AMATH 352
Summer 2012

Book: Applied Linear Algebra
by
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Preliminaries:
- Moodle webpage
- Piazza message board – all HW questions.
- Will use Matlab – quick intro as course progresses.

Homework:
- Part written – Part coding
- All code should be uploaded on moodle for each homework.
- Weekly

Grading:
1 midterm
1 final – take-home coding

40% – homework
30% – each exam.

ICL:
- Communications B022 – Drop-in computing
- B027 – Office/Lab Hours

Motivation

Linear algebra is arguably the most algorithmic subject in mathematics. The goal of this course is to obtain a fundamental understanding of the subject so that if we can reduce a harder problem to linear algebra we know how to solve the full problem.

<u>Example</u>: Find ~~the~~ a quadratic polynomial $p(x)$ such that

$$p(-1) = 0$$
$$p(1) = 0$$
$$p'(0) = 0$$

$$p(x) = ax^2 + bx + c$$

$$\left.\begin{aligned} a - b + c &= 0 \\ a + b + c &= 0 \\ 0 + b + 0 &= 0 \end{aligned}\right\} \quad p(x) = x^2 - 1$$

or $p(x) = \alpha(x^2-1)$

for any $\alpha \in \mathbb{R}$

$\in$ means "element of"

The solutions of this problem can be characterized by linear algebra.

The essence of linear algebra is solving linear systems and characterizing when they can be solved.

The solution of linear systems

What is a linear system? We will talk about this in a more abstract setting later but for now we define a linear system to be a system of equations of the form

$$\begin{cases} x + 2y + z = 2 \\ 2x + 6y + z = 7 \\ x + y + 4z = 3 \end{cases} \qquad (1)$$

where no variable are multiplying each other (e.g. xy, zx, x^3, ...).

We know from precalculus/algebra that adding a multiple of one row to another does not change the solution. Multiply row 1 by -2 and add this to row 2:

$$(1) \quad -2r_1 + r_2 \to r_2 \quad \Rightarrow \begin{cases} x + 2y + z = 2 \\ 0 + 2y - z = 3 \\ x + y + 4z = 3 \end{cases}$$

$$-r_1 + r_3 \to r_3 \quad \Rightarrow \begin{cases} x + 2y + z = 2 \\ 0 + 2y - z = 3 \\ 0 - y + 3z = 1 \end{cases}$$

Now we eliminate y from the 3rd equation:

$$\tfrac{1}{2} r_2 + r_3 \to r_3 \qquad \begin{cases} x + 2y + z = 2 \\ 0 + 2y - z = 3 \\ 0 + 0 + \tfrac{5}{2} z = \tfrac{5}{2} \end{cases}$$

This is what we call a triangular system. It can always be solved provided x appers in the first equation, y in the second, ... More precisely, that the elements on the diagonal are non-zero:

$$\begin{cases} z = 1 \\ 2y - 1 = 3 \quad \Rightarrow \quad y = 2 \\ x + 2 \cdot 2 + 1 = 2 \quad \Rightarrow \quad x = -3. \end{cases}$$

This process is called <u>back substitution</u>.

<u>Matrices and Vectors</u>

We now introduce what is essentially just notation to simplify everything that follows.

A <u>matrix</u> is a rectangular array of numbers

$$\begin{pmatrix} 1.1 & 0 & 3 \\ 2 & 1.2 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 1 \end{pmatrix}$$

In general,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

is an $m \times n$ matrix. m rows and n columns.

A row vector is a $1 \times n$ matrix and a column vector is an $m \times 1$ matrix. A general linear system is of the form

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \ddots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_n \end{cases}$$

and we call A the coefficient matrix.

Matrix Arithmetic

We have three basic operations with matrices.

1) matrix addition,
2) scalar multiplication,
3) matrix multiplication.

Matrix Addition

Matrix addition is defined element wise for matrices of the same size.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 4 & 4 \end{pmatrix}$$

Matrix subtraction is defined in the obvious way, or we can just appeal to scalar multiplication below.

