# Lesson Week 7

Definition of the Fourier Transform

In the previous three lessons, we discussed the Fourier Series, which is for periodic signals. This lesson will cover the Fourier Transform which can be used to analyze aperiodic signals. (Later on, we'll see how we can also use it for periodic signals.) The Fourier Transform is another method for representing signals and systems in the frequency domain.

Definition of the Fourier Transform
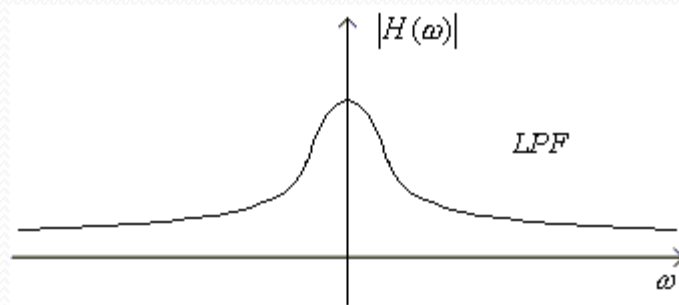
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

is the continuous time Fourier transform of $f(t)$.

It is an extension of the Fourier Series. The Fourier transformation creates $F(\omega)$ in the FREQUENCY domain. We will see that $F(\omega)$ can be seen as a "continuous coefficient" of a Fourier Series if we let the period of a periodic function go to infinity so that the resulting function becomes aperiodic in the limit.
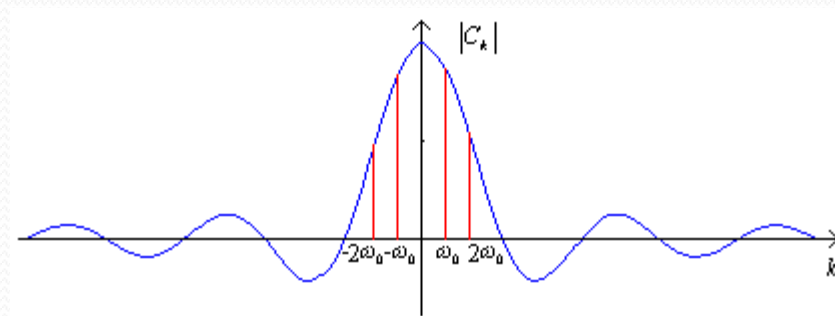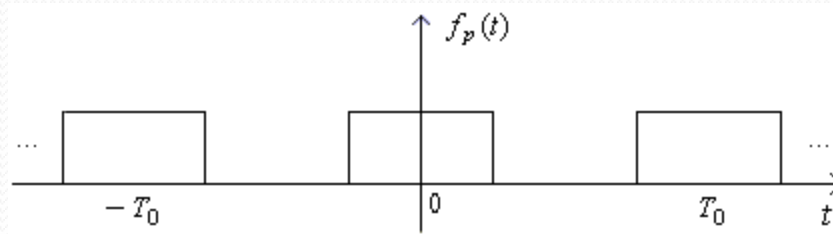
Let $f_p(t)$ be a periodic function. Therefore, we can express it with a Fourier series:

$$f_p(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}, \quad C_k = \frac{1}{T_0} \int_{T_0} f_p(t) e^{-jk\omega_0 t} dt$$

As usual,

$$T_0 = \frac{2\pi}{\omega_0} \text{ is the period of } f_p(t).$$

Assuming that $f_p(t)$ is a periodic rectangular pulse train, let's plot its magnitude spectrum C_k vs. $\omega=k\omega_o$

As shown, each C_k is the frequency component of $f_p(t)$ at the frequency $\omega = k \omega_o$.

Now, if we let T_o, the period of $f_p(t)$, go to infinity (meaning that $f_p(t)$ becomes aperiodic), then the fundamental frequency ω_o will go to zero. Then the lines in the plot will get closer and closer together and merge into a continuous spectrum.

