

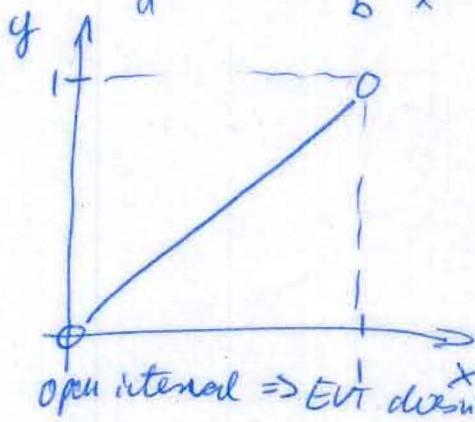
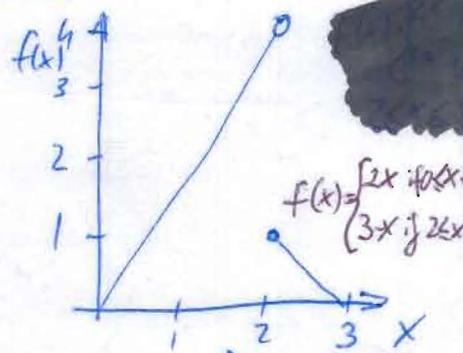
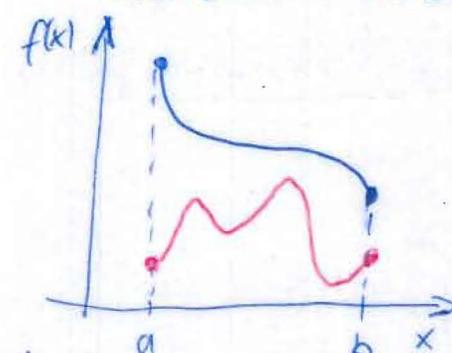
Chapter 5 Applications of Differentiation \rightarrow Optimization

Global maximum/minimum: If function f is defined on D and $c \in D$, then f has a global max/min at $x=c$,

if $f(c) \geq f(x) \forall x \in D$
 $f(c) \leq f(x) \forall x \in D$.

Extreme value Theorem:

If function f is continuous on $[a, b]$, with $-\infty < a < b < +\infty$, then f has a global max & min on $[a, b]$.



\Rightarrow discontinuous (has no global max! but it has a global min)

open interval \Rightarrow EVT doesn't apply

Local Extremes

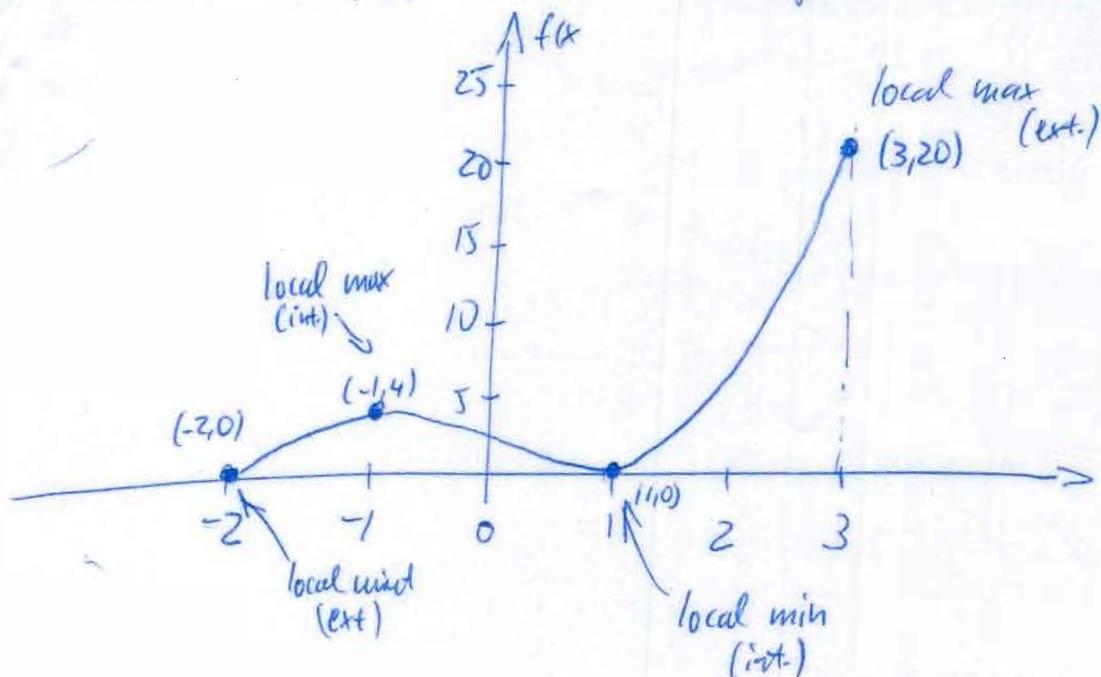
f has a local max at $x=c$ if there exist a $\delta > 0$ such (depend on δ)