In general if A and B are $m \times n$ matrices then

$$A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}, \quad B = (b_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

$$A + B = (a_{ij} + b_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}.$$

Scalar Multiplication

If $\alpha \in \mathbb{R}$ then

$$\alpha A = (\alpha a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Ex.

$$(-3)\begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -9 \\ -12 & -3 \end{pmatrix}$$

Matrix Multiplication

We first define a vector product between a $1 \times n$ row vector and a $1 \times n$ column vector.

$$(a_1 \quad a_2 \ldots a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \sum_{k=1}^{n} a_k b_k$$

Ex

$$(1 \quad 2 \quad 3) \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 3 + 4 + 3 = 10$$

The matrix product $C = AB$ for A being $m \times n$ and B, $n \times p$ is defined using vector products of the rows of A and the columns of B:

$$C = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{\substack{1 \le i \le m \\ 1 \le j \le p}}$$

This is much easier to see through an example:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1/2 & 3 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4\,1/2 & 2\,1/2 \\ 3 & 2 \end{pmatrix}$$

$$3 \times 3 \xleftrightarrow[\text{match}]{} 3 \times 2$$

We can now write linear systems in a much more convenient format:

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\ \vdots \qquad \qquad \vdots \qquad \qquad \ddots \qquad \vdots \qquad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m \end{cases}$$

$$\Rightarrow \quad A x = b \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

and A is the coefficient matrix.

Remarks: Sometimes a convient way to interpet matrix multipliation is by:

$$\underset{m \times n}{A} \underset{n \times p}{B} = A (b_1 \ b_2 \ \cdots \ b_p) = (A b_1 \ A b_2 \ \cdots \ A b_p)$$

where b_j is the jth column of B.

Define the square identity matrix

$$I = \underset{n \times n}{I_n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & 1 & \\ & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Basic Matrix Arithmetic

Matrix Addition:

Commutativity	$A+B = B+A$
Associativity	$(A+B)+C = A+(B+C$
Zero Matrix	$A+0 = A = 0+A$
Inverse	$A+(-A) = 0$

Scalar Multiplication:

Associativity	$c(dA) = (cd)A$
Distributivity	$c(A+B) = cA+cB$
Unit	$1A = A$
zero	$0A = 0$

Matrix Multiplication:

Associativity	$(AB)C = A(BC)$
Distributivity	$A(B+C) = AB+AC$
	$(A+B)C = AC+BC$
Identity	$IA = A = AI$
Zero Matrix	$A0 = 0 = 0A.$

Matlab Intro

- Plotting

```
x = linspace(-1, 1, 300);
plot(x, sin(x));
plot(x, cos(x), '--k');    ← changes plot
hold on                    ← plot together.
plot(x, sin(x));
```

- `x^2.` vs. `x.^2`

  ↑ interprets as `x·x` with matrix multiplication

  ↑ element wise multiplication.

- `x*x` → error

  `x.*x` → element wise.

- A → n x m matrix

  B → m x p matrix

  ⇒ use `A*B` ~~`A.*B`~~

- A → n x n → use `A^2` for the square.

For Loops:

syntax

```
sum = 0;
for j = 1:100
    sum = sum + 1;
end
sum
```

If Statements:

```
sum = 0;
for j = 1:100
    if j > 50
        sum = sum + 1;
    end
end
sum
```

Indexing Matrix Elements

$A \leftarrow n \times n$ matrix $(a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$

$A(i,j)$ returns the element a_{ij}

$A(:,j)$ returns the jth column

$A(i,:)$ returns the ith row

Double loop example:

```
A = rand(10,10);
for i=1:length(A)
    for j=1:length(A)
        if i > j
            A(i,j)=0;
        end
    end
end
```

row > column

$$\begin{pmatrix} & & \\ 0 & & \end{pmatrix}$$

=> upper triangular matrix.

Other commands:

- eye(n) ← identity matrix

Gaussian Elimination - Regular Case.

We replace the system

$$Ax = b$$

with an augmented matrix:

$$M = (A \mid b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Ex For the system

$$\begin{aligned} 10x + 2y &= 3 \\ 4x + y &= 1 \end{aligned} \Rightarrow \left(\begin{array}{cc|c} 10 & 2 & 3 \\ 4 & 1 & 1 \end{array}\right)$$

Ex

$$\begin{aligned} x + 2y + z &= 2 \\ 2x + 6y + z &= 7 \\ x + y + 4z &= 3 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 6 & 1 & 7 \\ 1 & 1 & 4 & 3 \end{array}\right)$$

Let's follow solving this system to point out the key features.

