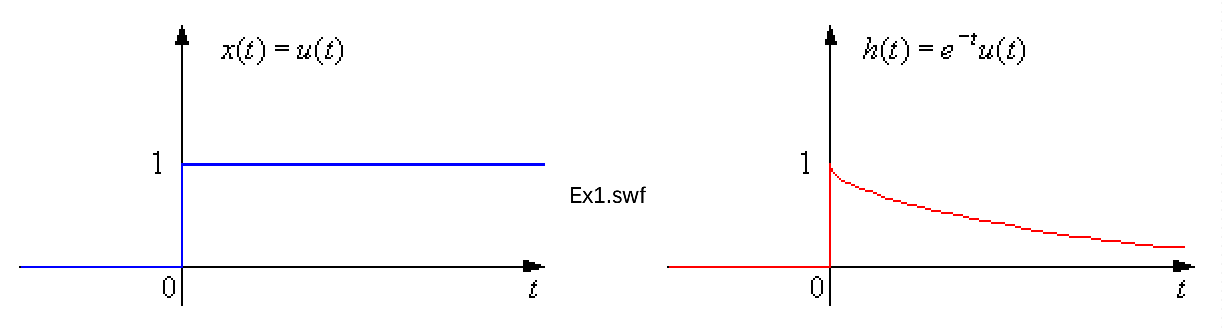
Lesson Week 3

More Examples of Continuous-Time Convolution

Example 2 This example shows why it is easier to flip the "easy" function rather than the "hard" function. Try it both ways!

Find $u(t) * e^{-t}u(t)$



Play the flash object ex1 posted to see how the convolution is performed

From before:
Sifting property of impulse:

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$$

"Sifts out value of function $x(t)$ where impulse is“

Now, what if we convolve $x(t)$ with an impulse?

Exemple Find $x(t) * \delta(t - t_o) = y(t)$

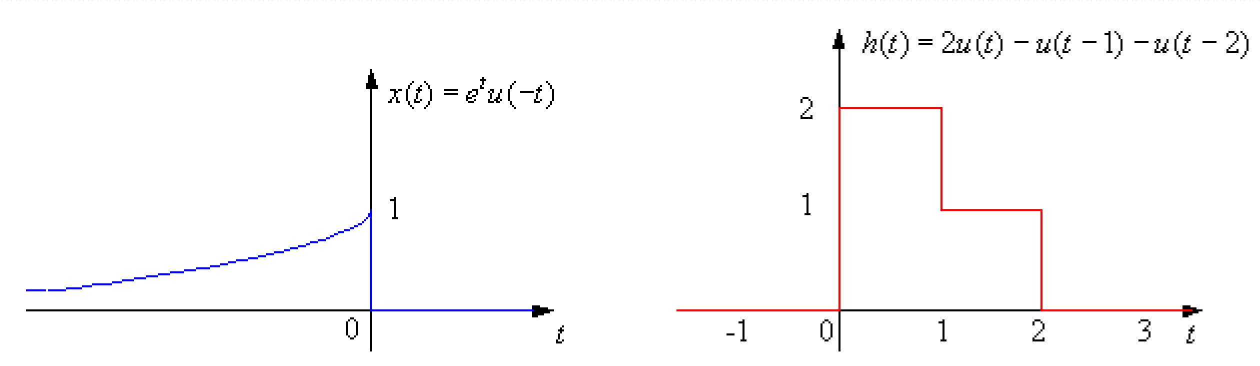
Play the flash object ex2 posted to see how the convolution is performed

Now, $\delta(t - \tau - t_o)$ is an impulse at $\tau = t - t_o$. So, by SIFTING property, we sift out the value of $x(t)$ where the impulse is. This is just $x(t - t_o)$.
So, convolve $x(t)$ with a shifted impulse, and you get $x(t)$ shifted to where impulse is.

