

## Combinations of continuous functions:

If  $a$  is a constant and functions  $f$  &  $g$  are continuous at  $x=c$ , then the following functions are also continuous at  $x=c$ :

$a \cdot f$ ,  $f+g$ ,  $f \cdot g$ ,  $\frac{f}{g}$  w/  $g(c) \neq 0$

- Polynomial
- Rational
- Power
- Trigonometric
- Exponential functions of form  $a^x$ ;  $a > 0$  &  $a \neq 1$
- Logarithmic functions of form  $\log_a x$ ,  $a > 0$  &  $a \neq 1$

are continuous wherever they are defined!

Ex. 6: Check for which values of  $x \in \mathbb{R}$  are the following functions continuous:

a)  $f(x) = 2x^3 - 3x + 1$  - polynomial, so it is cont.  $\forall x \in \mathbb{R}$

b)  $f(x) = \frac{x^2 + x + 1}{x - 2}$  - rational but undefined  $x=2$ , so it is cont.  $\forall x \in \mathbb{R} \setminus \{2\}$

c)  $f(x) = x^{\frac{1}{4}}$  - power function defined  $\forall x \geq 0$

d)  $f(x) = 3 \sin x$  - trigonometric defined for  $\forall x \in \mathbb{R}$

e)  $f(x) = \tan x$  - trig. but undefined  $\forall x \in \frac{\pi}{2} + k\pi$  ( $k \in \mathbb{Z}$ )

f)  $f(x) = 3^x$  - exponential defined  $\forall x \in \mathbb{R}$

g)  $f(x) = 2 \ln(x+1)$  - log  $\rightarrow$  undefined  $\forall x \in \mathbb{R}$  as long as  $x+1 > 0$   
 $\Downarrow$   
 $x > -1$

## Compound functions:

$$\lim_{x \rightarrow c} (f \circ g)x = \lim_{x \rightarrow c} f[g(x)] = f\left[\lim_{x \rightarrow c} g(x)\right] = f[g(c)] = f(c)$$

Ex. 7:

a)  $h(x) = e^{-x^2}$ ;  $\begin{cases} g(x) = -x^2 \\ \text{polynomial} \\ \text{cont. at } x \in \mathbb{R} \end{cases}$ ;  $f(x) = e^x$ ;  $\begin{cases} \text{exp.} \\ \text{cont. at } x \in \mathbb{R} \end{cases}$   $\Rightarrow h(x)$  is cont. at  $x \in \mathbb{R}$

b)  $h(x) = \sin \frac{\pi}{x}$ ;  $\begin{cases} g(x) = \frac{\pi}{x} \\ \text{cont. but} \\ \text{undefined/discont.} \\ \text{at } x=0. \end{cases}$ ;  $f(x) = \sin x$ ;  $\begin{cases} \text{cont. at } x \in \mathbb{R} \setminus \{0\} \end{cases}$

c)  $h(x) = \frac{1}{1+2x^{\frac{1}{3}}}$   $\Rightarrow$  rational  $\rightarrow$  cont., whenever  $1+2x^{\frac{1}{3}} \neq 0$

$$\frac{x^{\frac{1}{3}} \neq -\frac{1}{2}}{|x \neq -\frac{1}{8}|}$$

## Limits at infinity:

Ex. 1:  $\lim_{x \rightarrow \infty} \frac{x}{x+1}$

/ divide both the numerator & the denominator by  $x$

$$\lim_{x \rightarrow \infty} \frac{1}{1+\frac{1}{x}} = \frac{\lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n}} = \frac{1}{1+0} = \underline{\underline{1}}$$

$$\lim_{x \rightarrow \infty} \frac{x^2+2x-1}{x^3-3x+1} = \frac{\lim_{n \rightarrow \infty} \left( \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x^3} \right)}{\lim_{n \rightarrow \infty} \left( 1 - \frac{3}{x^2} + \frac{1}{x^3} \right)} = \frac{\lim_{n \rightarrow \infty} \frac{1}{x} + \lim_{n \rightarrow \infty} \left( \frac{\sqrt[3]{2}}{x} \cdot \frac{\sqrt[3]{2}}{x} \right) - \lim_{n \rightarrow \infty} \left( \frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x} \right)}{\lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \left( \frac{3}{x^2} \cdot \frac{1}{x} \right) \cdot \lim_{n \rightarrow \infty} \left( \frac{1}{x} \cdot \frac{1}{x} \right)}$$

$$= \frac{0}{1} = \underline{\underline{0}}$$

$$\lim_{x \rightarrow \infty} \frac{2x^3-4x+7}{3x^3+7x^2-1} = \lim_{n \rightarrow \infty} \left( \frac{2 - \frac{4}{x^2} + \frac{7}{x^3}}{3 + \frac{7}{x} - \frac{1}{x^3}} \right) = \frac{2-0+0}{3+0-0} = \underline{\underline{\frac{2}{3}}}$$