that: $f(c) \geq f(x) \forall x \in (c-\delta, c+\delta) \cap D$

f has a local min at $x=c$ if there exists a $\delta > 0$ such

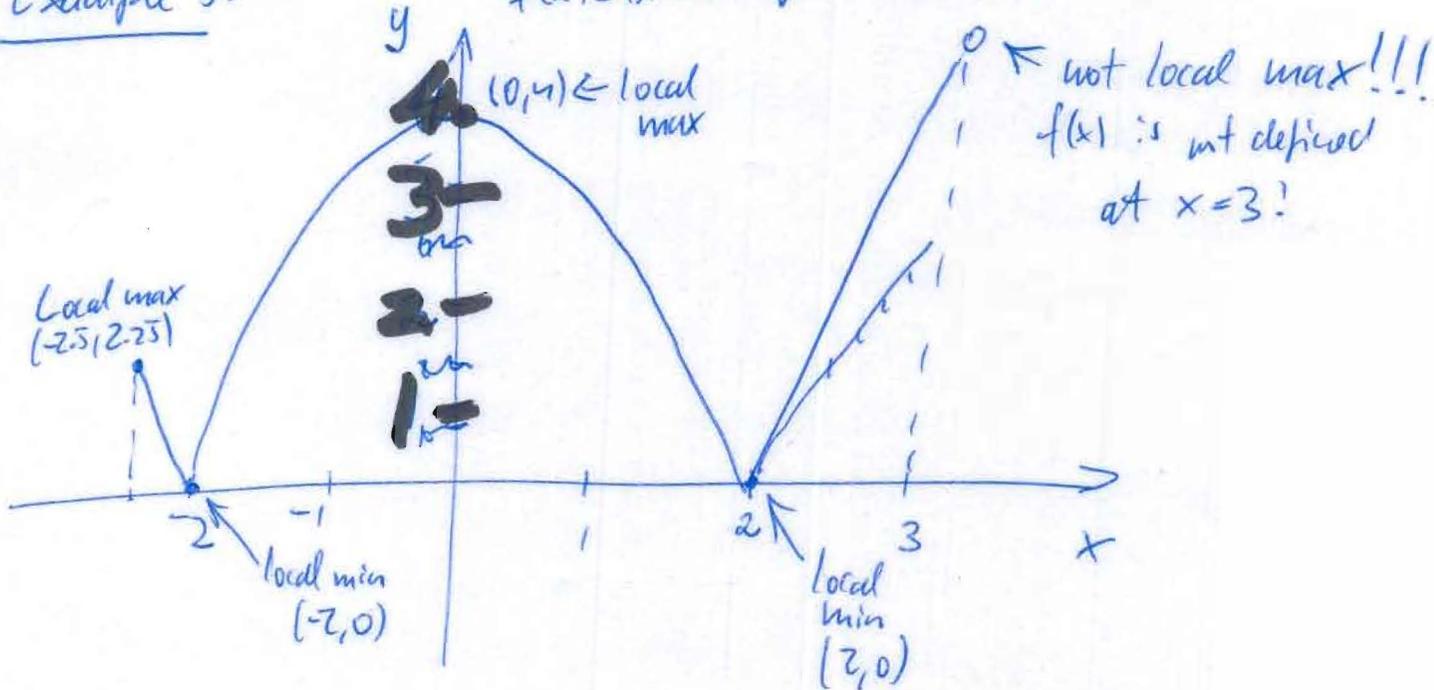
that: $f(c) \leq f(x) \forall x \in (c-\delta, c+\delta) \cap D$

Example 4: $f(x) = (x-1)^2(x+2)$ for $-2 \leq x \leq 3$



Example 5:

$f(x) = |x^2 - 4|$ for $-2.5 \leq x < 3$



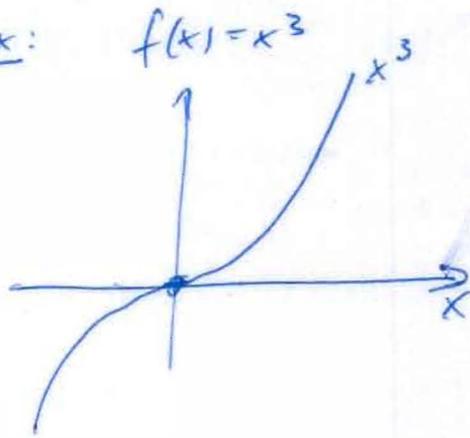
$f(x)$ is not defined on $[-2.5, 3]$ (It is defined on $[-2.5, 3)$, and hence EVT doesn't apply: While it has a global min, it doesn't have a global max.