As $\omega_o \to 0$, the distance between lines goes to 0, so let's write $\Delta\omega$ for ω_o in our Fourier Series representation of $f_p(t)$:

$$f_p(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} C_k e^{jk\Delta\omega t}$$

Next, we can replace C_k in our Fourier Series representation of $f_p(t)$ by using the formula we derived in Lesson 13 for the Fourier Series coefficients:

$$f_p(t) = \sum_{k=-\infty}^{\infty} \left[\frac{1}{T_0} \int_{T_0} f_p(t) e^{-jk\omega_0 t} dt \right] e^{jk\Delta\omega t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \left[\int_{-\frac{\pi}{\Delta\omega}}^{\frac{\pi}{\Delta\omega}} f_p(\tau) e^{-jk\Delta\omega \tau} d\tau \right] e^{jk\Delta\omega t}$$

Next, Let $T_0 \to \infty$, $(\Delta\omega \to 0)$ and define $f(t) = \lim\limits_{T_0 \to \infty} f_p(t)$:

$$f(t) = \frac{1}{2\pi} \lim\limits_{\Delta\omega \to 0} \sum_{k=-\infty}^{\infty} \left[\int_{-\frac{\pi}{\Delta\omega}}^{\frac{\pi}{\Delta\omega}} f(\tau)\, e^{-jk\Delta\omega\tau} d\tau \right] e^{jk\Delta\omega t} \Delta\omega$$

As $\Delta\omega \to 0$, we write it as $d\omega$ and the sum becomes an integral (and $k\Delta\omega$ approaches ω as $T_o \to \infty$ where ω is a continuous variable).

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau} d\tau \right] e^{j\omega t} d\omega$$

Let $\quad F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad$ be the Fourier Transform of $f(t)$.

So $F(\omega)$ replaces the C_k as $\Delta\omega \to 0$ and is a continuous function of ω.

<u>Finally, we have derived our Fourier transform pair:</u>

$$\text{ANALYSIS } F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\text{SYNTHESIS } f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Notice the similarity between the two formulas except for the sign change in the exponent and the multiplicative factor in front of the synthesis formula.

Because the Fourier Transform is an integral over an infinite range, we must consider whether or not the integral converges. Sufficient conditions for the existence of the Fourier Transform are the Dirichlet conditions. That is, the Fourier Transform exists if:

1. On any finite interval
   (a) $f(t)$ is bounded
   (b) $f(t)$ has a finite number of minima and maxima
   (c) $f(t)$ has a finite number of discontinuities

2. $f(t)$ is absolutely integrable, that is

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

This can be seen because we know that $|e^{-j\omega t}| = 1$:

$$|F(\omega)| = \left| \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right| \le \int_{-\infty}^{\infty} |f(t) e^{-j\omega t}| dt$$

$$\int_{-\infty}^{\infty} |f(t)| |e^{-j\omega t}| dt = \int_{-\infty}^{\infty} |f(t)| dt$$
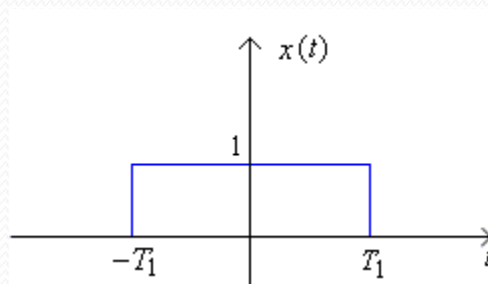
Therefore, if $f(t)$ is absolutely integrable, then its Fourier Transform exists.

3. Basically, if you can generate a signal in a laboratory, since it has finite energy, it will have a Fourier Transform.

Now, let's compute our first Fourier Transforms:

Example 1 Compute the Fourier Transform of

$$x(t) = \begin{cases} 1 & -T_1 \leq t \leq T_1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

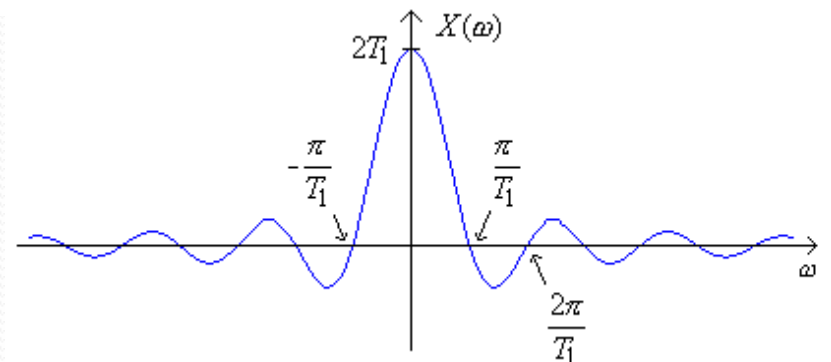$$= \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = \frac{j}{\omega}e^{-j\omega t}\Big|_{-T_1}^{T_1}$$