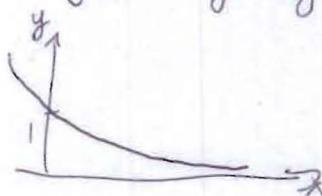
$$\lim_{x \rightarrow \infty} \frac{x^4+2x-5}{x^2-x+2} = \lim_{n \rightarrow \infty} \left( \frac{1 + \frac{2}{x^3} - \frac{5}{x^4}}{\frac{1}{x^2} - \frac{1}{x^3} + \frac{2}{x^4}} \right) = \frac{1+0-0}{0-0+0} = \underline{\underline{\frac{1}{0}}} \rightarrow \text{E f/p}$$

## General Rule:

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg(p) < \deg(q) \\ L \neq 0 & \text{if } \deg(p) = \deg(q) \\ \pm\infty & \text{if } \deg(p) > \deg(q) \end{cases}$$

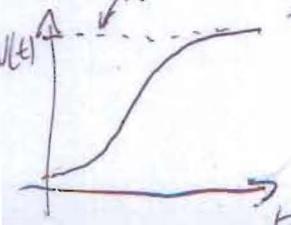
## Exponential functions:

$$\lim_{n \rightarrow \infty} e^{-x} = 0$$



## Ex. 3

### Logistic Growth:



$$N(t) = \frac{K}{1 + \left( \frac{K}{N_0} - 1 \right) e^{-rt}} \quad t \geq 0$$

$$\lim_{t \rightarrow \infty} \frac{K}{1 + \left( \frac{K}{N_0} - 1 \right) e^{-rt}} = K \quad \text{because } \lim_{t \rightarrow \infty} e^{-rt} = 0$$

carrying capacity

## Sandwich Theorem:

If  $f(x) \leq g(x) \leq h(x) \quad \forall x \in (a, b)$ , where  $c \in (a, b)$ .

AND

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$$

THEN

$$\lim_{x \rightarrow c} g(x) = L$$

Ex.:  $g(x) = e^{-x} \cdot \cos(10x) \quad \forall x \geq 0$

Since  $-1 \leq \cos(10x) \leq 1$  and  $\lim_{n \rightarrow \infty} e^{-x} = 0$

$$-e^{-x} \leq e^{-x} \cos(10x) \leq (e^{-x}) \rightarrow \text{tend to } 0$$

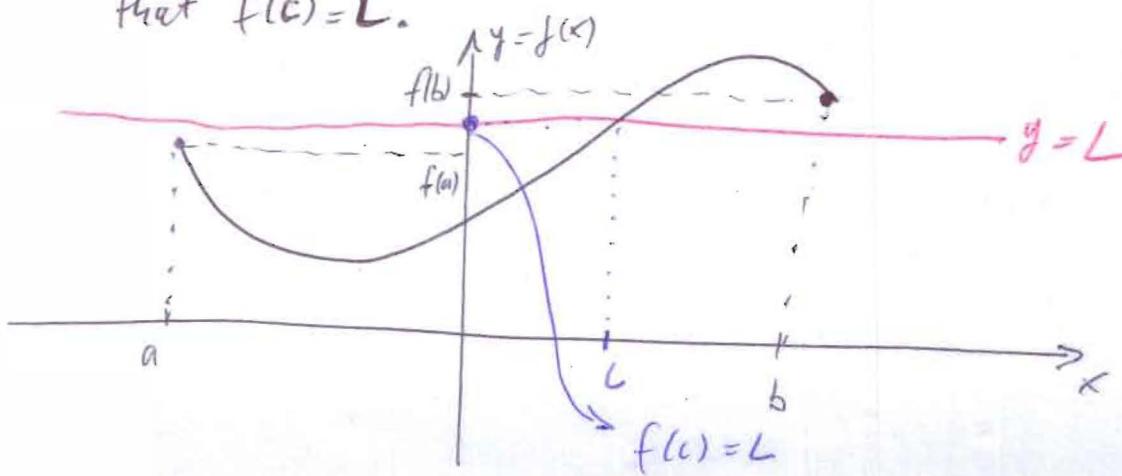
↓  
must tend to 0

## Trigonometric limits:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \& \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