So $$x(t) * \delta(t - t_o) = x(t - t_o)$$

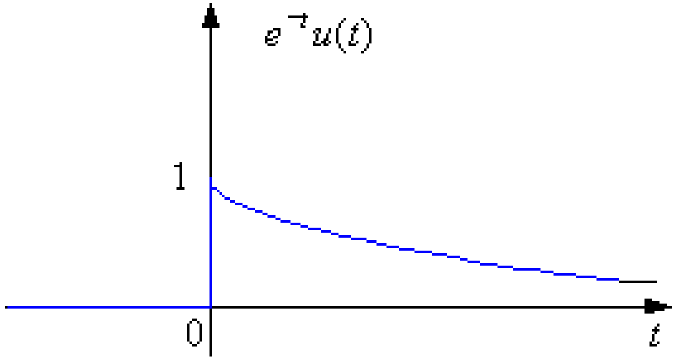
We'll see this a lot. One important place we'll see this is when we discuss SAMPLING or discretizing a continuous time signal.

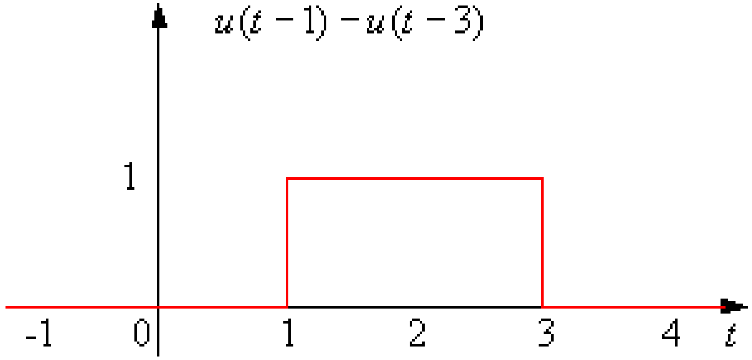
Example Find $h(t)*x(t)$ where $x(t) = e^t u(-t)$ and $h(t) = 2u(t) - u(t-1) - u(t-2)$.



Play the flash object ex3 posted to see how the convolution is performed

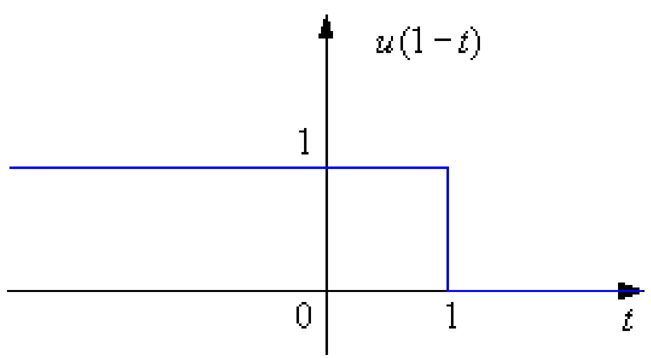
Example Find $e^{t}u(-t)*[\,u(t\text{-}1) - u(t\text{-}3)]$

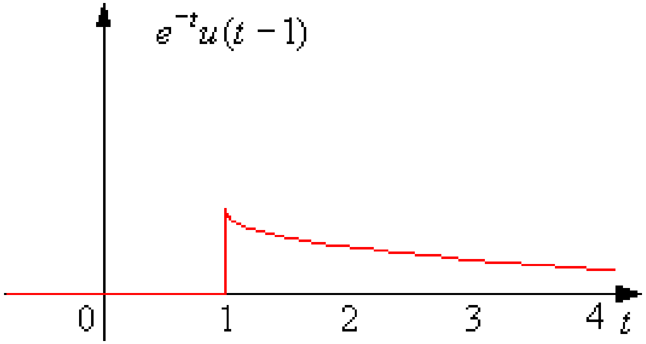




(Answer is not available yet)