$$= -\frac{1}{j\omega}\left(e^{-j\omega T_1} - e^{j\omega T_1}\right)$$
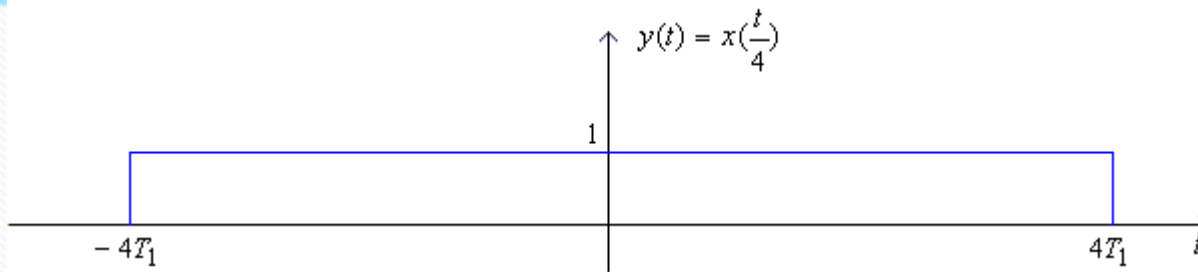
$$= \frac{-2j\sin(\omega T_1)}{-j\omega}$$

$$= 2T_1\,\mathrm{sinc}(\omega T_1)$$
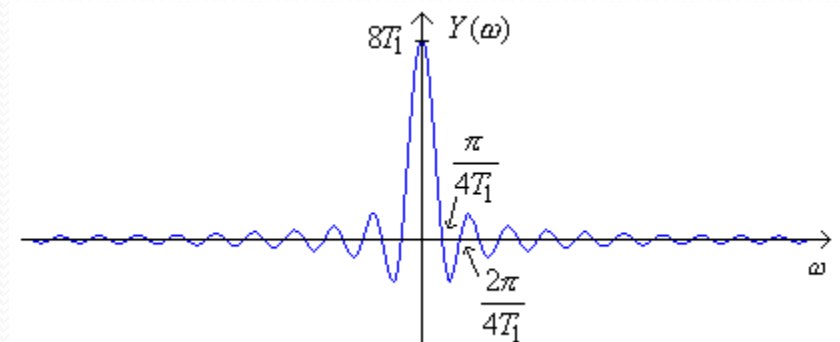
Note: $\mathrm{sinc}(t) = \dfrac{\sin(t)}{t}$

is an <u>IMPORTANT</u> continuous time function that we will see many times.
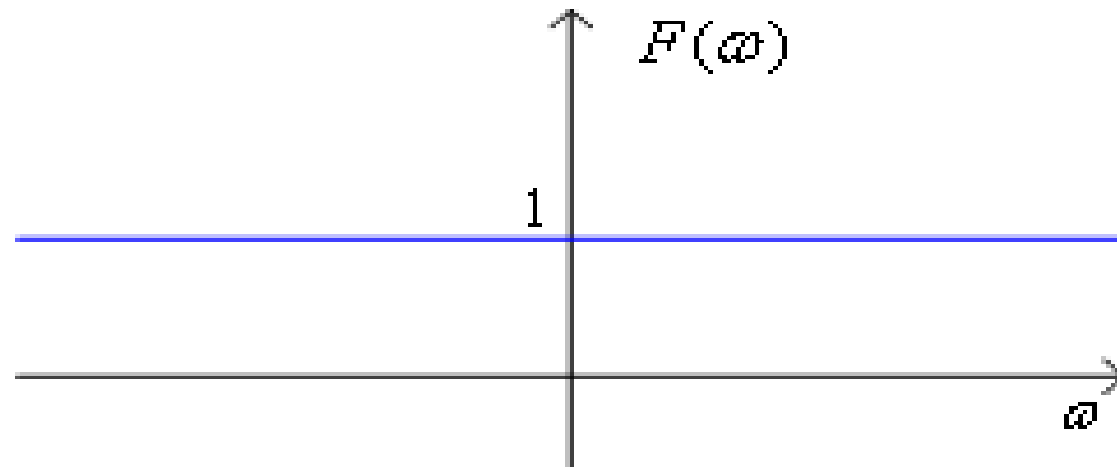
Example 2 Compute the Fourier Transform of