## Intermediate Value Theorem:

Suppose  $f(x)$  is continuous on  $[a, b]$ . If  $f(a) < L < f(b)$  or  $f(b) < L < f(a)$  for  $L \in \mathbb{R}$ , then there exists at least one number  $c \in (a, b)$  such that  $f(c) = L$ .



Ex 1  $f(x) = 3 + \sin x \quad \forall 0 \leq x \leq \frac{3\pi}{2}$

- Show that there exists at least one point  $c$  in  $(0, \frac{3\pi}{2})$ , such that  $f(c) = \frac{5}{2}$ .

- To use the Intermediate Value Theorem, we first note that  $f(x)$  is continuous on  $[0, \frac{3\pi}{2}]$ .

- Also  $f(0) = 3 + \sin 0 = 3 + 0 = 3$

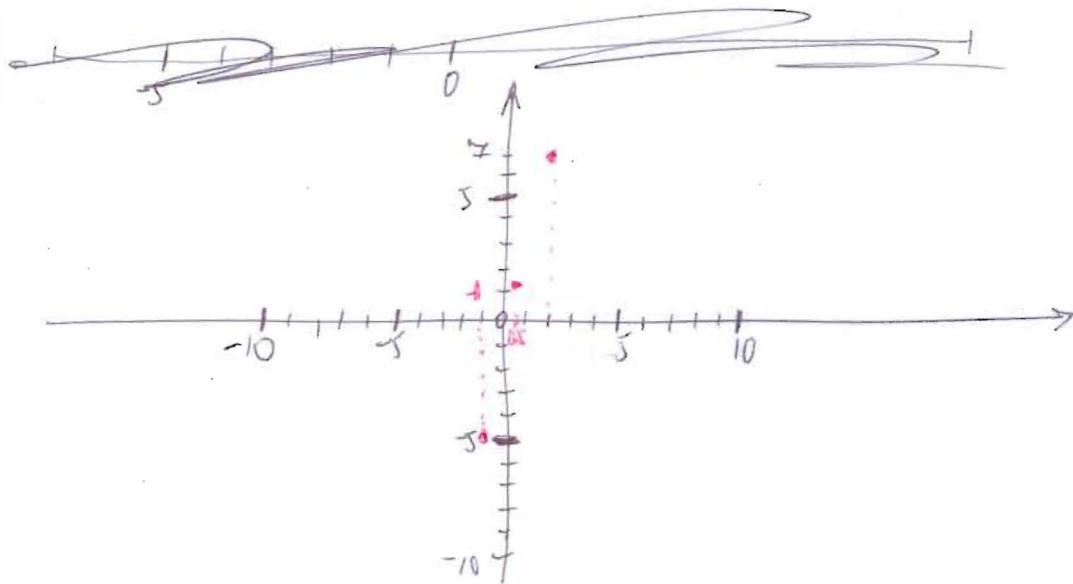
$$f\left(\frac{3\pi}{2}\right) = 3 + \sin \frac{3\pi}{2} = 3 - 1 = 2$$

- Since  $2 < \frac{5}{2} < 3$  and  $f(x)$  is cont.,  ~~$f(x) = \frac{5}{2}$~~  exists such that  $f(c) = \frac{5}{2}$ .

Ex 2: Find root of  $f(x) = x^5 - 7x^2 + 3 = 0$  by bisection.

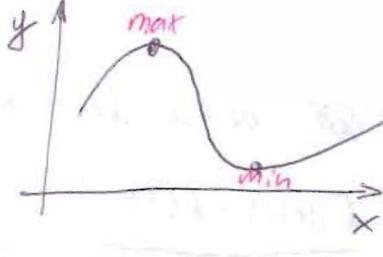
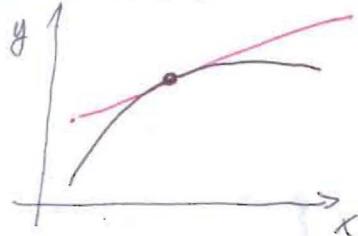
- Since  $f(x)$  is polynomial, it is continuous at  $x \in \mathbb{R}$ .

