Combinations and Permutations

Combinations -- number of different samples of containing r elements that can be selected from n elements

Permutations – number of different ***ordered*** samples containing r elements that can be selected from n elements

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Combinations Rule

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Denoted as “n choose r”

where “!” is the factorial symbol

$$r! = r(r-1)(r-2)\cdots(3)(2)(1)$$

$$0! = 1$$

May also be denoted as ${}_nC_r$

Permutations Rule

$${}_nP_r = \frac{n!}{(n-r)!}$$

Thus, ${}_nC_r = \dfrac{{}_nP_r}{r!}$

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Combinations Rule

“100 choose 5”

$$\binom{100}{5} = \frac{100!}{5!(100-5)!}$$
$$= \frac{100\times 99\times 98\times 97\times 96\times 95!}{5\times 4\times 3\times 2\times 1\times 95!}$$
$$= 5\times 33\times 49\times 97\times 96$$
$$= 75{,}287{,}520$$

$$\binom{100}{95} = \frac{100!}{95!(100-95)!}$$
$$= \frac{100\times 99\times 98\times 97\times 96\times 95!}{95!\times 5\times 4\times 3\times 2\times 1}$$
$$= 5\times 33\times 49\times 97\times 96$$
$$= 75{,}287{,}520$$

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Example 1

5 cards are drawn at random from a well shuffled deck of 52 cards

Find the probability of drawing 4 aces

Solution

Total number of ways to choose 5 cards from 52 =

$$\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52\times 51\times 50\times 49\times 48\times 47!}{5!47!}$$
$$= \frac{52\times 51\times 50\times 49\times 48}{5\times 4\times 3\times 2\times 1}$$
$$= 2{,}598{,}960$$

This gives us the denominator; what about the numerator?

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Example 1 (continued)

____ ____ ____ ____ ____

$$\frac{\binom{4}{4}\times 48}{\binom{52}{5}} = \frac{1\times 48}{2{,}598{,}960} = 0.000018 \text{ or } 0.0018\%$$

Example 2

Find the probability of drawing 2 queens and 3 kings.

$$\frac{\binom{4}{2}\binom{4}{3}}{\binom{52}{5}} = \frac{6\times 4}{2{,}598{,}960} = 9.6\times 10^{-6}$$

Key Formulas for Probability

Complementary Events

$$P(A)+P(\overline{A})=1$$

Additive Rule

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

Mutually Exclusive Events

$$P(A\cap B)=0$$

Additive Rule for Mutually Exclusive Events

$$P(A\cup B)=P(A)+P(B)$$

Key Formulas for Probability

Conditional Probability

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

Multiplicative Rule

$$P(A\cap B)=P(B)P(A|B)=P(A)P(B|A)$$

Independent Events

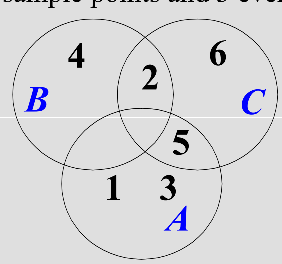
$$P(A|B)=P(A)$$

Multiplicative Rule for Independent Events

$$P(A\cap B)=P(A)P(B)$$

Example 3

Six sample points and 3 events



$P(1) = .3;\ P(2) = P(6) = .2;\ P(3) = P(4) = P(5) = .1$

Find the following (exercise for home):

$P(A\cap B)$, $P(B\cap C)$, $P(A\cup C)$, $P(A\cup B\cup C)$, $P(\overline{B})$, $P(\overline{A}\cap B)$, $P(B|C)$, and $P(B|A)$

Example 3 (continued)

Six sample points and 3 events

- Are A and B independent? Mutually exclusive? Why?

- Are B and C independent? Mutually exclusive? Why?

Example 4

A company that manufactures computer chips uses two different manufacturing processes. Process 1 produces nondefective chips 98.5% of the time and process 2 produces nondefective chips 97.1% of the time. Process 1 is used 60% of the time. What is the probability that a randomly chosen chip was produced by process 2 and is defective?

Let A: {chip is defective} and
B: {chip was produced by process 2}

Want to find $P\left(A \cap B\right)$.

Example 5

You have an unfair coin with
P(heads) = 2/3 and P(tails) = 1/3

You also have two urns with colored marbles:

Urn 1: 3 blue, 5 red

Urn 2: 7 blue, 6 red

Conduct the following experiment: toss the coin; if heads draw a ball at random from Urn 1; if tails, draw a ball at random from Urn 2.

Question: What is the probability of drawing a blue marble?

Example 6

The probability that an Avon salesperson sells beauty products to a prospective customer on the first visit to the customer is 0.4. If the salesperson fails to make the sale on the first visit, the probability that the sale will be made on the second visit is 0.65. The salesperson never visits a prospective customer more than twice. What is the probability that the salesperson will make a sale to a particular customer?

Bayes Rule

Requires the Rule of Total Probability

Suppose that events $A_1, A_2, \ldots, A_k$ are mutually exclusive and exhaustive, that is exactly one of the events must occur. Then for any event B,

$$P(B) = \sum_{j=1}^{k} P(A_j) P(B | A_j)$$

Bayes Rule (continued)

Region	% of US Population	% Seniors
NE	19.0	13.8
Mid W	23.1	13.0
South	35.5	12.8
West	22.4	11.1
	100.0	

Suppose a U.S. resident is selected at random

$S =$ event the resident selected is a senior

$R_1 =$ event the resident selected lives in NE

$R_2 =$ event the resident selected lives in Mid W

$R_3 =$ event the resident selected lives in South

$R_4 =$ event the resident selected lives in W

Bayes Rule (continued)

Region	% of US Population	% Seniors
NE	19.0	13.8
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South	35.5	12.8
West	22.4	11.1
	100.0	

$$P(S) = \sum_{j=1}^{4} P(R_j) P(S | R_j) =$$

Bayes Rule (continued)

Suppose that events $A_1, A_2, \ldots, A_k$ are mutually exclusive and exhaustive, that is exactly one of the events must occur. Then for any event B,

$$P(A_i | B) = \frac{P(A_i) P(B | A_i)}{\sum_{j=1}^{k} P(A_j) P(B | A_j)}$$

where A_i can be any one of the events $A_1, A_2, \ldots, A_k$

Bayes Rule (continued)

Return to the regions/senior example and ask: What percentage of seniors are NE residents?

$$P(R_1|S) = \frac{P(R_1)P(S|R_1)}{\sum_{j=1}^{4} P(R_j)P(S|R_j)} =$$

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Bayes Rule (continued)

Prior probability

$P(R_1)$ is a prior probability because is represents the probability that the person selected lives in the NE *before* knowing whether the person is a senior

Posterior probability

Suppose that the person selected is determined to be a senior—using this information we can revise the probability that the person lives in the NE ➔ $P(R_1|S)$

This represents the probability that the person selected lives in the NE *after* we learn that the person is a senior

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Example 7 (from Weiss, Introductory Statistics, 8th edition, page 212)

The national Sporting Goods Association collects and publishes data on participation in selected sports activities. For Americans 7 years old or older, 17.4% of males and 4.5% of females play golf. And, according to the U.S. Census Bureau's Current Population Reports, of Americans 7 years old or older, 48.6% are male and 51.4% are female. From among those who are 7 years old or older, one is selected at random. Find the probability that the person selected

a) plays golf

b) plays golf, given that the person is male

c) is a female, given that the person plays golf

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