$$y(t) = x(\frac{t}{4})$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t} dt$$

$$= \int_{-4T_1}^{4T_1} e^{-j\omega t} dt = \frac{-1}{j\omega} e^{-j\omega t}\Big|_{-4T_1}^{4T_1}$$

$$= -\frac{1}{j\omega}\left(e^{-j\omega 4T_1} - e^{j\omega 4T_1}\right)$$

$$= \frac{-2j\sin(\omega 4T_1)}{-j\omega}$$

$$= \frac{-8T_1\sin(\omega 4T_1)}{-\omega 4T_1}$$
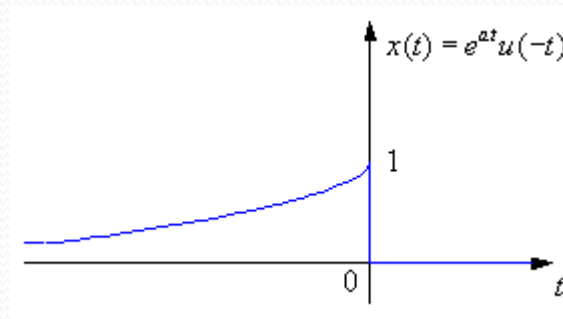
$$= 8T_1\,\mathrm{sinc}\,(\omega 4T_1)$$

Example 3 Compute the Fourier Transform of $\delta(t)$

$$y(\omega) = \int_{-\infty}^{\infty} \delta(t) \, e^{-j\omega t} dt = 1$$

$$F(\omega) = \int_{-\infty}^{\infty} C\delta(t - t_0) \, e^{-j\omega t} dt$$

$$= C e^{+j\omega t_0}$$

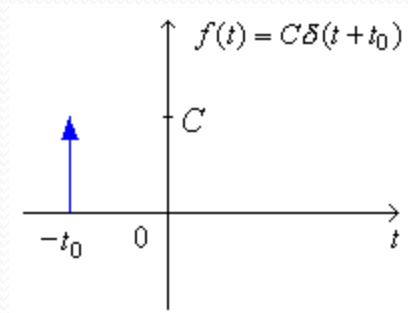**Example 4** Compute the Fourier Transform of

$$X(\omega) = \int_{-\infty}^{0} e^{(a-j\omega)t} dt$$

$$= \frac{1}{a - j\omega} e^{(a-j\omega)t} \Big|_{-\infty}^{0}$$

$$= \frac{1}{a - j\omega}(1-0) = \frac{1}{a - j\omega} \Rightarrow |X(\omega)| = \sqrt{\frac{1}{a^2 + \omega^2}}$$

Example 5 Find $F(\omega)$ for $f(t) = C\delta(t + t_o)$

$f(t) = C\delta(t + t_0)$

$$F(\omega) = \int_{-\infty}^{\infty} C\delta(t + t_0) e^{-j\omega t} \, dt$$

$$= C e^{+j\omega t_0}$$

Example 6 Find $f(t)$ by taking the Inverse Fourier Transform of $F(\omega)$. (This is the Fourier Transform of an ideal low pass filter.)

$$F(\omega) = \begin{cases} 1, & |\omega| < \omega_b \\ 0, & |\omega| > \omega_b \end{cases}$$

