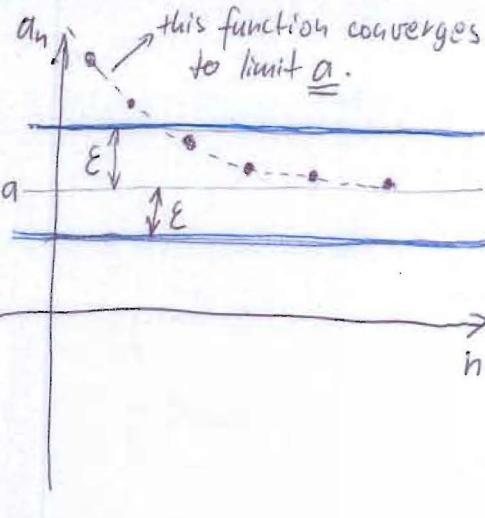


## Formal Definition of limits:

Sequence  $\{a_n\}$  has limit  $a$ , i.e.,  $\lim_{n \rightarrow \infty} a_n = a$ , if for every  $\epsilon > 0$ , there exists an integer  $N$  such that:

$$|a_n - a| < \epsilon \quad \forall n > N$$

If limit exists, the sequence is convergent. It is divergent otherwise.



Ex. 9. • Use definition of limits (above) to prove that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

- We need to show that for every  $\epsilon > 0$ , there exists an integer  $N$  such that:

$$\left| \left( \frac{1}{n} \right) - 0 \right| < \epsilon \quad \forall n > N$$

↓ since  $n \in \mathbb{Z}^+$

$$\frac{1}{n} < \epsilon$$

↓ I rearranged the inequality to a more convenient form.

$$n > \frac{1}{\epsilon}$$

- If we pick ( $N \leq \frac{1}{\epsilon}$  but  $N+1 > \frac{1}{\epsilon}$ ), then  $n \geq N+1 > \frac{1}{\epsilon}$

↑ To prove our claim that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ ,

we only need to find one  $N$  that

makes sure that  $n > \frac{1}{\epsilon}$  is valid for any  $n > N$ .

$\underline{n > \frac{1}{\epsilon}}$  QED

## Limit Laws:

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} c a_n = c \lim_{n \rightarrow \infty} a_n$$

constant

$$\lim_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \quad \text{as long as } \lim_{n \rightarrow \infty} b_n \neq 0$$

$$\underline{\text{Ex. 10.: } \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n} = 1 + 0 = 1}$$

$$\underline{\text{Ex. 11.: } \lim_{n \rightarrow \infty} \frac{4n^2 - 1}{n^2} = \lim_{n \rightarrow \infty} 4 - \lim_{n \rightarrow \infty} \frac{1}{n^2} = 4 - \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = 4 - 0 \cdot 0 = 4}$$

## Limits of Exponential Growth: Ex. 12.

$$a_n = a_0 \cdot R^n \quad \forall n \in \mathbb{Z}^+$$

$$\lim_{n \rightarrow \infty} a_n = \begin{cases} 0 & \text{if } 0 < R < 1 \\ a_0 & \text{if } R = 1 \\ \infty & \text{if } R > 1 \end{cases}$$

Finding limits through recursions ( $a_n$  is not given explicitly as a function of  $n$ )

Ex. 13  $a_{n+1} = \frac{1}{4}a_n + \frac{3}{4}$  with  $a_0 = 2$  Recursive definition of  $\{a_n\}$

\* Try to see first how this sequence behaves:

$$a_0 = 2$$

$$a_1 = \frac{1}{4} \cdot 2 + \frac{3}{4} = \frac{5}{4}$$

$$a_2 = \frac{1}{4} \cdot \frac{5}{4} + \frac{3}{4} = \frac{5}{16} + \frac{12}{16} = \frac{17}{16}$$

$$a_3 = \frac{1}{4} \cdot \frac{17}{16} + \frac{3}{4} = \frac{17}{64} + \frac{48}{64} = \underline{\underline{\frac{65}{64}}}$$

\* We notice a pattern:

$$a_n = \frac{4^n + 1}{4^n}$$

\* Let's prove that this formula is equivalent to the recursion given above:

\* Step 1:  $a_0 = \frac{4^0 + 1}{4^0} = \frac{1 + 1}{1} = 2$  OK ✓

We do these steps because we know that we need  $a_{n+1}$  and  $a_n$  in the recursion!