Example Find $u(1-t) * e^{-t}u(t-1)$





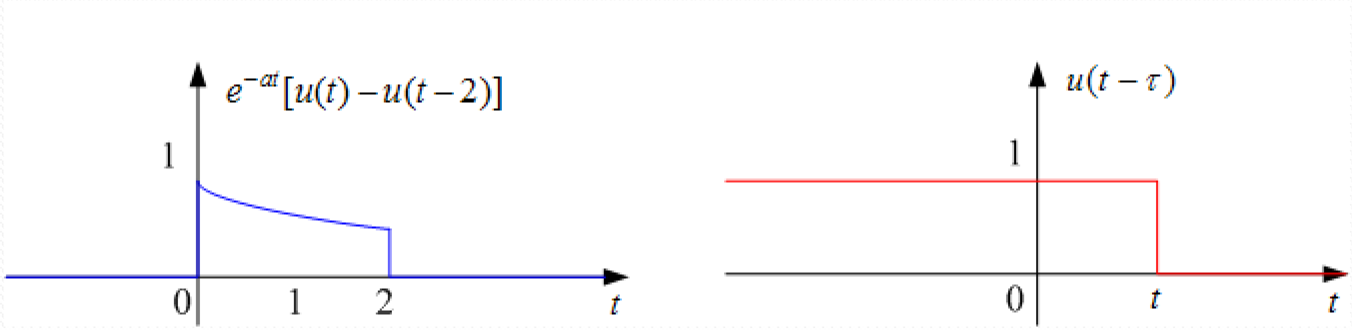
MORE EXAMPLES OF CONTINUOUS-TIME CONVOLUTION, PROPERTIES OF CONVOLUTION

The Step Response is the response of an LTI system to a unit step function. In other words, the input to the system is simply the unit step function: $x(t) = u(t)$.

This is equivalent to simply integrating the input from the infinite past up to time t.

$$s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

Example Let $h(t) = e^{-at}\,[\,u(t) - u(t - 2)]$. Find $s(t)$, the step response to $h(t)$.



(1) $t < 0$, no overlap

$$s(t) = 0$$

(2) $0 \le t \le 2$

$$s(t) = \int_0^t e^{-a\tau} d\tau = -\frac{1}{a} e^{-a\tau} \Big|_0^t = \frac{1}{a}(1 - e^{-at})$$

(3) $t > 2$

$$s(t) = \int_0^2 e^{-a\tau} d\tau = -\frac{1}{a} e^{-a\tau} \Big|_0^2 = \frac{1}{a}(1 - e^{-2a})$$

$$x(t) = \sum_k a_k \Phi_k(t)$$

$$y(t) = \sum_k a_k \Psi_k(t)$$

Superposition (or Divide-and-Conquer):

We can directly apply superposition to find the output of LTI systems if $x(t)$ can be expressed as a linear combination of basis functions $\Phi_k(t)$.

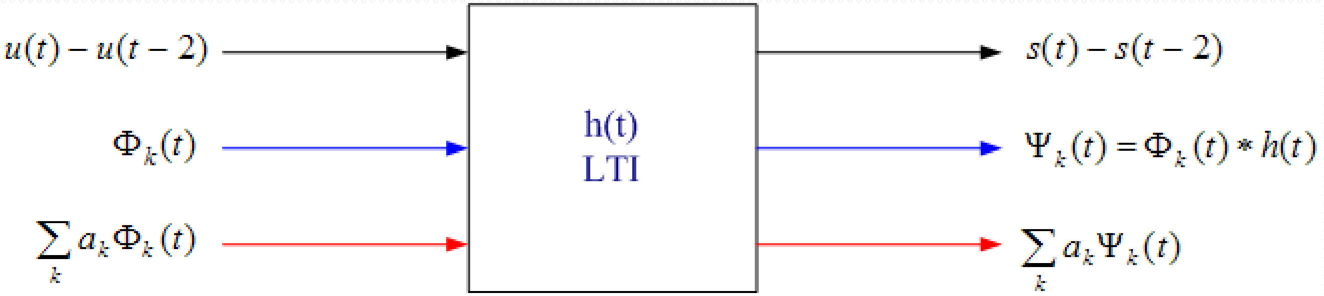
The $\Phi_k(t)$ are some convenient set of functions, for example unit impulses, unit step functions, or complex exponentials.

