

Product Rule:

If $h(x) = f(x) \cdot g(x)$ and both $f(x)$ & $g(x)$ are differentiable at x , then:

$$h'(x) = f'(x)g(x) + f(x) \cdot g'(x)$$

$$(uv)' = u'v + v'u \quad \text{OR} \quad \text{if we set } u=f(x) \text{ \& } v=g(x)$$

Practise:

Ex. 1: $f(x) = (3x+1) \cdot (2x^2-5)$

Set $u = 3x+1$; $v = 2x^2-5$

$u' = 3$; $v' = 4x$

Then: $f'(x) = (uv)' = u'v + v'u = 3(2x^2-5) + 4x(3x+1) =$
 $= 18x^2 + 4x - 15$

(This answer could have been found by multiplying out $f(x)$ and differentiating a polynomial function.)

Ex. 2: $f(x) = (3x^3 - 2x)^2$

Set $u = v = 3x^3 - 2x$

Then $(uv)' = u'v + u'v = (3x^3 - 2x) \cdot (9x^2 - 2) + (9x^2 - 2)(3x^3 - 2x)$

$= 2(3x^3 - 2x)(9x^2 - 2)$

Ex. 3: Population growth rate: $\frac{dN}{dt} = f(N)$ in time t .
 $N = N(t)$

• Per Capita Growth Rate: $\frac{1}{N} \cdot \frac{dN}{dt} = g(N)$
where $g(N) = \frac{f(N)}{N}$

• Show that $g'(0) = \lim_{N \rightarrow 0^+} \frac{d}{dN} f(N)$ assuming that $g(N)$ is differentiable & the right-hand limits exist for $g(N)$ & $g'(N)$.

- since $f(N) = g(N) \cdot N$, ~~we~~ we first take the derivative of $g(N) \cdot N$ using the product rule:

$$\frac{d}{dN} f(N) = N g'(N) + g(N) \cdot 1 = \underline{g(N) + N g'(N)}$$

- Then, we calculate:

$$\lim_{N \rightarrow 0^+} [g(N) + N g'(N)] = \lim_{N \rightarrow 0^+} g(N) + \lim_{N \rightarrow 0^+} \overbrace{N g'(N)}^{=0} = \underline{g(0)}$$

Ex. 4: Apply the product rule repeatedly: $y = (2x+1)(x+1)(3x-4)$

- Let $u = (2x+1)(x+1)$ & $v = 3x-4$

$$(uv)' = u'v + v'u = \left[(2x+1)(x+1) \right] \frac{d}{dx} (3x-4) + 3(2x+1)(x+1)$$

- Let $w = 2x+1$ & $z = x+1$

$$(wz)' = w'z + z'w = 2(x+1) + 2x+1 = 2x+2+2x+1 = \underline{4x+3}$$

- Plus in $(wz)'$ to get

$$(uv)' = (4x+3)(3x-4) + 3(2x+1)(x+1) = \underline{18x^2 + 2x - 9}$$

Quotient Rule: The derivative of a rational function in particular.

If $h(x) = \frac{f(x)}{g(x)}$ & $g(x) \neq 0$ and both $f(x)$ & $g(x)$ are differentiable at x ,

then:
$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} \quad \text{if } u = f(x) \text{ & } v = g(x)$$

Ex. 5: $y = \frac{x^3 - 3x + 2}{x^2 + 1} \quad \forall x \in \mathbb{R} \quad (x^2 + 1 \neq 0)$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} = \frac{(3x^2 - 3)(x^2 + 1) - 2x(x^3 - 3x + 2)}{(x^2 + 1)^2} = \frac{x^4 + 6x^2 - 4x - 3}{(x^2 + 1)^2}$$

Ex. 6: Differentiate the Mound Growth Function:

$$f(R) = \frac{aR}{k+R} \quad \text{with } R \geq 0$$

$\underbrace{\quad}_{\text{positive constants.}}$

Setting $u = aR$ & $v = k+R$, we apply the quotient rule:

$$\left(\frac{u}{v}\right)' = \frac{a(k+R) - aR}{(k+R)^2} = \frac{ak}{(k+R)^2} \quad \rightarrow \text{GO TO MAIN RULE}$$

Power Rule (for negative integer exponents):

$$\text{If } f(x) = x^{-n}, \text{ then } f'(x) = -n x^{-n-1}$$

Prove using the quotient rule:

$$f(x) = x^{-n} = \frac{1}{x^n}$$

Setting $u = 1$ & $v = x^n$, we have $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} = \frac{-1 \cdot n \cdot x^{n-1}}{x^{2n}} =$

$$= -\frac{n \cdot x^{n-1}}{x^{2n}} = -n \cdot (x^{(n-1)-2n}) = -n \cdot x^{-n-1}$$

Ex. 7: a, $y = \frac{1}{x}$

$$\frac{d}{dx} y = -\frac{1}{x^2}$$

b, $g(x) = \frac{3}{x^4}$

$$\frac{d}{dx} g(x) = -\frac{4x^3 \cdot 3}{x^8} = -\frac{12x^3}{x^8} = -12x^{-5} = -\frac{12}{x^5}$$

Power Rule ($n \in \mathbb{R}$)