1st pivot

important this is one

$-2r_1 + r_2 \to r_2$: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 1 & 1 & 4 & 8 \end{array}\right)$

$-r_1 + r_3 \to r_3$: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & -1 & 3 & 1 \end{array}\right)$

$\frac{1}{2} r_2 + r_3 \rightarrow r_3$ $\quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 5/2 & 5/2 \end{array}\right)$ third pivot

second pivot

This algorithm (using just addition of rows) is called <u>regular Gaussian elimination</u>. (RGE)

We say a matrix is <u>regular</u> if RGE reduces it to an upper triangular matrix with non-zero digonal entries.

Only a few lines of code are needed to perform RGE on a given matrix, see text for pseudo code.

<u>Elementary Matrices and the LU Factorization</u>:

<u>Definition</u>

The elementary matrix E associated with an elementary row operation for m-rowed matrices is the matrix obtained by applying the row operation to the $m \times m$ identity matrix I_m.

- Two times the first row to the second:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad -2r_1 + r_2 \rightarrow r_2 : \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_1$$

This is convenient since this has the same effect as applying this operation to a given matrix:

$$E_1 \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} \checkmark$$

We now continue this example. The other operations needed are

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$$

so that

$$E_3 E_2 E_1 \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5/2 \end{pmatrix}.$$

$\leftarrow$ upper triangular matrix.

<u>Claim</u>:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = I$$

$\vdots$

Thus

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{M_1} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{M_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix}}_{M_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5/2 \end{pmatrix}$$

Then (Check this!)

$$M_1 M_2 M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1/2 & 1 \end{pmatrix} \leftarrow \text{Lower triangular.}$$

<u>Definition</u> A <u>special lower (upper) triangular</u> matrix is an $n \times n$ matrix that is lower (upper) triangular with ones on the diagonal.

This property is preserved under matrix multiplication.

$$\underset{\substack{\uparrow \\ \text{(special) lower triangular}}}{L_1 \, L_2} = L_3 \leftarrow \text{(special) lower triangular}$$

$$\underset{\substack{\uparrow \\ \text{(special) lower triangular}}}{U_1 \, U_2} = U_3 \leftarrow \text{(special) upper triangular.}$$

<u>Theorem</u> A matrix A is regular if and only if it can be factored

$$A = LU$$

where L is special lower triangular.

Ex Compute the LU factorization of

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}$$

Negative of the factors used in GE

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 0 & -3 & 2 \end{pmatrix}$$

$$\longrightarrow \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3/2 & 1 \end{pmatrix}}_{L}, \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}}_{U}$$

Remark: The LU factorization is Gassian Elimination with book-keeping.

Applications of LU factorization:

1) Forward and Backward substitution

Solve
$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3/2 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\begin{pmatrix} u \\ v \\ w \end{pmatrix}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

First solve

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3/2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$u = 1$$
$$u + v = 0 \Rightarrow v = -1$$
$$2u + 3/2\, v + w = 0 \Rightarrow w = -\frac{1}{2}$$

Then solve

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1/2 \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$z = \frac{1}{8}$$
$$-2y + 4z = -1 \Rightarrow y = 3/4$$
$$x + 2y - z = 1 \Rightarrow x = -3/8$$

2) Determinants (to come).

Permutations and Pivoting:

Problem:

Compute LU factorization of

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

We fail immediately.

To deal with this complication we add a new technique: the row interchange. Before we consider factorization let's try to solve equations.

$$\begin{aligned} y - z &= 1 \\ x + 2y - 2z &= 0 \\ x + y + z &= 0 \end{aligned} \rightarrow \left(\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & 2 & -2 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

- Interchange first two rows

$$\begin{aligned} r_1 &\to r_2 \\ r_2 &\to r_1 \end{aligned} : \left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$$-r_1 + r_3 \to r_3 : \left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 3 & 0 \end{array}\right)$$

$$r_2 + r_3 \to r_3 : \left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

Back substitution gives $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3/2 \\ 1/2 \end{pmatrix}$

<u>Definition</u>: A square matrix is called non-singular if it can be reduced to upper triangular form with non-zero elements on the diagonal — the pivots — by swapping row and adding rows.