\* Step 2: Calculate:  
 $(a_{n+1}) = \frac{4^{n+1} + 1}{4^{n+1}} = 1 + \frac{1}{4^{n+1}} = 1 + \frac{1}{4} \cdot \frac{1}{4^n}$

\* Step 3: Calculate:

$$(a_n) = \frac{4^n + 1}{4^n} - 1 + \frac{1}{4^n} \Rightarrow a_n - 1 = \frac{1}{4^n}$$

\* Step 4: Substitute:

$\frac{1}{4^n}$  in the calculation of  $a_{n+1}$  with  $a_n - 1$ :

$$a_{n+1} = 1 + \frac{1}{4}(a_n - 1) = \frac{1}{4}a_n + \frac{3}{4}$$

\* Calculate limit:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{4^n + 1}{4^n} = \lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \left(\frac{1}{4}\right)^n = 1 + 0 = 1$$

where  $0 < \frac{1}{4} < 1$

## Chapter 3 : Limits & Continuity :

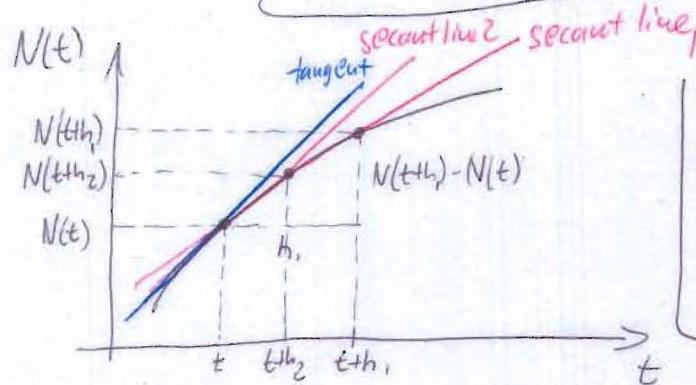
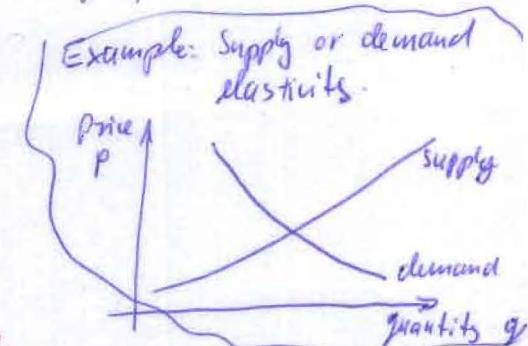
Limits : •  $\lim_{x \rightarrow c} f(x)$ , where  $x \in \mathbb{R}$  tends to a fixed value  $c$ .

\* changes over small time intervals: For example, the change in  $N(t)$  over  $[t, t+h]$ , where  $h > 0$  is

$$\Delta N = N(t+h) - N(t)$$

• And average growth rate:

$$\boxed{\frac{\Delta N}{\Delta t} = \frac{N(t+h) - N(t)}{h}}$$



$$\boxed{\lim_{h \rightarrow 0} \frac{N(t+h) - N(t)}{h} = \text{slope of tangent at } (t, N(t))}$$

Definitions:

$$\boxed{\lim_{x \rightarrow c} f(x) = L}$$

If  $L$  is finite, the limit exists and  $f(x)$  converges to  $L$ .

$$(f(x) \rightarrow L \text{ as } x \rightarrow c)$$

$\lim_{x \rightarrow c^+}$   $\rightarrow x$  approaches  $c$  from the right

$\lim_{x \rightarrow c^-}$   $\rightarrow x$  approaches  $c$  from the left

Ex. 1:  $\lim_{x \rightarrow 2} x^2 = 4$

Ex. 2: a,  $\lim_{x \rightarrow 2} g(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases} \Rightarrow \lim_{x \rightarrow 2} g(x) = 4 \neq g(2)$  ) The function doesn't have to be defined at a point to have a limit.

b,  $h(x) = x^2$ , where  $x \neq 2 \Rightarrow \lim_{x \rightarrow 2} h(x) = x^2 = 4$

Ex. 3:  $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$ , where  $x \neq 3$

$$\lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3)} = x+3 = 6$$

Rules: If  $\lim_{x \rightarrow c} f(x) = +\infty$  &  $|f(x)|$  increases w/o bound as  $x \rightarrow c$  | In other words

Ex. 8 :-

$$\lim_{x \rightarrow 0} \sin \frac{\pi}{x}$$

$$\lim_{x \rightarrow 0^+} \frac{\pi}{x} = +\infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{\pi}{x} = -\infty$$

at  $+\infty$  or  $-\infty$ , the sin function oscillate between  $-1 \& +1$ .