Fermat's Theorem: If f has a local extremum at an interior point c , & $f'(c)$ exists, then $f'(c) = 0$.

Example 6: Why doesn't $\tan x$ have a local extremum at $x=0$?
 $\frac{d}{dx} \tan x = \sec^2 x$, with $\frac{d}{dx} \tan x \Big|_{x=0} = \sec^2 0 = 1 \Rightarrow$ not a local extremum

• Fermat's Theorem is only a necessary but not sufficient condition for local extrema.

Counter Ex:



$$f'(x) = 3x^2$$

$$\frac{d}{dx} x^3 \Big|_{x=0} = 3 \cdot 0^2 = 0 \text{ BUT}$$

$x=0$ is not a local ext.

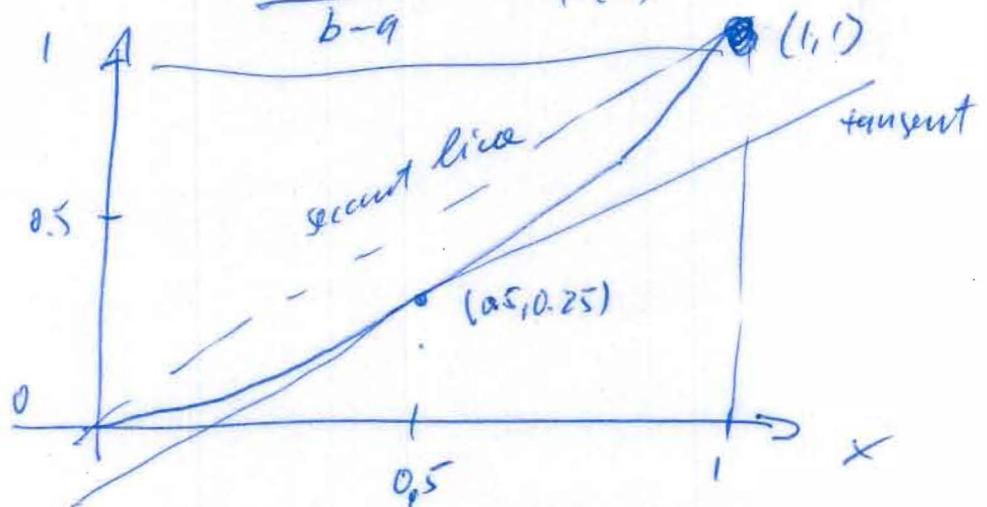
Fermat's Theorem only provides candidates for local ext.

• For Local Extrema Candidates, we also have to look at points where the function is not differentiable. E.g.: $x=2, 2$ in Ex 5.

• Check the endpoints!

The Mean Value Theorem = If f is cont on $[a,b]$ and differentiable on (a,b) , then there exists at least one number $c \in (a,b)$ such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



Monotonicity: Function f is strictly increasing on interval I if

$$f(x_1) < f(x_2) \quad \forall x_2 > x_1$$

strictly decreasing

$$f(x_1) > f(x_2) \quad \forall x_1 > x_2$$

} if } the f is monotonic!

5.3 Extrema, Inflection Points

Candidates for Local Extrema:

- Find all numbers c where $f'(c) = 0$
- Find all numbers c where $f'(c)$ doesn't exist.
- Find the end points of the domain of f .

Example:

$$f(x) = |x^2 - 4| \quad \text{for } -3 \leq x \leq 2.5$$

⇓

$$f(x) = \begin{cases} x^2 - 4 & \text{for } 2.5 > x \geq 2 \text{ and for } -3 \leq x \leq -2 \\ -x^2 + 4 & \text{for } -2 \leq x \leq 2 \end{cases}$$

- Find all numbers c where $f'(c) = 0$:

$$2x = 0$$

and

~~$$-2x = 0$$~~

$$-2x = 0$$

$$x = 0$$

$$x = 0$$

but NOT OK since

OK, since

$$0 \notin [3, -2] \cup [2, 2.5]$$

$$x = 0 \in [-2, 2]$$

$x = 0$ is a candidate $(0, 4)$

- Find all numbers where $f'(c)$ is undefined:

~~$$f(x) = \lim_{x \rightarrow -2^-} (2x) = -4 \neq \lim_{x \rightarrow -2^+} (-2x) = 4 = f(x)$$~~

$$f'(x) \in \mathbb{Q} \text{ at } x = -2$$

$x = -2 \rightarrow \underline{(-2, 0)}$ is a candidate

$$f'(x) = \lim_{x \rightarrow 2^-} (-2x) = -4 \neq \lim_{x \rightarrow 2^+} (2x) = 4$$

$$f'(x) \in \mathbb{Q} \text{ at } x = 2$$

$x = 2 \rightarrow \underline{(2, 0)}$ is a candidate

Find the endpoints of the domain of f :

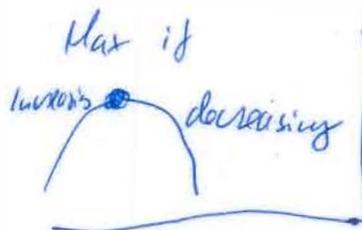
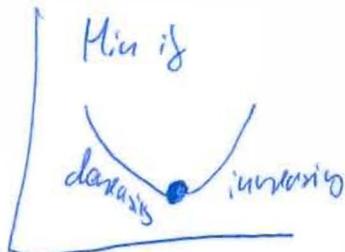
$x = -3 \Rightarrow f(x) = 5$ $x = -3 \in [-3, 2.5]$
 $(-3, 5)$

\Downarrow
 $(-3, 5)$ is a candidate

$x = 2.5 \Rightarrow$ is not candidate because $2.5 \notin [-3, 2.5]$

Local Extrema:
Candidates

- $(0, 4)$
- $(-2, 0)$
- $(2, 0)$
- $(-3, 5)$



$$f'(x) = \begin{cases} 2x & \text{for } 2 \leq x < 2.5 \text{ \& } -3 \leq x \leq -2 \\ -2x & \text{for } -2 \leq x \leq 2 \end{cases}$$

Candidate #1: $x = 0$

$$f'(x) \Big|_{x=0} = -2 \cdot 0 = 0$$

to the left: $f'(-1) = 2 \Rightarrow$ increasing

to the right: $f'(1) = -2 \Rightarrow$ decreasing

$\Rightarrow x = 0$ is a local max

Candidate #2: $x = -2$

to the left: $f'(-3) = -6 \Rightarrow$ decreasing

to the right: $f'(-1) = 2 \Rightarrow$ increasing

\Rightarrow local min ($x = -2$)

Candidate #3: $x = 2$

to the left: $f'(1) = -2 \Rightarrow$ decreasing

to the right: $f'(2.1) = 4.2 \Rightarrow$ increasing

$\Rightarrow x = 2$ local min

Candidate #4: $x = -3$

to the left: ~~nothing~~
 to the right: $f'(-2.5) = -5 \rightarrow$ decreasing } $\Rightarrow x = -3$ local max

o Candidates for global min

$$\left. \begin{array}{l} x = -2 \quad (-2, 0) \\ x = 2 \quad (2, 0) \end{array} \right\} \text{GLOBAL MIN}$$

o Candidates for global max

$$\left. \begin{array}{l} x = 0 \quad (0, 4) \\ x = -3 \quad (-3, 5) \end{array} \right\} \rightarrow \text{GLOBAL MAX}$$

Compare to

$\lim_{x \rightarrow 2.5^-} f(x) = 2.25$ because The Extreme Value Theorem doesn't apply since we don't have a closed interval: $[-3, 2.5]$

Second Derivative Test for Local Extrema:

Suppose f is twice differentiable on an open interval containing c

o If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at $x = c$

o If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at $x = c$.

Ex 1 Find all local & global extrema of

$$f(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 2 \quad x \in \mathbb{R}$$

$$f'(x) = 6x^3 - 6x^2 - 12x = 0$$

$$x^3 - x^2 - 2x = 0$$

$$x(x^2 - x - 2) = 0 \Rightarrow x = 0 \Rightarrow f(0) = 2 \quad (0, 2)$$

$$x^2 - x - 2 = 0$$

$$x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{matrix} 2 & (2, -14) \\ -1 & (-1, -0.5) \end{matrix}$$

$$\begin{array}{l} x^2 + px + q = 0 \\ x = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \end{array}$$

$$f''(x) = 18x^2 - 12x - 12 \Rightarrow f''(0) = -12 \Rightarrow \text{local max}$$

Compare to $\lim_{x \rightarrow \infty} f(x)$ $\Rightarrow f''(2) = 36 \Rightarrow \text{local min GLOB}$

$\& \lim_{x \rightarrow \infty} f(x)$ $\Rightarrow f''(-1) = 18 \Rightarrow \text{local min}$