$$F(\omega) = \begin{cases} 1, & |\omega| < \omega_b \\ 0, & |\omega| > \omega_b \end{cases}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$
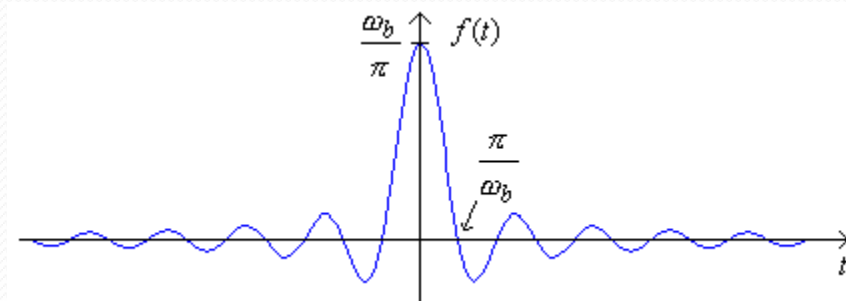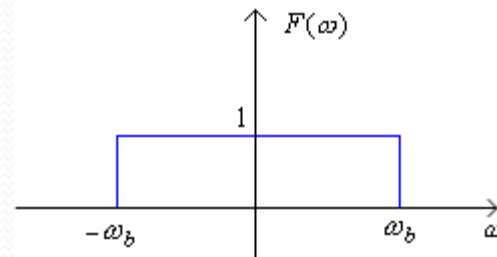
$$= \frac{1}{2\pi} \int_{-\omega_b}^{\omega_b} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \frac{1}{jt} e^{j\omega t} \Big|_{-\omega_b}^{\omega_b}$$

$$= \frac{1}{2\pi jt} \left(e^{jt\omega_b} - e^{-jt\omega_b} \right)$$

$$= \frac{2j \sin(\omega_b t)}{2\pi jt} = \frac{\omega_b}{\omega_b} \frac{\sin(\omega_b t)}{\pi t} = \frac{\omega_b}{\pi} \frac{\sin(\omega_b t)}{\omega_b t}$$

$$= \frac{\omega_b}{\pi} \operatorname{sinc}(\omega_b t)$$

Properties of the Fourier Transform (Linearity, Time Scaling, Time Shifting, and Duality)

The point of this lesson is that knowledge of the properties of the Fourier Transform can save you a lot of work. We will cover some of the important Fourier Transform properties here.

Linearity
Because the Fourier Transform is linear, we can write:
$$F[a\, x_1(t) + bx_2(t)] = aX_1(\omega) + bX_2(\omega)$$
where $X_1(\omega)$ is the Fourier Transform of $x_1(t)$ and $X_2(\omega)$ is the Fourier Transform of $x_2(t)$.

Time Scaling

$$x(t) \leftrightarrow X(\omega)$$

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Proof: $F[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$

Let $u = at, \, du = adt, \, dt = \frac{du}{a}$

$$F[x(at)] = \int_{-\infty}^{\infty} x(u) e^{-j\frac{\omega u}{a}} \frac{1}{a} du = \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

if $a>0$. If $a< 0$, then

$$F[x(at)] = \int_{+\infty}^{-\infty} x(u) e^{-j\frac{\omega u}{a}} \frac{du}{a} = -\frac{1}{a} X\left(\frac{\omega}{a}\right)$$

(since $u=at$). Therefore

$$\boxed{x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$$

Time Shifting

$$x(t) \leftrightarrow X(\omega)$$

$$x(t - t_0) \leftrightarrow X(\omega)e^{-j\omega t_0}$$

$$\text{Proof}: \quad F[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(u)e^{-j\omega(u+t_0)}du = e^{-j\omega t_0}X(\omega)$$

Duality

Note the <u>DUALITY</u> when you compare Examples 1 and 6 from Lesson 15.

Example 1 of Lesson 15 showed that the Fourier Transform of a block (or rect) function in time is a sinc in frequency. Example 6 of Lesson 15 showed that the Fourier Transform of a sinc function in time is a block (or rect) function in frequency.

In general, the Duality property is very useful because it can enable to solve Fourier Transforms that would be difficult to compute directly (such as taking the Fourier Transform of a sinc function). The Duality Property tells us that if $x(t)$ has a Fourier Transform $X(\omega)$, then if we form a new function of time that has the functional form of the transform, $X(t)$, it will have a Fourier Transform $x(\omega)$ that has the functional form of the original time function (but is a function of frequency). Mathematically, we can write:

$$x(t) \leftrightarrow X(\omega)$$
$$X(t) \leftrightarrow 2\pi\, x(-\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(b) e^{-j\omega b} db$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(b) e^{jbt} db$$

$$2\pi x(t) = \int_{-\infty}^{\infty} X(b) e^{jbt} db$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(b) e^{-jb\omega} db$$

Notice that the second term in the last line is simply the Fourier Transform integral of the function $X(t)$, i.e.

