Optimization Word Problem Walkthrough

When presented with a word problem involving the maximization or minimization of some quantity, try following these steps.

1. Name all relevant variables, and draw a picture if possible. For geometrical problems, don't forget to name side lengths, radii, areas, perimeters, or volumes as needed. At least make sure that, in the information given to you, each noun relevant to the problem corresponds to a variable. If you can't draw a picture, maybe give a short description for each variable. Ideally, you'd like to name just as many variables as you need, but it's usually okay if you start out with too many variables. If you start out with too few variables, it will become apparent in subsequent steps and you might need to create a variable on the spot.

2. Identify the quantity (which you may have already named in step 1) that we wish to have maximum or minimum. Look for key words such as "biggest", "smallest", "most", and "least". After identifying this quantity, see if you can come up with a *formula* for that quantity that uses the other named variables. This quantity is also called the objective function, because our "objective" has to do with its value being smallest or biggest.

3. Identify the constraint in the problem. Without any constraints, it probably wouldn't make sense to ask the question we're asking. For example, if I just told you to find the dimensions of a rectangular fence that make the area enclosed largest, you would have to admit that there is no upper limit—in theory—to how big the area could be. You could make the fenced-in area as big as you wanted. But if I told you we only have 100 ft. of fencing, the problem now has a constraint, and it is reasonable to look for a finite answer. Once you have identified the constraint, do just as you did in step 2: come up with a *formula* for it using other named variables in the problem. For the constraint, we can do better than a formula: we can write down an equation since we are often given a certain value for the constrained quantity. For example, if I know the perimeter of a fence must be 100 ft., and in the formula for the perimeter is $P = 2x + 2y$, then I can write down the equation $100 = 2x + 2y$.

4. In the constraint equation, solve for one of the variables. There will usually just be two variables. If there are more than two, there are probably other constraints in the problem (i.e., other relationships between the variables). Go back and try to identify any constraining relationships between the variables so that you can solve for one in terms of the other.

5. Take the solved-for variable from step 4 and substitute it into the objective function formula from step 2. Now the objective function should only be a function of a single variable. The idea is that even though at first it looks like lots of variables could change independently of each other initially, there is really only one variable that is completely free to change, and it is the independent variable on the right-hand-side of the objective function.

6. To optimize the objective function, take its first derivative f'(x) and set it equal to zero. Solving for x in the equation f'(x) = 0 gives the critical points.

7. For each critical point, test to see whether it is a local maximum or local minimum. Compare the objective function values at the local extrema with those at the endpoint to make sure the value(s) you've found are actually the global maxima/minima.

8. Supposing you've found one or more x-values that give you the desired maximum or minimum, you may need to use your equations from steps 2 and 4 to solve for other variables in the problem. For example, if the problem asks for the minimum area but you've solved for only a side length, you'd need to find the other side length using step 4 and the area using step 2.