- Let  $f(a) < 0$  &  $f(b) > 0$ . Then, ~~( $a, b$ )~~ there must be a number  $c \in (a, b)$  such that  $f(c) = 0$ .

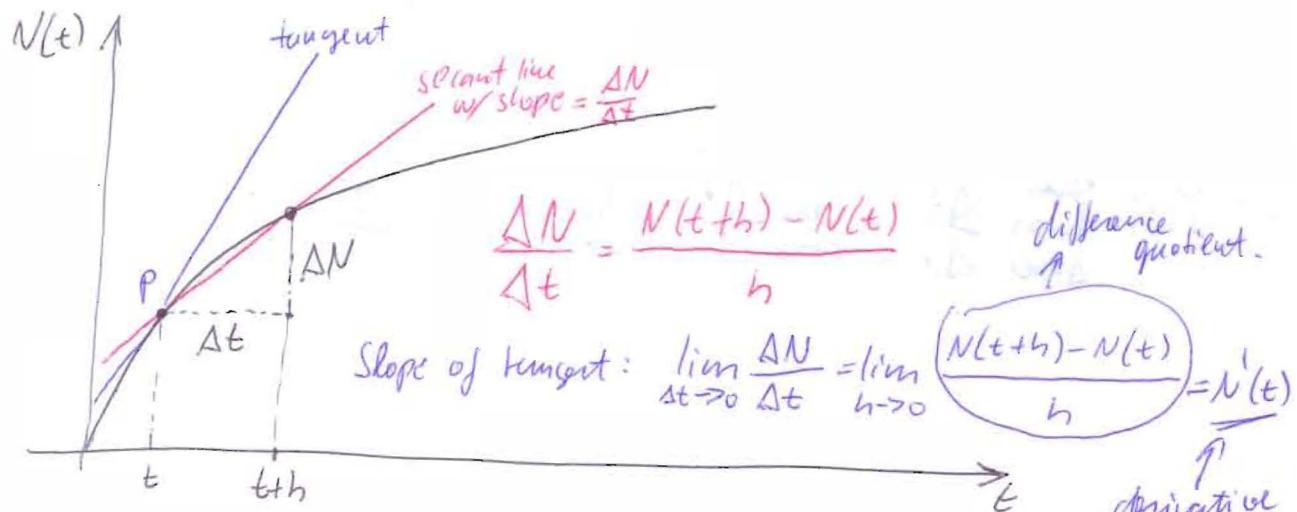


## Chapter 4 : Differentiation :

- Differential Calculus: Allows us to find tangent lines and optima of func.



- Definition of Derivative:



Definition:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (\text{if it exists})$$

If  $f'(x)$  exists,  $f$  is said to be differentiable at  $x$ .

Function  $f(x)$  is differentiable at  $(a, b)$  if it is differentiable at  $\forall x \in (a, b)$ .

The derivative at a given point  $x=c$  :

$$\text{Notation: } y = \frac{dy}{dx} = f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x) \quad \leftarrow \text{Leibniz}$$

$y \leftarrow$  Newton

$$f'(c) = \left. \frac{df}{dx} \right|_{x=c}$$

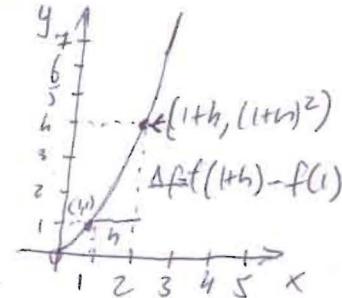
When you emphasize that the derivative is taken at  $x=c$ .