Theorem The system $Ax = b$ has a unique solution x for every choice of b if and only if A is nonsingular.

Generalizing the LU factorization:

How do we represent row interchanges as matrix multiplication? Let's apply the row interchange to the identity matrix as we did before:

$$\begin{matrix} r_1 \to r_2 \\ r_2 \to r_1 \end{matrix} : \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = P_1$$

(Check this!)

$$\Rightarrow \quad P_1 \underbrace{\begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & -2 \\ 1 & 1 & 1 \end{pmatrix}}_{A} = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \checkmark$$

has LU factorization!

We find

$$P_1 A = LU \quad \leftarrow \text{permuted LU factorization.}$$

But how to compute?

Ex:

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & -1 \\ -3 & -5 & 6 & 1 \\ -1 & 2 & 8 & -2 \end{pmatrix}$$

1)

$$\tilde{U} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 4 & 7 & -2 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Need non-zero pivot

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2)

$$\tilde{u} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 4 & 7 & -2 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Switch same rows

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3)

$$\tilde{u} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -5 & -6 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4) $\mathcal{U} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Then (check this!)

$$\underline{PA = LU}$$

How do we use this to solve

$$Ax = b \, ?$$

1) Premultiply by P:

easy to compute.

$$\underbrace{PA}_{LU} x = Pb$$

$$L\mathcal{U}x = Pb \quad (\mathcal{U}x = w)$$

2) Solve $\quad Lw = Pb \quad$ by Forward substitution.

3) solve $\quad \mathcal{U}x = w \quad$ by Back substitution.

Matrix Inverses

Definition Let A be an $n \times n$ matrix. The matrix X $(n \times n)$ satisfying

$$XA = I = AX$$

is called the inverse of A, denoted $X = A^{-1}$.

Ex: 2×2 case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

Solve

$AX = I$ for x, y, z, w.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} ax + bz &= 1 \\ cx + dz &= 0 \end{aligned} \Rightarrow \begin{aligned} x + \tfrac{b}{a} z &= \tfrac{1}{a} \\ x + \tfrac{d}{c} z &= 0 \end{aligned} \Rightarrow \begin{aligned} x + \tfrac{b}{a} z &= \tfrac{1}{a} \\ 0 + \left(\tfrac{d}{c} - \tfrac{b}{a}\right) z &= -\tfrac{1}{a} \end{aligned}$$

So,

$$z = -\frac{1}{a} \cdot \frac{1}{\frac{d}{c} - \frac{b}{a}} = -\frac{c}{ad - bc}$$

$$x = \frac{1}{a} - \frac{b}{a}\left(-\frac{c}{ad - bc}\right) = \frac{1}{a}\left(\frac{ad - \cancel{bc}}{ad - bc} + \frac{\cancel{bc}}{ad - bc}\right)$$
$$= \frac{d}{ad - bc}$$

Continuing, we find

$$Y = -\frac{b}{ad-bc}, \quad W = \frac{a}{ad-bc}.$$

Thus

$$X = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

A^{-1} exists whenever $ad-bc \neq 0$. We define the determinant

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc.$$

We will generalize this to higher-dimensions later.

<u>Recall</u>: A square matrix is non-singular if it has a $PA = LU$ factorization.

<u>Theorem</u>: A square matrix has an inverse if and only if it is non-singular.

<u>Properties of Matrix inverses</u>

- The inverse, if it exists is unique.
- If A^{-1} exists then $(A^{-1})^{-1} = A$.
- $(AB)^{-1} = B^{-1}A^{-1}$

Finding the inverse of the 2×2 was a bit cumbersome. A process called Gauss-Jordan elimination allows one to find the inverse (if it exists) in a different way.

Represent $\quad X = (\vec{x}_1, \vec{x}_2 \dots \vec{x}_n)$

and $\quad I = (\vec{e}_1, \vec{e}_2 \dots \vec{e}_n)$

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots$$

Then

$$AX = I \iff (A\vec{x}_1 \; A\vec{x}_2 \dots A\vec{x}_n) = (\vec{e}_1 \; \vec{e}_2 \dots \vec{e}_n)$$

We obtain n equations

$$A\vec{x}_i = \vec{e}_i, \quad i = 1, \dots, n.$$

Q: Can we solve all of these equations at once? Yes!

