Basic Linear Programming Concepts

Lecture 2 (3/29/2017)

Definition

- "Linear Programming (LP) is a mathematical method to allocate scarce resources to competing activities in an optimal manner when the problem can be expressed using a linear objective function and linear inequality constraints."

An Liner Program (LP)

Objective function coefficient: represents the contribution of one unit of each variable to the value of the obj. func.

$$Max \sum_{i=1}^{n} c_i X_i = c_1 X_1 + c_2 X_2 + \ldots + c_n X_n$$

Decision variables

$$subject\ to:$$

RHS constraint coefficient: they represent limitations

$$\sum_{i=1}^{n} a_{j,i} \cdot X_i \le b_j \qquad for\ j = 1, 2, \ldots, m$$

$$X_i \ge 0 \qquad for\ i = 1, 2, \ldots, n$$

Non-negativity constraint

Formulating Linear Programs

1. Identify the decision variables;
2. Formulate the objective function;
3. Identify and formulate the constraints; and
4. Write out the non-negativity constraints

The Lumber Mill Problem

A lumber mill can produce pallets or high quality lumber. Its lumber capacity is limited by its kiln size. It can dry **200 mbf** per day. Similarly, it can produce a maximum of **600 pallets per day**. In addition, it can only process **400 logs per day** through its main saw. Quality lumber sells for **$490 per mbf**, and pallets sell for **$9 each**. It takes **1.4 logs** on average **to make one mbf** of lumber, and **four pallets** can be made **from one log**. Of course, different grades of logs are used in making each product. Grade 1 lumber logs cost **$200 per log**, and pallet-grade logs cost only **$4 per log**. Processing costs per mbf of quality lumber are **$200 per mbf**, and processing costs per pallet are only **$5**. How many pallets and how many mbf of lumber should the mill produce?

Solving the Lumber Mill Problem

1. Defining the decision variables: What do I need to know?

   P = the number of pallets to produce each day, and

   L = the number of mbf of lumber to produce each day

2. Formulate the objective function

$$Max\ Z = c_p \cdot P + c_L \cdot L$$

2. Formulating the objective function

$$c_p = \$9 - \$5 - 1/4 \cdot \$4 = \underline{\$3 / pallet}$$

$$c_L = \$490 - 1.4 \cdot \$200 - \$200 = \underline{\$10 / mbf}$$

$$Max\ Z = 3 \cdot P + 10 \cdot L$$

3. Formulating the constraints

$L \le 200\ (mbf\ /\ day)$ Kiln capacity constraint

$P \le 600\ (pallets / day)$ Pallet capacity constraint

$1/4 \cdot P + 1.4 \cdot L \le 400\ (logs / day)$

Log capacity constraint

3. Non-negativity constraints

$P \geq 0$ and $L \geq 0$

The complete formulation:

$$Max\ Z = 3 \times P + 10 \times L$$

$$Subject\ \ to:$$

$$L \leq 200$$

$$P \leq 600$$

$$1/4 \cdot P + 1.4 \cdot L \leq 400$$

$$P \geq 0$$

$$\mathrm{L} \geq 0$$

A logging Problem

A logging company must allocate logging equipment between two sites in the manner that will maximize its daily net revenues. They have determined that the net revenue of a cord of wood is $1.90 from site 1 and $2.10 from site 2. At their disposal are two skidders, one forwarder and one truck. Each kind of equipment can be used for 9 hours per day, and this time can be divided in any proportion between the two sites. The equipment needed to produce a cord of wood from each site varies as show in the table below:

Site	Skidder	Forwarder	Truck
1	0.30	0.30	0.17
2	0.40	0.15	0.17

Solving the Logging Problem

1. Defining the decision variables: What do I need to know?

   X_1 = the number of cords per day to produce from site 1;
   X_2 = the number of cords per day to produce from site 2.