## Geometric Interpretation:

Example:  $f(x) = x^2 \quad \forall x \in \mathbb{R}$   $f'(1) = ?$

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{1+2h+h^2 - 1}{h} = \lim_{h \rightarrow 0} (2+h) =$$

$$= \lim_{h \rightarrow 0} 2 + \lim_{h \rightarrow 0} h = 2$$



Def. of Tangent Line:

If the derivative of function  $f(x)$  exists at  $x=c$ , then the tangent line at  $x=c$  is the line going through point  $(c, f(c))$  with slope  $\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = f'(c)$

Point-Slope Form:  $y - y_0 = m(x - x_0)$

$$\boxed{y - f(c) = f'(c)(x - c)} \quad \begin{array}{l} \text{the point is } (c, f(c)) \\ \text{the slope is } f'(c) \end{array}$$

Example #1: The derivative of the constant Function:  $f(x) = a$  horizontal line

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{a-a}{h} = \lim_{h \rightarrow 0} 0 = 0$$

Example #2: The derivative of the Linear Function:  $f(x) = mx + b$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{m(x+h) + b - (mx + b)}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{mx + mh + b - mx - b}{h} = \lim_{h \rightarrow 0} m = m$$

Example #3: The derivative of the Reciprocal Function:  $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \rightarrow 0} \frac{\frac{x - x-h}{x(x+h)}}{h} =$$

$$= \lim_{h \rightarrow 0} -\frac{1}{x^2 + hx} \cdot \frac{1}{h} = \lim_{h \rightarrow 0} -\frac{1}{x^2 + hx} = -\frac{1}{x^2}$$

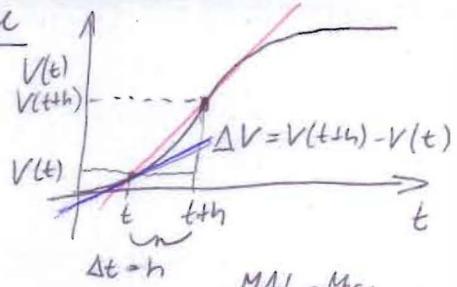
Derivative = Instantaneous Rate of Change

As opposed to the Average Rate of Change, which is characterized by the Difference Quotient:

$$\frac{\Delta V}{\Delta t} = \frac{V(t+h) - V(t)}{h}$$

COMPARE

$$V'(t) = \lim_{h \rightarrow 0} \frac{V(t+h) - V(t)}{h}$$



MAI = Mean Annual Increment

In Forest

Differentiability and Continuity:

Example:  $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h} \begin{cases} 1 & h > 0 \\ -1 & h < 0 \end{cases}$$

Since  $\lim_{h \rightarrow 0^+} \frac{|h|}{h} = 1 \neq \lim_{h \rightarrow 0^-} \frac{|h|}{h} = -1$ , function  $f(x) = |x|$  is not differentiable at  $x=0$

(even though it is continuous)

Rule: Continuity is a necessary but not sufficient condition for differentiability

Rule: If  $f$  is differentiable at  $x=c$ , then  $f$  is also continuous at  $x=c$

## DIFF

## TECHNIQUES

Power Rule:

$$\frac{d}{dx} (x^n) = nx^{n-1} \quad \forall n \in \mathbb{Z}^+$$

• Prove the rule for  $n=2$

$$\begin{aligned} \frac{d}{dx} (x^2) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x+h)}{h} = \lim_{h \rightarrow 0} (2x+h) = \\ &= 2x \end{aligned}$$

• To prove the rule for any positive integer  $n$ , the Binomial Theorem is used.

$$\begin{aligned} \frac{d}{dx} (x^n) &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \rightarrow 0} \frac{\cancel{x}^n \cancel{x}^{n-1} h + \frac{n(n-1)}{2 \cdot 1} x^{n-2} h^2 +}{h} + \\ &\quad + \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} x^{n-3} h^3 + \dots + \frac{n(n-1) \dots (n-k+1)}{k(k-1) \dots 2 \cdot 1} x^{n-k} h^k + \dots + \\ &\quad + \cancel{hx^{n-1} + h^n} \cancel{h} = \\ &= \lim_{h \rightarrow 0} \frac{h \left[ \cancel{nx^{n-1}} + \frac{n(n-1)}{2} x^{n-2} \cdot h + \dots + h^{n-1} \right]}{h} = \lim_{h \rightarrow 0} nx^{n-1} = \underline{\underline{nx^{n-1}}} \end{aligned}$$

Constant Rule / Summation Rule:

If  $a$  is constant and  $f(x)$  &  $g(x)$  are differentiable at  $x$ , then

$$\boxed{\frac{d}{dx} [af(x)] = a \cdot \frac{d}{dx} f(x)}$$

$$\boxed{\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)}$$

Practise:

$$y = 5x^4 - 17x^2 + x^3 - 7x + 6$$