We consider the augmented matrix

$$(A \mid \vec{e}_1 \; \vec{e}_2 \cdots \vec{e}_n)$$

and perform row operations to reduce this to

$$(\vec{e}_1 \; \vec{e}_2 \cdots \vec{e}_n \mid X) \Rightarrow X = A^{-1}.$$

Ex Find the inverse of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

Step 1: Augment

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

Step 2: Row operations $\rightarrow$ upper triangular

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right)$$

Step 3: Row Operation: Scalar mutiplication $\rightarrow$ 1's on diagonal.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right)$$

Step 4: Eliminate upper triangular elements

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -1/2 & 0 & 1/2 \end{pmatrix}.$$

Back to the Permuted LU factorization

ex

$$A = \tilde{L}\,\hat{\mathcal{U}} \qquad \tilde{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \hat{\mathcal{U}} = \begin{pmatrix} 1 & 1 & 5 \\ 0 & 0 & 4 \\ 0 & 2 & 3 \end{pmatrix}$$

$$P : \begin{matrix} r_2 \to r_3 \\ r_3 \to r_2 \end{matrix}$$

Want

$$\cancel{A = \tilde{L}\, \underbrace{P\hat{\mathcal{U}}}_{\mathcal{U}}}$$ ← but this is not true

use matrix inverse

$$A = \tilde{L}\, P^{-1} \underbrace{P\hat{\mathcal{U}}}_{\mathcal{U}}$$ ← this is true but what is $\tilde{L}P^{-1}$?

$$\tilde{L}P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$ ← not lower triangular!

What about

$$P\tilde{L}P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ← lower triangular

$$PA = P\tilde{L}P^{-1}\, P\hat{\mathcal{U}}$$
$$= L\,\mathcal{U}$$

Remarks: When it comes to numerical analysis we almost never compute the inverse of a matrix. The LU factorization is as good as having the inverse but requires less work to find!

LDV factorization

This is a small extension of LU factorization.

Theorem: A matrix is regular if and only if it admits a factorization

$$A = LDV.$$

L - special lower tri
U - special upper tri

How does this work?

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & -2 & 0 \end{pmatrix} \mapsto LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We further factor U:

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{V \\ \text{"rename"}}}$$

$A = LDV$

The same can be done for

$$PA = LDU$$

in the non regular but non-singular case.

Transposes:

If $A = (a_{ij})_{\substack{1\le j\le n \\ 1\le i\le m}}$ is an $n \times m$ matrix

then $A^T = (a_{ji})_{\substack{1\le i\le m \\ 1\le j\le n}}$ is an $m \times n$ matrix.

This is called the transpose of the matrix A, it is an interchange of rows and columns.

- $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v^T = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$

- If A is square

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$, $(cA)^T = c\,A^T$
- $(AB)^T = B^T A^T$ (like inverses)

In MATLAB:

```
A = rand(3,3);
B = A'; % B is the transpose of A.
```

Counting Arithmetic Operations:

I previously mentioned that in practice we never compute the inverse matrix when solving

$$A x = b.$$

We now examine the computation complexity of some algorithms. This will clarify why we don't use the matrix inverse.

If we multiply an $n \times n$ matrix A and an $n \times 1$ vector v each component of the resulting vector requires n multiplications and $n-1$ additions

$$x = A v \qquad x_i = \sum_{j=1}^{n} A_{ij} v_j$$

We must do this n times, once for each entry.

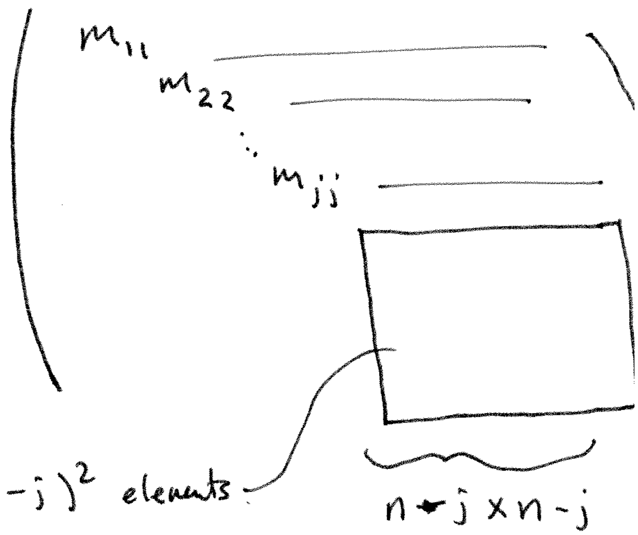
n^2 multiplications and $n(n-1)$ additions.

