

## Power Functions:

Form:  $f(x) = x^r$ , where  $r \in \mathbb{R}$

Example: Allometry - study of the scaling relations b/w biological variables such as organ sizes.

LIDAR !!!

$$y = k \cdot x^r, \text{ where } r \in \mathbb{R} \setminus \{0\}$$

proportionality factor

cell biomass

$$y = 3x^{0.794}$$

all volume

## Exponential Functions:

Growth:  $N(t) = 2^t$  <sup>time</sup>, where  $t \geq 0$  (at  $t=0, N(0) = 2^0 = 1 \Rightarrow$  one individual)

population size (this population doubles its size after unit time elapses)

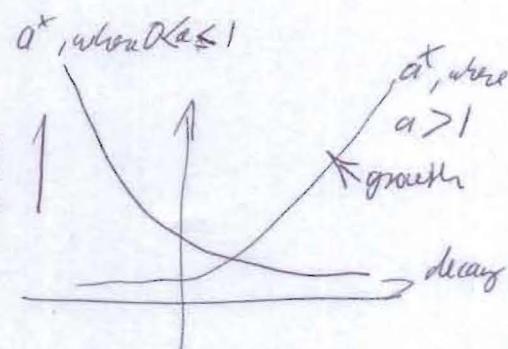
$$\frac{N(t+1)}{N(t)} = \frac{2^{t+1}}{2^t} = 2 = \underline{\underline{Z}}$$

General Form:  $(N(t) = N_0 Z^t)$ , where  $N_0$  is the initial pop. size

## Exponential growth & decay:

$$f(x) = a^x, \text{ where } x \in \mathbb{R}$$

base can be  $e = 2.71...$



## Ex-10: Radioactive decay

$C^{14} - N^{14}$  decays at a constant rate.

$C^{12}$  DOES NOT DECAY

PLANT DIES  $\rightarrow C^{12}$  remains constant but  $C^{14}$  decays

$\frac{C^{14}}{C^{12}}$  goes down over time

There's a constant ratio/equilibrium b/w  $C^{14}$  &  $C^{12}$  atmospheric carbon

$$W(t) = W_0 \cdot e^{-\lambda t}$$

decay, where  $t \geq 0$   
No amount of  $C^{14}$  at time  $t$

- How much time does  $C^{14}$  take to halve:

~~$$W(t) = W_0 \cdot e^{-\lambda t}$$~~

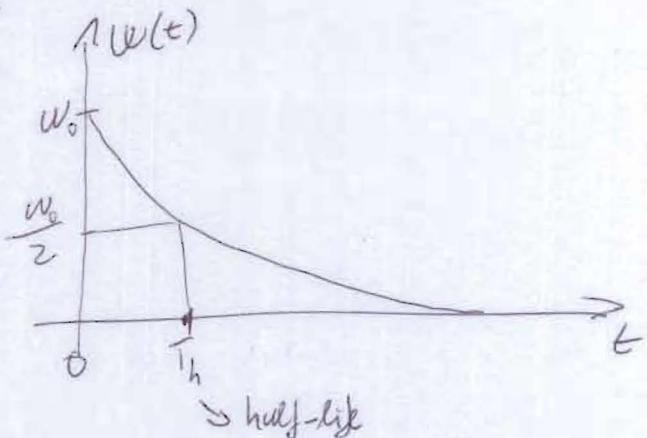
$$\frac{1}{2} W_0 = W_0 \cdot e^{-\lambda T_h} \quad \left\{ \begin{array}{l} W(T_h) = W_0 \cdot e^{-\lambda \cdot T_h} \\ W(T_h) = \frac{1}{2} W_0 \end{array} \right.$$

$\Downarrow$

$$\frac{1}{2} = e^{-\lambda T_h}$$

$$\frac{\ln \frac{1}{2}}{T_h} = -\lambda T_h$$

$$T_h = -\frac{\ln \frac{1}{2}}{\lambda} = -(\ln 1 - \ln 2) \cdot \frac{1}{\lambda} = \frac{\ln 2}{\lambda} \approx \underline{5730 \text{ years}}$$



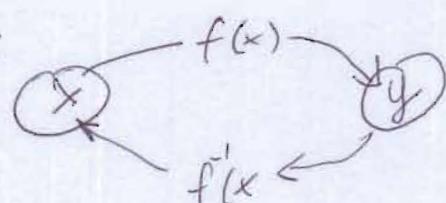
Ex. 11: Excavation site: wood found! It contains 23% as much  $C^{14}$  as living material. When was the wood cut?