2. Formulate the objective function

$$Max\ Z = c_1 \cdot X_1 + c_2 \cdot X_2$$

2. Formulating the objective function

$$Max\ Z = 1.9 \cdot X_1 + 2.1 \cdot X_2$$

3. Formulating the constraints

$$0.30 \cdot X_1 + 0.40 \cdot X_2 \leq 18\ (skidder - hrs / day)$$

Skidder constraint

$$0.30 \cdot X_1 + 0.15 \cdot X_2 \leq 9\ (forwarder - hrs / day)$$

Forwarder constraint

$$0.17 \cdot X_1 + 0.17 \cdot X_2 \leq 9\ (truck - hrs / day)$$

Truck constraint

3. Non-negativity constraints

$$X_1 \geq 0 \text{ and } X_2 \geq 0$$

The complete formulation:

$$Max\ Z = 1.90 \cdot X_1 + 2.10 \cdot X_2$$

$$Subject\ to:$$

$$0.30 \cdot X_1 + 0.40 \cdot X_2 \leq 18$$

$$0.30 \cdot X_1 + 0.15 \cdot X_2 \leq 9$$

$$0.17 \cdot X_1 + 0.17 \cdot X_2 \leq 9$$

$$X_1 \geq 0$$

$$X_2 \geq 0$$

The complete formulation of the Logging Problem

$$Max\ Z = 1.90 \cdot X_1 + 2.10 \cdot X_2$$

$$Subject\ to:$$

$$0.30 \cdot X_1 + 0.40 \cdot X_2 \leq 18$$

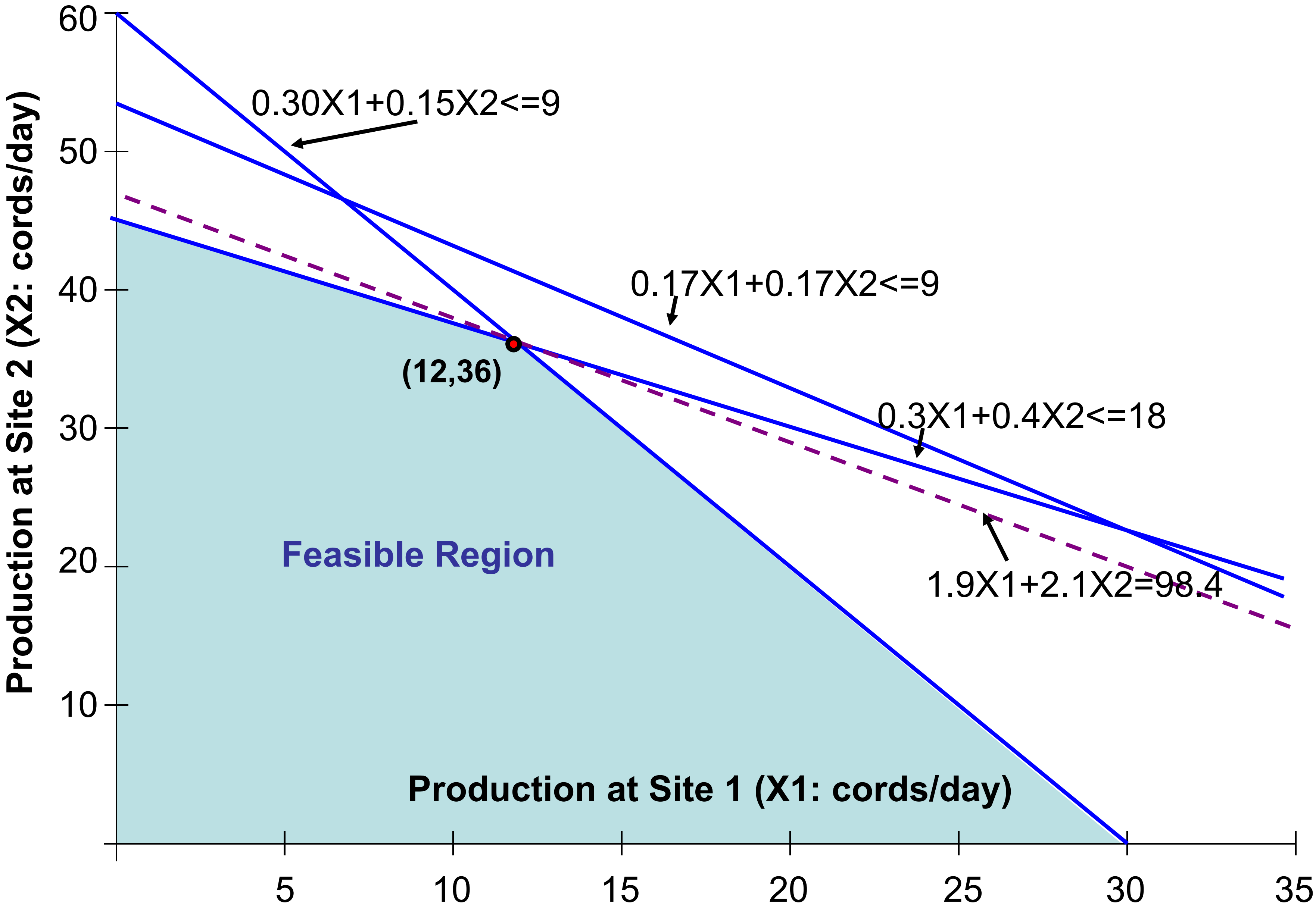
$$0.30 \cdot X_1 + 0.15 \cdot X_2 \leq 9$$