Ex. 8: a, $y = \sqrt{x} = x^{1/2}$

$$\frac{d}{dx} y = \frac{1}{2} x^{-1/2} = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$d_y \frac{h(t)}{h(s)=5} = \frac{1}{5} \Rightarrow h'(s) = \underline{\underline{t^{-1/3}}}$$

Ex. 9: $f(x) = \sqrt{x}(x^2-1)$

Let $u = \sqrt{x}$ & $v = x^2 - 1$

$$(uv)' = u'v + v'u = \frac{1}{2\sqrt{x}} \cdot (x^2-1) + 2x \cdot \sqrt{x} = \frac{x^2-1}{2\sqrt{x}} + 2x\sqrt{x}$$

$$= \frac{x^2-1 + 2x\sqrt{x} \cdot 2\sqrt{x}}{2\sqrt{x}} = \frac{x^2-1 + 4x^2}{2\sqrt{x}} = \frac{5x^2-1}{2\sqrt{x}}$$

SAME

OR

$f(x) = \sqrt{x}(x^2-1) = x^{5/2} - x^{1/2}$ and use the Power Rule

$$f'(x) = \frac{5}{2}x^{3/2} - \frac{1}{2}x^{-1/2} = \frac{5}{2}x^{3/2} - \frac{1}{2\sqrt{x}} = \frac{5x^{3/2}\sqrt{x} - 1}{2\sqrt{x}} = \frac{5x^2-1}{2\sqrt{x}}$$

4.4. The Chain Rule & Higher Derivatives (Composite functions)

Rule: If g is differentiable at x , and f is differentiable at $y=g(x)$,
then $f[g(x)]$ is differentiable at x :

$$(f \circ g)'(x) = f'[g(x)]g'(x) \quad \text{OR} \quad \frac{d}{dx}[(f \circ g)(x)] = \frac{df}{du} \cdot \frac{du}{dx}$$

Ex. 1: $h(x) = (3x^2 - 1)^2 \Rightarrow \overset{u}{g(x)} = 3x^2 - 1 \Rightarrow g'(x) = 6x$ | INNER F.
 $\Rightarrow f(u) = u^2 \Rightarrow f'(u) = 2u$ | OUTER F.

Apply the Chain Rule:

$$h'(x) = 2(3x^2 - 1) \cdot 6x = \underline{\underline{12x(3x^2 - 1)}}$$

Ex. 2: $h(x) = (2x+1)^3 \Rightarrow \overset{u}{g(x)} = 2x+1 \Rightarrow g'(x) = 2$

$$f(u) = u^3 \Rightarrow f'(u) = 3u^2$$

Chain Rule: $h'(x) = 3(2x+1)^2 \cdot 2 = \underline{\underline{6(2x+1)^2}}$

Ex. 3: $h(x) = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}} \Rightarrow \overset{u}{g(x)} = x^2+1 \Rightarrow g'(x) = 2x$

$$\Rightarrow f(u) = \frac{1}{2}u^{\frac{1}{2}} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$$

Chain Rule:

$$h'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \underline{\underline{\frac{x}{\sqrt{x^2+1}}}}$$

Ex. 4: $h(x) = \sqrt[7]{2x^2+3x} \Rightarrow g(x) = 2x^2+3x \Rightarrow g'(x) = 4x+3$

$$f(u) = u^{\frac{1}{7}} \Rightarrow f'(u) = \frac{1}{7}u^{-\frac{6}{7}} = \frac{1}{7}u^{-\frac{6}{7}}$$

Chain Rule: $h'(x) = \frac{1}{7}(2x^2+3x)^{-\frac{6}{7}} \cdot (4x+3) = \underline{\underline{\frac{4x+3}{7(2x^2+3x)^{\frac{6}{7}}}}}$

$$h(x) = \frac{x^4}{x+1} = \frac{x^4}{(x+1)^2} = \frac{x^4}{(x+1)^3}$$

$$h(x) = \frac{f(x)}{g(x)}$$

Step 1: ~~$f(x) = \frac{1}{g(x)}$~~ $j(x) = \frac{1}{x} \Rightarrow j[g(x)] = \frac{1}{g(x)}$

Chain Rule: $(j \circ g)'(x) = \frac{1}{[g(x)]^2} \cdot g'(x) = -\frac{g'(x)}{[g(x)]^2}$

Product Rule for $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$

$$f'(x) \cdot \frac{1}{g(x)} + (-1) \frac{g'(x)}{[g(x)]^2} \cdot f(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

~~Generalized Power Rule:~~

~~$f(x)$ is differentiable & $r \in \mathbb{R}$.~~

~~$$\frac{d}{dx} [f(x)]^r = r f(x)^{r-1} \cdot f'(x)$$~~

Nested Chain Rule (Ex 10) $h(x) = (\sqrt{x^2+1} + 1)^2 \Rightarrow g(x) = \sqrt{x^2+1} + 1 \Rightarrow v(x) = x^2+1$
 $\Downarrow w(v) = \sqrt{v}$

$$\Downarrow f(u) = u^2$$

Chain Rule: $h'(x) = 2\sqrt{x^2+1} + 2 \cdot g'(x) = 2(\sqrt{x^2+1} + 1) \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x =$

$$= 2(\sqrt{x^2+1} + 1) \frac{x}{\sqrt{x^2+1}}$$