$$\frac{W_{\text{now}}}{W_0} = 0.23 \quad , \text{ where } W_{\text{now}} = W_0 \cdot e^{-\lambda t}$$

$$\frac{0.23}{W_0} = e^{-\lambda t}$$

$$(\ln 0.23 = -\lambda t) \Rightarrow \boxed{t = -\frac{\ln 0.23}{\lambda}}$$

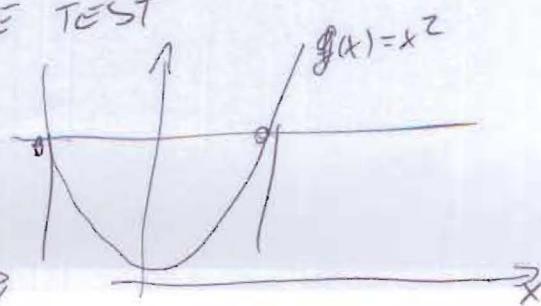
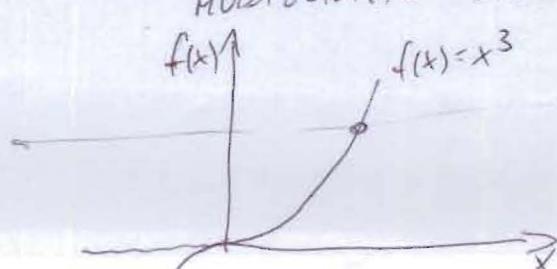
Inverse functions:



- $f^{-1}(x)$  reverses the effect of  $f(x)$
- Not every function has an inverse

- $f(x) = x^3$  has an ~~no~~ inverse because  $x_1^3 \neq x_2^3$  if  $x_1 \neq x_2$ !!!
- $f(x) = x^2$  does not have an inverse because  $x_1^2 = x_2^2$  if  $x_1 = -x_2$

HORIZONTAL LINE TEST



Ex. 12: Find inverse function of  $f(x) = x^3 + 1$

Step 1: Does  $f(x)$  have an inverse? Check if it's one-to-one!

Is  $f(x_1) \neq f(x_2)$  if  $x_1 \neq x_2$ ?

It is the same to say that if  $f(x_1) = f(x_2)$  then  $x_1 = x_2$ !

$$x_1^3 + 1 = x_2^3 + 1$$

$$\Downarrow$$

$$x_1^3 = x_2^3$$

$$\sqrt[3]{x_1^3} = \sqrt[3]{x_2^3}$$

$$\Downarrow$$

$$x_1 = x_2$$

Step 2:

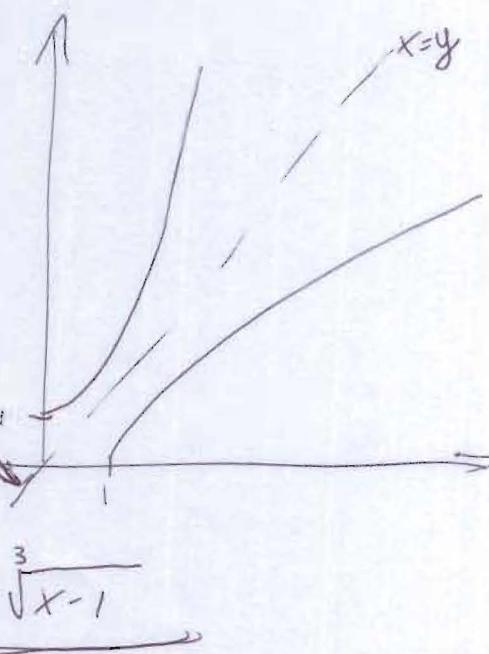
$$y = x^3 + 1$$

$$\Downarrow$$

$$x^3 = y - 1$$

$$x = \sqrt[3]{y-1}$$

$$\Rightarrow y = \sqrt[3]{x-1}$$



Logarithmic Functions: Inverse of exponential functions

$$g(x) = x^2$$

$$\text{If } f(x) = a^x, \text{ then } f^{-1}(x) = \log_a x$$

$$\text{If } f(x) = e^x, \text{ then } f^{-1}(x) = \ln x$$

Rules

$$a^{\log_a x} = x \quad \forall x > 0$$
$$\log_a a^x = x \quad \forall x \in \mathbb{R}$$