Example If an input is written as: $x(t) = \sum_k a_k \Phi_k(t)$

using superposition, we can write its output as: $y(t) = \sum_k a_k \Psi_k(t)$

Where $\Psi_k(t) = S[\Phi_k(t)] = h(t) * \Phi_k(t) = \Phi_k(t) * h(t)$

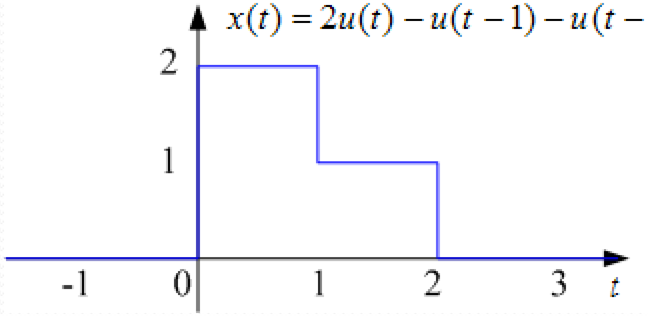
Again, using superposition, we can write:

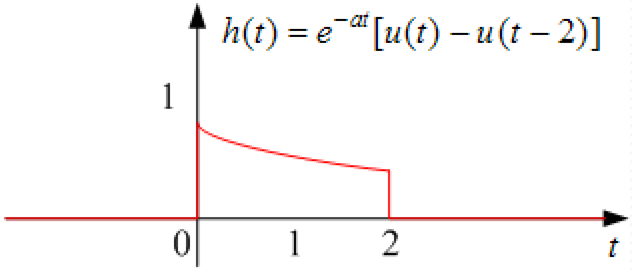


Example Find the output of the system where

$$x(t) = 2u(t) - u(t - 1) - u(t - 2) \text{ and}$$

$$h(t) = e^{-at} [\ u(t) - u(t - 2)\ ].$$





(Answer is not available yet)

We saw that for this system function $h(t)$,

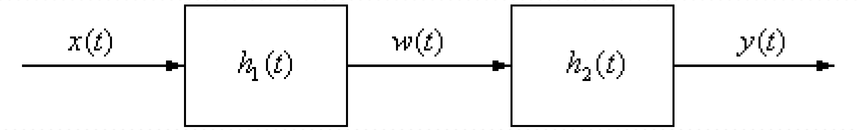
$$s(t) = \frac{1}{a}[1 - e^{-at}][u(t) - u(t-2)] + \frac{1}{a}[1 - e^{-2a}]u(t-2)$$

Therefore $y(t) = 2\, s(t) - s(t - 1) - s(t - 2)$, i.e. finding the output is very simple using superposition.

PROPERTIES OF CONVOLUTION

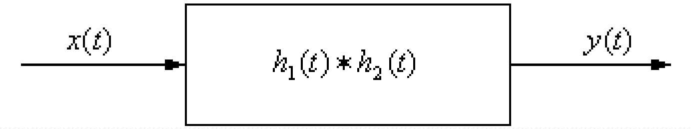
Convolution is commutative, associative, and distributive. Keeping this in mind may simplify some convolutions for you.

Cascade Interconnections



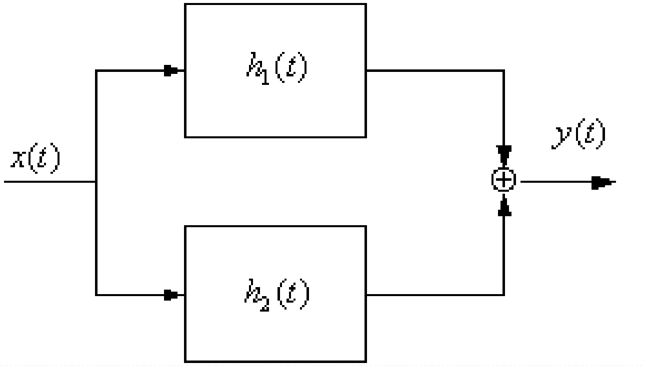
$$w(t) = x(t) * h_1(t), \quad y(t) = w(t) * h_2(t)$$
$$= [\, x(t) * h_1(t) \,] * h_2(t)$$
$$= x(t) * [h_1(t) * h_2(t)], \text{ by associativity of convolution}$$

Therefore the impulse response $h(t)$ for this overall system is $h_1(t) * h_2(t)$.