Gaussian Elimination

- Assume the regular case
- At row j we must eliminate entries in the $n-j$ rows below it
- This involves one division

$$m_{ij}/m_{jj}$$

for each of the $n-j$ row



- Each of the $(n-j)^2$ elements an addition and a multiplication

$$\sum_{j=1}^{n} (n-j)(n-j+1) = \frac{n^3-n}{3} \quad \text{multiplications}$$

$$\sum_{j=1}^{n} (n-j)^2 = \frac{2n^3-3n^2+n}{6}$$

additions

RGE

```
Start
  for j=1 to n
    if m_jj = 0 stop
    else for i=j+1 to n
      set l_ij = m_ij/m_jj
      add -l_ij times row j
        to row i
    next i
  next j
end
```

This is the same number of operations required to compute the LU factorization. Let's examine forward and backward substitution.

$$Lc = b \qquad (L \text{ special lower tri}).$$

$$c_j = b_j - \underbrace{\sum_{k=1}^{j-1} l_{jk} c_k}_{\substack{j-1 \text{ additions} \\ j-1 \text{ multiplications}}}$$

$$\sum_{j=1}^{n} (j-1) = \frac{n^2-n}{2} \quad \left\{ \begin{array}{l} \text{additions and} \\ \text{multiplications} \end{array} \right.$$

Solving

$$Ux = c$$

requires n additional divisions

$$x_j = \frac{1}{u_{jj}} \left(c_j - \sum_{k=j+1}^{n} u_{jk} x_k \right).$$

$$\Rightarrow \left\{ \begin{array}{ll} \dfrac{n^2+n}{2} & \text{multiplications} \\[2ex] \dfrac{n^2-n}{2} & \text{additions} \end{array} \right.$$

<u>Computing A^{-1}</u>

To find A^{-1} we must solve

$$Ax = e_i, \quad i = 1, \dots, n$$

To do this we must compute LU and then perform forward and Backward substitution n times.

We summarize the findings in the table.

Algorithm	Operations	Leading Coefficient
Matrix-Vector multiplication	M: n^2 A: $n^2 - n$	n^2
RGE	M: $\frac{n^3-n}{3}$ A: $\frac{2n^3-3n^2+n}{6}$	$\frac{1}{3}n^3$
Forward Subs	M: $\frac{n^2-n}{2}$ A: $\frac{n^2-n}{2}$	$\frac{1}{2}n^2$
Back Subs	M: $\frac{n^2+n}{2}$ A: $\frac{n^2-n}{2}$	$\frac{1}{2}n^2$
A^{-1}	RGE + n x (Forward + Backward)	$\frac{1}{3}n^3 + n\left(\frac{1}{2}n^2 + \frac{1}{2}n^2\right)$ $= \frac{4}{3}n^3$
Solving $Ax=b$ w/ RGE	RGE + 1 x (Forward + Backward)	$\frac{1}{3}n^3$
Solving $Ax=b$ with A^{-1}	RGE + n x (Forward + Backward) + 1 x Matrix multiplication	$\frac{4}{3}n^3$