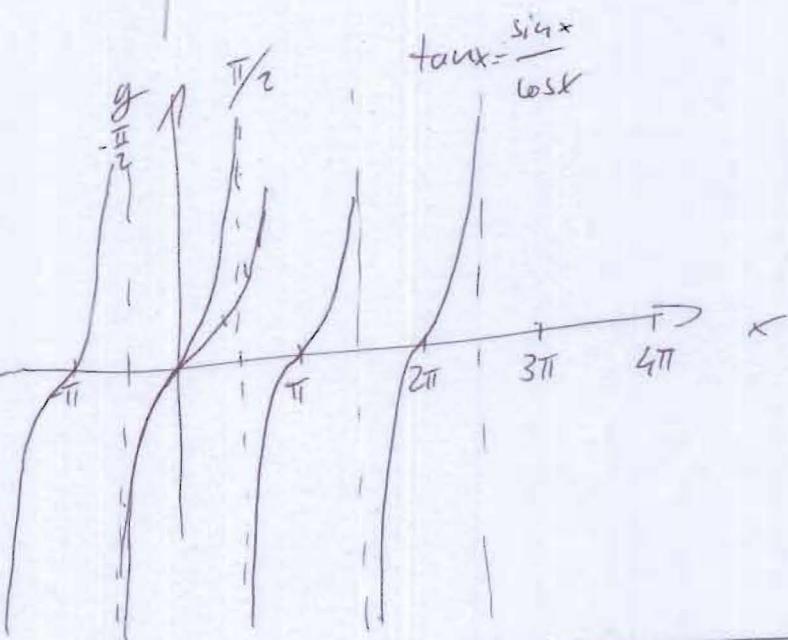
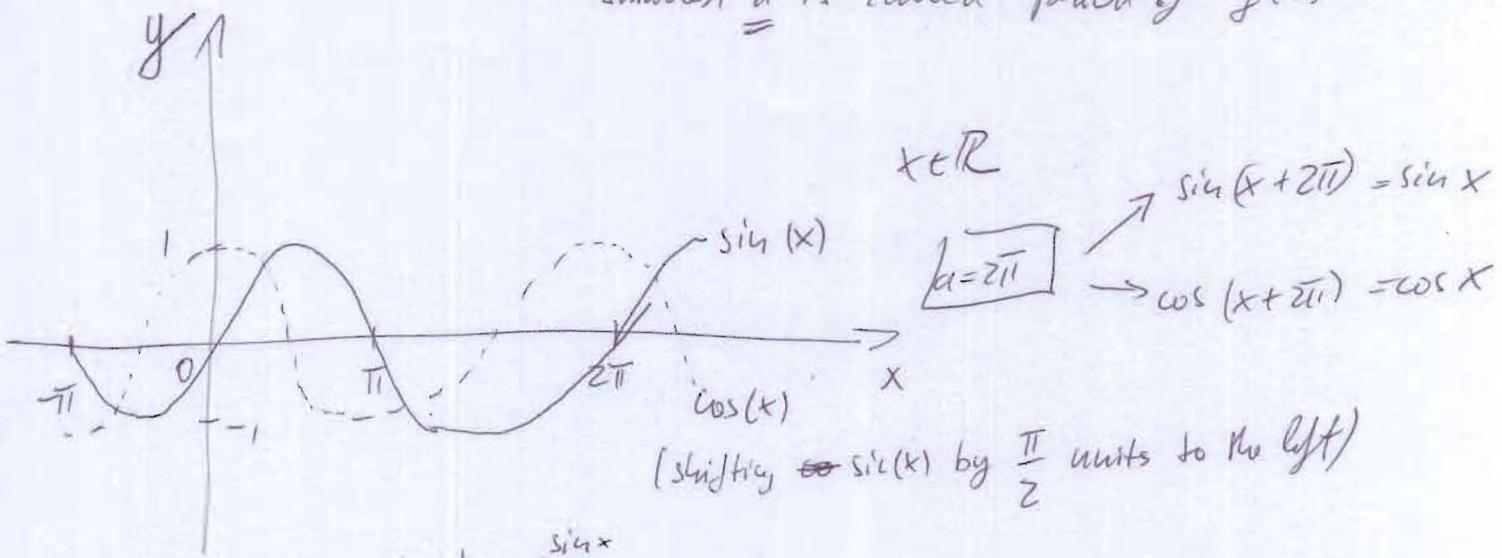
$$a^x = \exp(x \ln a) = e^{x \ln a}$$

$$\log_a x = \frac{\ln x}{\ln a}$$

$$e^{x \cdot \ln a} = (e^{\ln a})^x = a^x$$

## Trigonometric functions: Examples of periodic functions

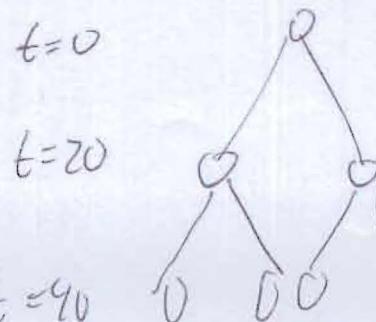
$f(x)$  is periodic if  $f(x+a) = f(x)$  for some  $a > 0$   
 Smallest  $a$  is called period of  $f(x)$