$$0.17 \cdot X_1 + 0.17 \cdot X_2 \leq 9$$

$$X_1 \geq 0$$

$$X_2 \geq 0$$

Graphical solution to the Logging Problem



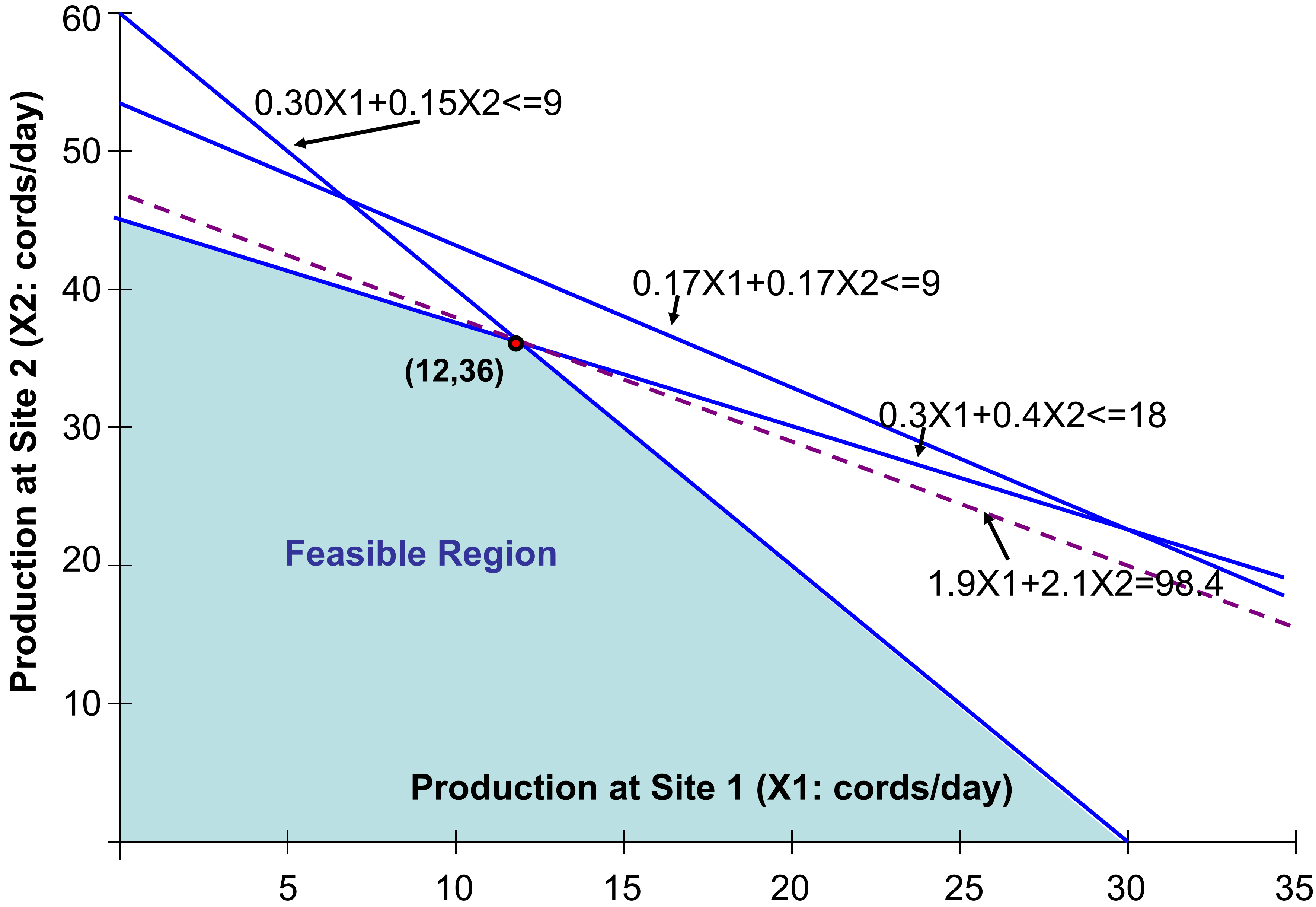
Key points from the graphical solution

- Constraints define a polyhedron (polygon for 2 variables) called *feasible region*
- The objective function defines a set of parallel *n*-dimensional hyperplanes (lines for 2 variables) – one for each potential objective function value
- The optimal solution(s) is the last corner or face of the feasible region that the objective function touches as its value is improved