Assume we know A^{-1} and $A = LU$

Solving $LUx = b$ $\sim n^2$ operations

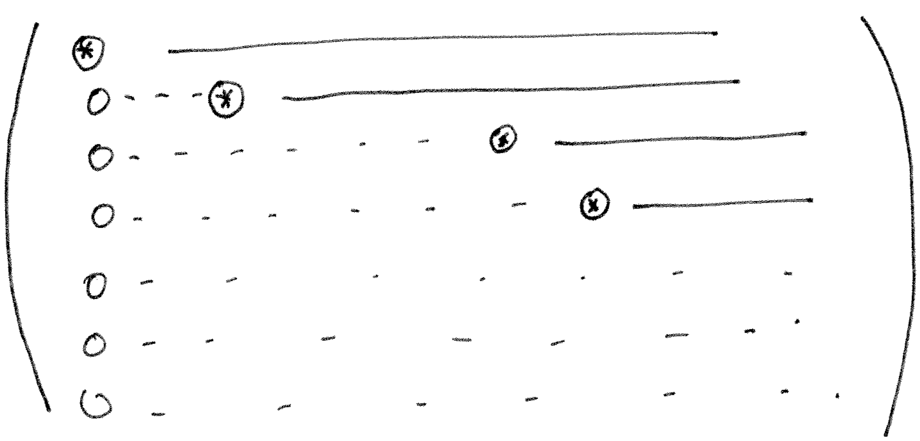
Computing $x = A^{-1}b$ $\sim n^2$ operations

From a computing perspective LU is just as good as having A^{-1}!

Row echelon Form

Definition An $m \times n$ matrix is said to be in row echelon form if it has the form

$U =$



⊛ are the pivots.

- Any matrix can be reduced to this form by row operations.
- The rank of a matrix is the number of Pivots.
- An $n \times n$ matrix is non-singular if and only if its rank is equal to n.

Important Theorem:

A system $Ax = b$ of m linear equations has either

1) exactly one solution
2) infinitely many solutions
3) no solution

- Cannot have exactly two solutions.

Ex

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 2 & 2 & 1 & b \\ 3 & 2 & 2 & c \end{array}\right)$$

$-2r_1 + r_2 \to r_2$
$-3r_1 + r_3 \to r_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 2 & -1 & b-2a \\ 0 & 2 & -1 & c-3a \end{array}\right)$$

$-r_2 + r_3 \to r_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 2 & -1 & b-2a \\ 0 & 0 & 0 & c-b-a \end{array}\right)$$

If $c-b-a \neq 0 \Rightarrow$ no solution.

If $c-b-a = 0$
we have

$$x + z = a$$
$$2y - z = b - 2a$$

a solution is

$$x = a - \alpha$$
$$y = \frac{1}{2}(b - 2a + \alpha)$$
$$z = \alpha$$

for any real number α.

Determinants

The <u>determinant</u> of a square matrix is the uniquely defined scalar quantity that satisfies

1) Adding a multiple of one row to another doesn't change the determinant. $(2r_1 + r_2 \to r_2)$
2) Row interchanges change the sign.
$$r_1 \to r_2$$
$$r_2 \to r_1$$
3) Multiplying a row by a scalar (including zero) multiplies the determinant by that scalar.

4) The determinant of an upper triangular matrix is equal to the product of its diagonal entries

$$\det U = u_{11} u_{22} \cdots u_{nn}.$$

Lemma:

Any matrix with an all zero row has zero determinant.

Facts:

- If $A = LU$

$$\text{then } \det A \underset{\substack{\uparrow \\ \text{adding rows}}}{=} \det U = u_{11} u_{22} \cdots u_{nn}$$

- If $PA = LU$ and P uses k row interchanges

$$\det A = (-1)^k \det U = \det P \det U$$

- A is non singular if and only if $\det A \neq 0$

Ex Compute the determinant of

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 4 & 0 & 0 \end{pmatrix}$$

$-2r_1 + r_2 \to r_2$

$-4r_1 + r_3 \to r_3$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & -2 \\ 0 & -4 & -12 \end{pmatrix} = U_1 \qquad \det U_1 = \det A$$

$r_2 \to r_3$

$r_3 \to r_2$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & -4 & -12 \\ 0 & 0 & -2 \end{pmatrix} = U_2 \qquad \det U_2 = -\det U$$

$$= -\det A$$

$$\det U_2 = 8$$

$$\Rightarrow \boxed{\det A = -8}$$

<u>Other facts</u>:

- $\det A^{-1} = \dfrac{1}{\det A}$

- $\det A^T = \det A$ ← determinants of lower triangular matrices.

- $\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} a_{22} - a_{21} a_{12}$

- $\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$

$$- a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$+ a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

- If I ask you to compute the determinant of a 4×4 or higher, you should look for a trick.