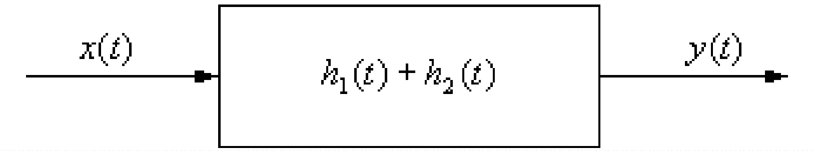
We can change the order in which the convolutions are performed due to commutativity. For a cascade of M systems there are $M!$ possible system orderings.

Parallel interconnection



$$y(t) = x(t) * h_1(t) + x(t) * h_2(t) = x(t) * [h_1(t) + h_2(t)]$$

$$\Rightarrow h(t) = h_1(t) + h_2(t)$$



Parallel systems is a large area of research today.

We saw that input/output properties of an LTI system are completely determined by the system's impulse response $h(t)$. We also saw that the output $y(t) = x(t) * h(t)$, that is, the output of the system is simply the convolution of the input with the system's impulse response.

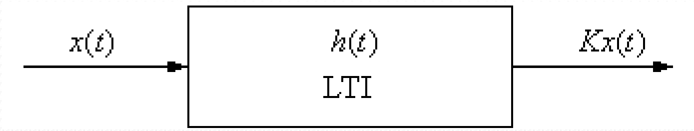
In this section, we will express other known system attributes in terms of conditions on the impulse response $h(t)$.

Systems with memory

In a memoryless system, the output $y(t)$ is a function of the input $x(t)$ at the time instant t alone. It does not depend on either past or future inputs.
An LTI system that is memoryless can only have this form:

$$y(t) = x(t) * h(t) = Kx(t)$$

Here, K is the system gain and it must be constant or else the system would vary with time.


$x(t)$
$h(t)$
LTI
$Kx(t)$

For $y(t)= Kx(t)$, the impulse response $h(t)$ must be of the form of a unit impulse weighted by a constant K:

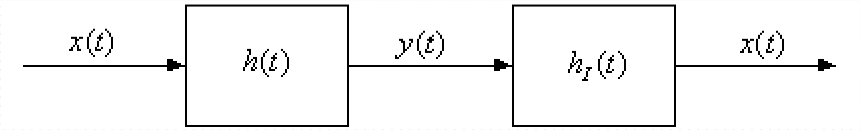
$$h(t) = K\delta(t)$$

Example What if $h(t) = K\delta(t - d)$, $d \neq 0$?
In this case, $y(t) = x(t) * h(t) = Kx(t - d)$.
The time shift d means that there is memory in the system and that the output $y(t)$ would depend on $x(t - d)$, not on $x(t)$.

Example Is a system described by $h(t) = u(t) - u(t - 1)$ memoryless?

The system has memory.

Invertible Systems



$$h(t) * h_I(t) = \delta(t)$$

A system is invertible if we can find $h_I(t)$ so that the original input $x(t)$ can be recovered from the output $y(t)$. For this to hold, the system must be *one-to-one*. We will see how to do this when we study transforms.

