Section 7.1

Another interpretation of matrix multiplication:

Definition: Let V and W be vectorspaces. A function $L: V \to W$ is called linear if it obeys two basic rules:

$$L[\vec{v}+\vec{w}] = L[\vec{v}] + L[\vec{w}], \quad L[c\vec{v}] = cL[\vec{v}]$$

for all $\vec{v}, \vec{w} \in V$ and c, a scalar.

Ex Let $L[x] = ax+b$. Then $L: \mathbb{R} \to \mathbb{R}$.

Is L linear?

$$L[x_1+x_2] = ax_1 + ax_2 + b$$

$$L[x_1] + L[x_2] = ax_1 + ax_2 + 2b.$$

Not linear!

Matrix representation of linear functions:

Every linear function between two finite-dimensional vector spaces has a matrix representation.

Let $\{\vec{v}_1, \dots, \vec{v}_m\}$ be a basis for V

and $\{\vec{w}_1, \dots, \vec{w}_n\}$ be a basis for W.

We consider

$$L[\vec{v}_1] = \vec{w}_1^* = c_{11}\vec{w}_1 + \dots + c_{n1}\vec{w}_n$$

$$L[\vec{v}_2] = \vec{w}_2^* = c_{12}\vec{w}_1 + \dots + c_{n2}\vec{w}_n$$

$$\vdots$$

$$L[\vec{v}_m] = \vec{w}_m^* = c_{1m}\vec{w}_1 + \dots + c_{nm}\vec{w}_n$$

for a vector $\vec{v} \in V$ such that

$$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_m \vec{v}_m$$

we use $\vec{v} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}$ to denote this vector.

likewise for $\vec{w} = \beta_1 \vec{w}_1 + \dots + \beta_m \vec{w}_m$

$$\vec{w} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow L[\vec{v}_1] = \begin{pmatrix} c_{11} \\ \vdots \\ c_{n1} \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \Rightarrow L[\vec{v}_2] = \begin{pmatrix} c_{12} \\ \vdots \\ c_{n2} \end{pmatrix}$$

Recall that

$$A\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix},$$

returns the 1st column of A and

$$A\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}$$ the second column.

So for a general vector $\vec{v} = \alpha_1\vec{v}_1 + \dots \alpha_m\vec{v}_m \in V$

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & & \ddots & \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix}\begin{pmatrix}\alpha_1\\\vdots\\\alpha_m\end{pmatrix} = \begin{pmatrix}\beta_1\\\vdots\\\beta_n\end{pmatrix}$$

produces the vector

$$\vec{w} = L[\vec{v}] = \beta_1\vec{w}_1 + \cdots + \beta_n\vec{w}_n.$$

This is why we only consider matrices!

Conclusion: L applied to each of the basis vectors gives each column of the matrix.