$$F[X(t)] = \int_{-\infty}^{\infty} X(b) e^{-jb\omega} db$$

Therefore we get the Duality Property:

$$x(t) \leftrightarrow X(\omega)$$

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

Example 1 Using the Fourier Transform integral equation, directly find the Fourier Transform of

$$x(t) = e^{-at}u(t), a > 0$$

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a + j\omega}$$

Example 2 Using the results of Example 1 and the Duality Property, find the Fourier Transform of

$$y(t) = \frac{1}{a + jt}$$

$$e^{at}u(-t) \longleftrightarrow \frac{1}{a - j\omega}$$

$$\frac{1}{a - jt} \longleftrightarrow 2\pi e^{-a\omega}u(\omega)$$

In this lesson, we will cover additional properties of the Fourier Transform. The most useful one is the Convolution Property. It tells us that convolution in time corresponds to multiplication in the frequency domain. Therefore, we can avoid doing convolution by taking Fourier Transforms! In many cases, this will be much more convenient than directly performing the convolution.

The Convolution Property

The convolution property states that:

$$x(t) * h(t) \leftrightarrow X(\omega)H(\omega)$$

Let us show that. To start, let

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = x(t) * h(t)$$

Then we can take the Fourier Transform of $y(t)$ and plug in the convolution integral for $y(t)$ (notice how we've marked the integrals with dt and $d\tau$ to keep track of them):

$$Y(\omega) = \int_{-\infty, dt}^{\infty} y(t)e^{-j\omega t} dt = \int_{-\infty, dt}^{\infty} \int_{-\infty, d\tau}^{\infty} x(\tau)h(t - \tau)d\tau \, e^{-j\omega t} dt$$

Now, let's switch the order of the two integrals (this is valid in all but the most pathological cases):

$$Y(\omega) = \int_{-\infty,d\tau}^{\infty} x(\tau) \int_{-\infty,dt}^{\infty} h(t-\tau)e^{-j\omega t} dt\, d\tau \quad \text{let } u = t - \tau$$

$$= \int_{-\infty,d\tau}^{\infty} x(\tau) \int_{-\infty,du}^{\infty} h(u)e^{-j\omega(u+\tau)} du\, d\tau$$

$$= \int_{-\infty,d\tau}^{\infty} x(\tau)e^{-j\omega\tau} d\tau \int_{-\infty,du}^{\infty} h(u)e^{-j\omega u} du$$

$$= X(\omega)H(\omega)$$

Therefore,

$$y(t) = x(t) * h(t) \leftrightarrow Y(\omega) = X(\omega)H(\omega)$$

We've just shown that the Fourier Transform of the convolution of two functions is simply the product of the Fourier Transforms of the functions. This means that for linear, time-invariant systems, where the input/output relationship is described by a convolution, you can avoid convolution by using Fourier Transforms. This is a very powerful result.

Multiplication of Signals

Our next property is the Multiplication Property. It states that the Fourier Transform of the product of two signals in time is the *convolution* of the two Fourier Transforms.

$$x_1(t)\, x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

$$x_1(t)\, x_2(t) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(u) * X_2(\omega - u)\, du$$

$$\text{Proof}: \quad F[x_1(t) x_2(t)] = \int_{-\infty}^{\infty} x_1(t) x_2(t) e^{-j\omega t}\, dt$$

Now, write $x_1(t)$ as an inverse Fourier Transform.

$$F[x_1(t) x_2(t)] = \int_{-\infty, dt}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty, da}^{\infty} X_1(a) e^{-jat}\, da \right] x_2(t) e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty, da}^{\infty} X_1(a) \int_{-\infty, dt}^{\infty} x_2(t) e^{-j(\omega - a)t}\, dt\, da$$

$$= \frac{1}{2\pi} \int_{-\infty, da}^{\infty} X_1(a) X_2(\omega - a)\, da$$

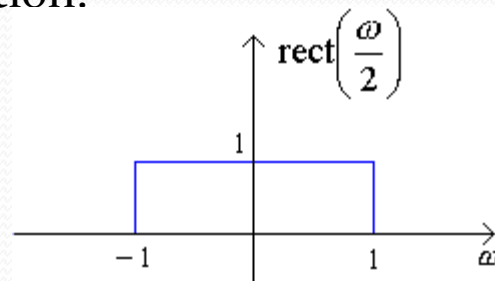$$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

