = (K\vec{x})^T \overline{\vec{x}}$$
$$= \lambda\, \vec{x}^T \overline{\vec{x}}$$

I claimed that $\vec{x}^T \overline{\vec{y}}$ is an inner product.

Thus

$$\vec{x}^T \overline{\vec{x}} > 0$$

We obtain

$$\overline{\lambda} = \lambda$$

If $\lambda = \alpha + i\beta$
$\overline{\lambda} = \alpha - i\beta$

$\lambda = \overline{\lambda} \Rightarrow \beta = 0.$

$\Rightarrow \lambda \in \mathbb{R}$!

b) The eigen vectors corresponding to distinct eigenvalues are orthogonal.

$$\vec{X}_1 \longleftrightarrow \lambda_1$$

$$\lambda_1 \neq \lambda_2$$

$$\vec{X}_2 \longleftrightarrow \lambda_2$$

Recall:

$$\vec{X}^T (K \vec{Y}) = (K\vec{X})^T \vec{Y}$$

(Assume $\vec{X}, \vec{Y}$ are in $\mathbb{R}^n$)

Let $\vec{X} = \vec{X}_1$, $\vec{Y} = \vec{X}_2$

$$\Rightarrow \quad \vec{X}_1^T (K \vec{X}_2) = (K\vec{X}_1)^T \vec{X}_2$$

$$\vec{X}_1^T (\lambda_2 \vec{X}_2) = (\lambda_1 \vec{X}_1)^T \vec{X}_2$$

$$\lambda_2 \vec{X}_1^T \vec{X}_2 = \lambda_1 \vec{X}_1^T \vec{X}_2$$

OR

$$(\lambda_2 - \lambda_1) \vec{X}_1^T \vec{X}_2 = 0$$

$$\underset{\text{not zero}}{(\lambda_2 - \lambda_1)} \quad \underset{\text{must be zero.}}{\vec{X}_1^T \vec{X}_2}$$

c) There is an orthonormal basis consisting of n orthonormal eigenvectors of K.

Proof: see text.

The Spectral Theorem

Theorem: Let A be a real, symmetric matrix. Then there exists an orthogonal matrix Q such that

$$A = Q \Lambda Q^{-1} = Q \Lambda Q^T.$$

Remark:

$$A = Q \Lambda Q^T$$

and

$$A = L D L^T$$

are completely different factorizations. The eigenvalues are not the pivots!

$$\Lambda \neq D.$$

Ex

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \qquad \begin{array}{l} \lambda_1 = 2 \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \\ \lambda_2 = 4 \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \end{array}$$

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = Q \Lambda Q^T$$

$$= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = L D L^T$$

Summary:

- A is diagonalizable (complete)

$$A = S \Lambda S^{-1}$$

- A is symmetric

$$A = Q \Lambda Q^{-1} = Q \Lambda Q^{T}$$

Extensions:

- Singular value Decomposition: A is of rank r

$$\underset{m \times n}{A} = \underset{m \times r}{P} \; \underset{r \times r}{\Sigma} \; \underset{r \times n}{Q^{T}}$$

- Jordan canonical form

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\lambda = 0$

$$(A - \lambda I)\vec{V}_1 \quad \Rightarrow \quad \vec{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(A - \lambda I)^2 \vec{V}_2 \Rightarrow (A)^2 \vec{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{V}_2$$

choose $\vec{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (generalized eigenvector)

When a matrix is not complete we can find generalized eigenvectors.

$$A = S J S^{-1}$$

columns are generalized eigenvectors

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$