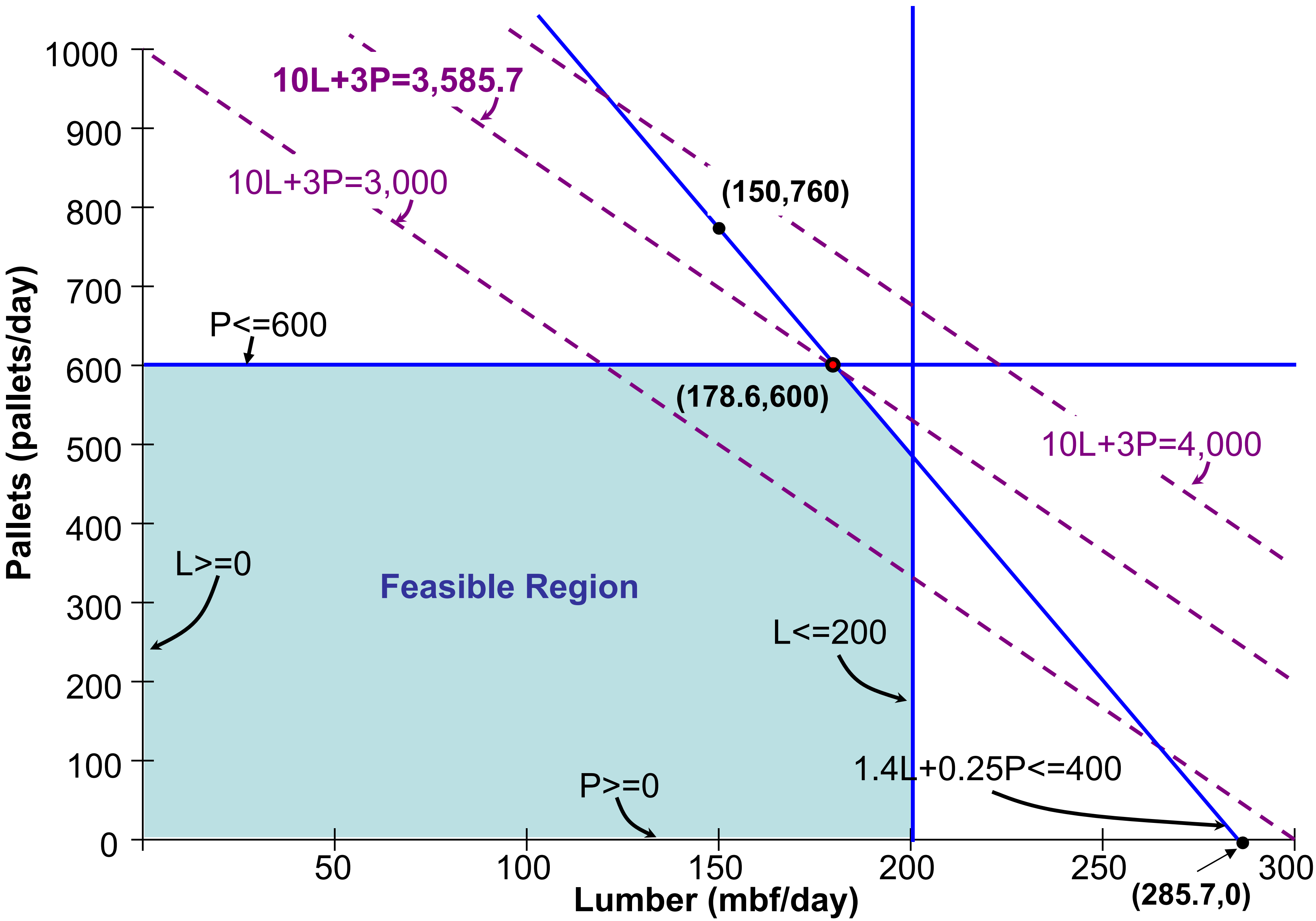
Solving an LP

- The Simplex Algorithm: searching for the best corner solution(s)
- Multiple versus unique solutions
- Binding constraints

Graphical solution to the Logging Problem



Graphical solution to the Lumber Mill Problem



Interpreting Computer Solutions to LPs

- Optimal values of the variables
- The optimal objective function value
- Reduced costs
- Slack or surplus values
- Dual (shadow) prices

Reduced costs

- Reduced costs are associated with the <u>variables</u>
- Reduced cost values are non-zero only when the optimal value of a variable is zero
- It indicates how much the objective function coefficient on the corresponding variable must be improved before the value of the variable will turn positive in the optimal solution

Reduced costs cont.

- Minimization (improved = reduced)
- Maximization (improved = increased)
- If the optimal value of a variable is positive, the reduced cost is always zero
- If the optimal value of a variable is zero and the corresponding reduced cost value is also zero, then multiple solutions exist

Slack or surplus

- They are associated with each <u>constraint</u>
- <u>Slack</u>: less-than-or-equal constraints
- <u>Surplus</u>: greater-than-or-equal constraints
- For binding constraints: the slack or surplus is zero