Limit Laws:

$$\left. \begin{aligned} \lim_{x \rightarrow c} af(x) &= a \lim_{x \rightarrow c} f(x) ; \quad \lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x), \\ \lim_{x \rightarrow c} \frac{f(x)}{g(x)} &= \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} ; \quad \lim_{x \rightarrow c} [f(x) \cdot g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) \\ \lim_{x \rightarrow c} & \end{aligned} \right\} \text{constant}$$

Ex. 10:  $\lim_{x \rightarrow 2} [x^3 + 4x - 1] = ?$   $\lim_{x \rightarrow 2} x \cdot \lim_{x \rightarrow 2} x \cdot \lim_{x \rightarrow 2} x + 4 \cdot \lim_{x \rightarrow 2} x - 1 = 2 \cdot 2 \cdot 2 + 4 \cdot 2 - 1 = 15$

Rules: If  $f(x)$  is polynomial, then

$$\lim_{x \rightarrow c} f(x) = f(c)$$

If  $f(x)$  is rational, then  $f(x) = \frac{p(x)}{q(x)}$ , where  $p(x) \& q(x)$  are both polynomials &  $q(c) \neq 0$ , then

$$\lim_{x \rightarrow c} f(x) = \frac{\lim_{x \rightarrow c} p(x)}{\lim_{x \rightarrow c} q(x)} = \frac{p(c)}{q(c)} = \cancel{f(c)}$$

Continuity: Function  $f(x)$  is continuous at  $x=c$  if  $\lim_{x \rightarrow c} f(x) = f(c)$

Checklist for functional continuity:

- 1, Is  $f(x)$  defined at  $x=c$ ?
- 2, Does  $\lim_{x \rightarrow c} f(x)$  exist?
- 3,  $\lim_{x \rightarrow c} f(x) = f(c) ?$

Ex. 1:

$$f(x) = 2x - 3 \quad \forall x \in \mathbb{R} \quad \text{at } x=1?$$

$f(x)$  is continuous at  $x=1$ .

- 1, Is  $f(x)$  defined at  $x=c$ ? Yes because  $x \in \mathbb{R}$  &  $f(1) = 2-3 = -1$
- 2, Does the limit exist? Yes, since  $\lim_{x \rightarrow c} x = c$ !
- 3,  $\lim_{x \rightarrow c} 2x - 3 = 2 \lim_{x \rightarrow c} x - \lim_{x \rightarrow c} 3 = 2c - 3 = c = f(x=1)$

Ex. 2:

$$f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3} & \text{if } x \neq 3 \\ a & \text{if } x = 3 \end{cases}$$

how much should  $a$  be for  $f(x)$  to be continuous at  $x=3$ .

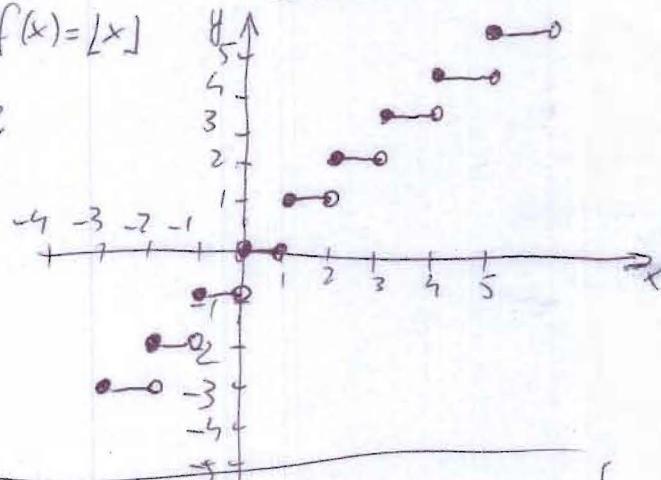
$$\lim_{x \rightarrow 3} f(x) = f(3); \quad \lim_{x \rightarrow 3} f(x) = \frac{(x+3)(x-2)}{(x-3)} = \lim_{x \rightarrow 3} (x+2) = 5$$

$$f(3) = 5 = a$$

Ex. 3: Floor function  $= f(x) = \lfloor x \rfloor$

$$\lim_{x \rightarrow k^+} f(x) = k \quad \forall k \in \mathbb{Z}$$

$$\lim_{x \rightarrow k^-} f(x) = k-1$$



\* Function  $f(x)$  is continuous from the right at  $x=c$  if

$$\lim_{x \rightarrow c^+} f(x) = f(c)$$

\* Function  $f(x)$  is continuous from the left at  $x=c$  if

$$\lim_{x \rightarrow c^-} f(x) = f(c)$$