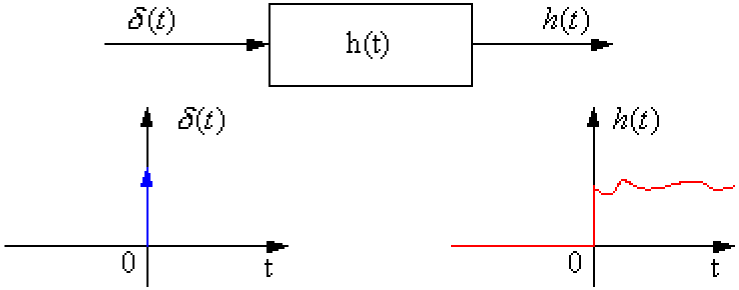
Causality

We know that for a causal system, the output depends only on past or present inputs and not on future inputs.

Equivalently, a causal system does not respond to an input until it occurs (the output is not based on the future).

In other words, a response to an input at $t = t_0$, would occur only for $t \geq t_0$ and not before t_0.

We know that $h(t)$ is the system response to $\delta(t)$, and that $\delta(t)$ occurs at $t = 0$.



A system is causal, if $h(t) = 0, \quad t < 0$

Another way to look at the causality condition: Let's examine the convolution equation, flipping $h(t)$ instead of $x(t)$:

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

Causality: if $h(t)$ is causal then $h(t - \tau) = 0$, $t - \tau < 0$ or $t < \tau$.
So,

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)d\tau$$

which shows us that the output $y(t)$ depends only on values of the input $x(\tau)$ for $\tau \leq t$, i.e. it only depends on the past and present.

Example Is $h_1(t) = u(t+1)$ causal?

The system is noncausal.

Is $h_2(t) = u(t-1)$ causal?

The system is causal.

Stability

We can tell if an LTI system is BIBO stable from its impulse response.
$|x(t)| \leq B_1$, for all t, to determine if the system is BIBO stable, we need to determine if its output remains bounded for all time:

$$|y(t)| = \left|\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau\right| \leq$$

$$\int_{-\infty}^{\infty} |x(t-\tau)h(\tau)|d\tau$$

$$= \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau \leq \int_{-\infty}^{\infty} B_1|h(\tau)|d\tau = B_1\int_{-\infty}^{\infty}|h(\tau)|d\tau$$

Therefore, $|y(t)| \leq B_1 \int_{-\infty}^{\infty}|h(\tau)|d\tau < \infty$ if $\int_{-\infty}^{\infty}|h(\tau)|d\tau < \infty$

That is, the system is BIBO stable iff the impulse response $h(t)$ is absolutely integrable:

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau = G < \infty$$

In this case, the output will be bounded by a second constant: $|y(t)| \leq B_1G = B_2$ and thus, the system is BIBO stable.

__Example__ Is $h(t) = u(t)$ stable?

$$y(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} x(\tau)\,d\tau$$

(This just gives the running integral of x(t).)

The system is not stable.

Example Given an impulse response $h(t) = e^{-at} u(t)$, $a > 0$, is the system *BI*BO stable? How about for $a < 0$?

$h(t)$ is stable for $a > 0$ and it is not stable for $a < 0$.

Unit Step Response

$$s(t) = h(t) * u(t)$$

$$s(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau) = \int_{-\infty}^{t} h(\tau)d\tau$$

We see that the step response $s(t)$ is just the running integral of the impulse response $h(t)$.

Thus, we can recover the impulse response $h(t)$ from the step response $s(t)$ by taking its derivative:

$$h(t) = s'(t) = \frac{d}{dt}s(t)$$

Example Given a step response

$$s(t) = \frac{1}{a}[1 - e^{-at}]\,u(t)$$

find the system's impulse response $h(t)$ by taking the derivative of $s(t)$

$$\frac{d}{dt}(s(t)) = \frac{d}{dt}\left(\frac{1}{a}[1 - e^{-at}]\right)u(t) + \frac{1}{a}[1 - e^{-at}]\frac{d}{dt}(u(t))$$

$$= e^{-at}u(t) + \frac{1}{a}[1 - e^{-at}]\delta(t)$$

$$= e^{-at}u(t) + 0$$

$$= e^{-at}u(t)$$