Therefore,
$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

Example 1 Find the inverse Fourier Transform of

$$X(\omega) = \begin{cases} 1 & -1 \le \omega \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Here is a plot of this function:

$$x(t) = \frac{1}{2\pi} \int_{-1}^{1} e^{j\omega t} d\omega = \frac{1}{2\pi(jt)} e^{j\omega t} \Big|_{-1}^{1} = \frac{e^{jt} - e^{-jt}}{2j} \frac{1}{\pi t} = \frac{\sin(t)}{\pi t} = \frac{1}{\pi} \operatorname{sinc}(t)$$

Example 2 Find the Fourier Transform of $x(t) = \mathrm{sinc}^2(t)$ (Hint: use the Multiplication Property).
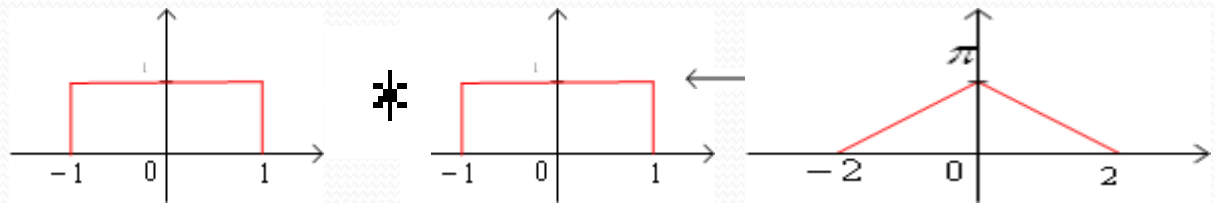
$$\frac{1}{\pi}\sin c(t) \leftrightarrow rect(\frac{\omega}{2})$$

$$\sin c(t) \leftrightarrow \pi\, rect(\frac{\omega}{2})$$

$$x_1(t) \times x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

$$\sin c^2(t) = \sin c(t) \times \sin c(t) = \frac{1}{2\pi}(\pi\, rect(\frac{\omega}{2}) * \pi\, rect(\frac{\omega}{2}))$$

The convolution of two rectangle functions is a triangle which is twice as wide.

Let's find the height of the triangle function.

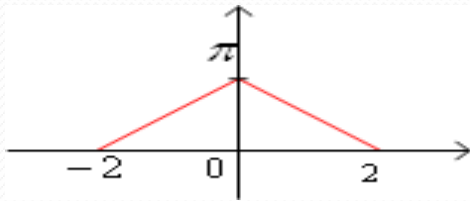$$X(0) = \frac{1}{2\pi} \int\limits_{-1}^{1} \pi^2 d\omega = \pi$$

Example 3 Find the Fourier Transform of $y(t) = \text{sinc}^2(t) * \text{sinc}(t)$. Use the Convolution Property (and the results of Examples 1 and 2) to solve this Example.
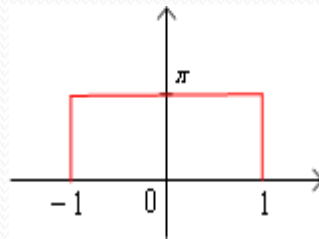
$$\text{sinc}(t) \leftrightarrow \text{rectangle function}$$

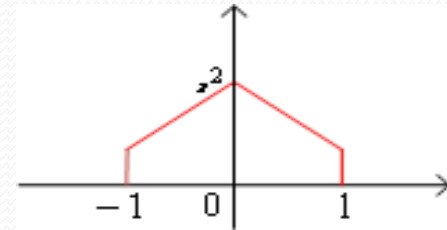$$\text{sinc}^2(t) \leftrightarrow \text{triangle function}$$

$$\text{sinc}^2(t) * \text{sinc}(t) \leftrightarrow \text{rectangle} \times \text{triangle}$$

Frequency Shifting or Modulation

$$F[e^{j\omega_0 t}x(t)] = \int\limits_{-\infty}^{\infty} e^{j\omega_0 t} x(t)e^{-j\omega t}dt$$

$$= \int\limits_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt = X(\omega - \omega_0)$$

$$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

This tells us that modulation (such as multiplication in time by a complex exponential, cosine wave, or sine wave) corresponds to a frequency shift in the frequency domain.