- <u>Slack</u>: amount of resource not being used
- <u>Surplus</u>: extra amount being produced over the constraint

Dual Prices

- Also called: shadow prices
- They are associated with each <u>constraint</u>
- The dual price gives the improvement in the objective function if the constraint is relaxed by one unit

Sample LP solution output

```
MAX
  3 P + 10 L
 st
 L <= 200
 P <= 600
 0.25 P + 1.4 L <= 400
End

 LP OPTIMUM FOUND AT STEP     2

        OBJECTIVE FUNCTION VALUE

        1)      3585.714

  VARIABLE        VALUE          REDUCED COST
    P              600.000000        0.000000
    L              178.571400        0.000000

       ROW   SLACK OR SURPLUS      DUAL PRICES
        2)           21.418570          0.000000
        3)             0.000000          1.214286
        4)             0.000000          7.142857

 NO. ITERATIONS=      2
```

The Fundamental Assumptions of Linear Programming

1. Linear constraints and objective function
   a. <u>Proportionality</u>: the value of the obj. func. and the response of each resource is proportional to the value of the variables
   b. <u>Additivity</u>: there is no interaction between the effects of different activities
2. <u>Divisibility</u> (the values of the decision variables can be fractions)
3. <u>Certainty</u>
4. <u>Data availability</u>