

# Final - QSci 291 - Winter 2014

March 18, 2014

The first two questions are worth 15 pts each, while the rest are 5 pts each. Put your name on the reverse side of each page.

- (1) Find the critical points for the following function and determine if they are global. Make sure that you explore all 3 pools of potential critical points:

$$y = |x^2 + 2x - 8|,$$

where  $x \in [-10, 10]$ .

- Rewrite function  $y$  to remove absolute values (piece-wise transformation):

$$y = \begin{cases} x^2 + 2x - 8 & \text{if } x \in [-10, -4] \cup [2, 10) \\ -x^2 - 2x + 8 & \text{if } x \in (-4, 2) \end{cases}$$

2 pts

- Identify critical points, where the first derivative is zero:

$$y' = \begin{cases} 2x+2 & \text{if } x \in (-10, -4) \cup (2, 10) \\ -2x-2 & \text{if } x \in (-4, 2) \end{cases}$$

Note that domain endpoints are excluded because at these points only one-sided limits exist. This means that the function is not differentiable at these points.

- $2x+2=0$   
 $x=-1 \notin (-10, -4) \cup (2, 10)$

- $-2x-2=0$   
 $x=-1 \in (-4, 2) \Rightarrow (-1, 9)$  is a critical point

2 pts

- Identify critical points where the first derivative is undefined:

- Identify critical points that are endpoints:  
 $x = -10$  ( $-10, 72$ )

2 pts

$(x = 10)$  is not a critical point because it is not part of the function's domain

- Check which of these 4 points are local max or min:

$(-1, 9)$  is local max because  $y'' = -2$  (negative)

$(-4, 0)$  is local min because at  $x = -4, 1$  and  $x = -3, 0 \Rightarrow y > 0$

(2, 0) is a local min because at  $x=1.9$  and  $x=2.1$ ,  $y > 0$

2.5 pts

(-10, 72) is a local max because at  $x=-9.9$ ,  $y < 72$

• Check which of these points are global extrema:

(-1, 9) is not global because  $(-10, 72)$  gives a higher value for  $y$

There is no global max!

(-4, 0) & (2, 0) are both global minima because  $\lim_{x \rightarrow 10} y = 112 > 0$

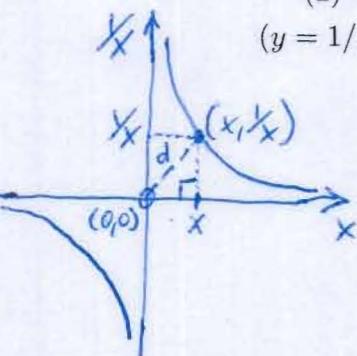
There are 2 global min:  $(x=-4, x=2)$

(-10, 72) is not global max because  $\lim_{x \rightarrow 10} y = 112 > 72$

2.5 pts

(2) Use optimization to determine how close the curve of the reciprocal function

( $y = 1/x$  with domain  $x \in \mathbb{R} \setminus \{0\}$ ) comes to the origin  $(0, 0)$ :



- Identify distance function by applying the Pythagorean Theorem to right-angle triangle with vertices  $(0,0)$ ,  $(x, 1/x)$  and  $(x, 0)$ :

$$d^2 = x^2 + \left(\frac{1}{x}\right)^2 = x^2 + x^{-2} \Rightarrow d = \sqrt{x^2 + x^{-2}}$$

5 pts

- Find critical points on  $d^2(x)$  [whatever  $x$  minimizes  $d^2(x)$  will minimize  $d(x)$  by setting  $[d^2(x)]' = 0$ :  $2x - 2x^{-3} = 0$ ]

$$x = x^{-3} = \frac{1}{x^3} \quad \begin{cases} 0 \rightarrow \text{is not a critical point} \\ \text{because it is not part of the domain.} \end{cases}$$

Critical points! 4 pts

- 1 pt → • Find critical points where  $[d^2(x)]' \in \mathbb{Q}$ : none because polynomial & rational functions are differentiable on their domain

- 1 pt → • Find endpoints: none because the distance function is defined on an open interval

- Check if  $x=1, x=-1$  are local min or max:  $[d^2(x)]'' = 2 + 6x^{-4}$ , which is  $\neq 0$

1 pt →

- Check if  $x=1, x=-1$  are global min:

$$\lim_{x \rightarrow \infty} (x^2 + x^{-2}) = \lim_{x \rightarrow \infty} x^2 + \lim_{x \rightarrow \infty} \frac{1}{x^2} = \infty + 0 = \infty > d^2(1) = d^2(-1) = 1$$

$$\lim_{x \rightarrow -\infty} (x^2 + x^{-2}) = \lim_{x \rightarrow -\infty} x^2 + \lim_{x \rightarrow -\infty} \frac{1}{x^2} = \infty + 0 = \infty > \sqrt{2}$$

2 pts →  $(1, \sqrt{2})$  and  $(-1, \sqrt{2})$  are both global min with  $\sqrt{2}$  being the distance.

(3) Find:

$$\lim_{x \rightarrow \infty} \frac{17 - 3x - 5x^3 + 4x^4}{x^5 - 2x} = 0 \text{ because the degree of the poly-}$$

nomial in the denominator is greater than that of the polynomial in the numerator.

2.5 pts

2.5 pts for explanation

(4) Calculate  $f'(x)$  for the following function:

Apply product rule and then chain  
rule twice:

$$f(x) = \log_3(\cos^2 x) 3^x$$

$$f'(x) = \frac{1}{\ln 3 \cdot \cos^2 x} \cdot 2 \cos x \cdot (-1) \sin x \cdot 3^x + 3^x \cdot \ln 3 \cdot \log_3(\cos^2 x) =$$

$$= -\frac{2 \sin x \cos x}{\ln 3 \cos^2 x} 3^x + \log_3(\cos^2 x) \cdot \ln 3 \cdot 3^x = 3^x \cdot \left[ -\frac{2 \sin x}{\ln 3 \cos x} + \ln 3 \cdot \log_3(\cos^2 x) \right]$$

5 pts

(5) Test if the following function is continuous at  $t = -1$ :

$$g(t) = \begin{cases} t^3 - 2t & \text{if } t \geq -1, \\ e^t & \text{otherwise.} \end{cases}$$

(2 pts)

- Step 1: Is  $g(t)$  defined at  $t = -1$ ? Yes,  $g(-1) = (-1)^3 - 2(-1) = -1 + 2 = 1$
  - Step 2: Does  $\lim_{t \rightarrow -1} g(t)$  exist? No, because  $\lim_{t \rightarrow -1^+} g(t) = 1 \neq \lim_{t \rightarrow -1^-} g(t) = e^{(-1)} = \frac{1}{e}$
- g(t) is not continuous at t = -1       $\neq \lim_{t \rightarrow -1^-} g(t) = e^{(-1)} = \frac{1}{e}$       (2 pts)
- 1 pt

(6) Find the explicit formula for sequence  $\{-1, 1/2, -1/3, 1/4, -1/5, \dots\}$ :

• If you set  $a_1 = -1$ , then  $a_n = (-1)^n \cdot \frac{1}{n}$

• If you set  $a_0 = -1$ , then  $a_n = (-1)^{(n+1)} \cdot \frac{1}{n+1}$

} both are correct.

5 pts