## CHAPTER 2

### 1. Discrete time population models (or growth models)

$$t=0$$



$$N(t) = 1$$

$$t=20$$

$$N(20) = 2$$

$$t=40$$

$$N(40) = 4$$

$t$	0	20	40	60	80	100
$N(t)$	1	2	4	8	16	32

if  $t = 20$  means

$t$ (20 years)	0	1	2	3	4	5	6
$N(t)$	1	2	4	8	16	32	64

$$N(t) = 2^t \quad \forall t=1,2,3,\dots,n$$

$\rightarrow$  exponential growth

$$\text{More generally: } N_t = N_0 \cdot 2^t \quad \forall t=1,2,\dots$$

initial population size

$$\text{Even more generally: } N_t = N_0 \cdot R^t \quad \forall t=1,2,\dots$$

growth constant

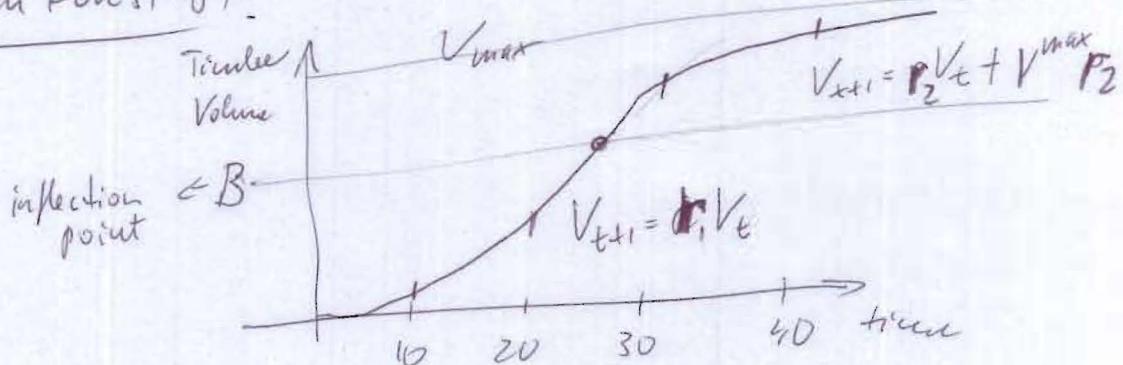
Recursion:

$$N_{t+1} = 2N_t \quad \text{equivalent}$$

$$N_{t+1} = R N_t \quad \rightarrow \text{defines population size recursively}$$

$$\therefore \frac{N_t}{N_{t+1}} = \left(\frac{1}{R}\right) \rightarrow \text{constant (for exponential growth)}$$

| Example from Forestry |:



| Sequences |

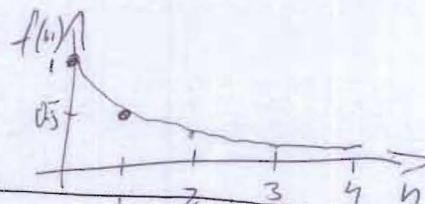
: Special functions:

with domain  $\mathbb{Z}^+ = \{0, 1, 2, \dots, n\}$

$f: \mathbb{Z}^+ \rightarrow \mathbb{R}$   
 $n \rightarrow f(n)$  (or  $t \rightarrow f(t)$  if  $t$  represents time)

Ex. 1:  $f: \mathbb{Z}^+ \rightarrow \mathbb{R}$

$$n \rightarrow f(n) = \frac{1}{n+1}$$



$$\boxed{a_0, a_1, a_2, \dots, a_n = f(n) \quad \text{or} \quad \{a_n : n \in \mathbb{Z}^+\}}$$

Sequence

$$\text{Ex. 4: } \underbrace{1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}}_{a_n}$$

$$a_2 = 1 \cdot \frac{1}{9} = \frac{1}{9}$$

$$a_n = a_0 R^n \quad \forall n=0, 1, 2, \dots \quad \leftarrow \text{EXPLICIT}$$

$$a_{n+1} = R a_n \quad \forall n=0, 1, 2, \dots \quad \leftarrow \text{RECURSIVE}$$

description of population growth.

Limits:  $\lim_{n \rightarrow \infty} a_n$  or  $\lim_{n \rightarrow \infty} a_n$  LONG TERM BEHAVIOR!

$$\text{Ex. 5: } \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 \quad (\text{limit exists})$$

$$\text{Ex. 6: } \lim_{n \rightarrow \infty} (-1)^n \in \{\emptyset\} \quad (\text{limit does not exist})$$

$$\text{Ex. 7: } \lim_{n \rightarrow \infty} 2^n \in \{\emptyset\} \quad (\text{limit does not exist})$$

~~Formal Definition~~

$$\text{Ex. 8: } \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1 \quad (\text{limit exists})$$

GUESSED

## Formal definition of limits:

Sequence  $\{a_n\}$  has limit  $a$ , i.e.,  $\lim_{n \rightarrow \infty} a_n = a$ , if  
for every  $\epsilon > 0$ , there exists an integer  $N$   
such that  $|a_n - a| < \epsilon \quad \forall n > N$

If limit exists  $\rightarrow$  the sequence is convergent.  
It is divergent otherwise

