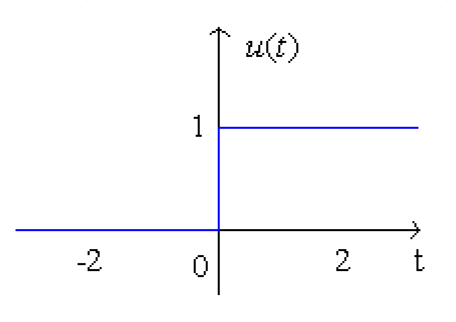
Lesson Week 2

Singularity Functions (Unit Step Function, Unit Impulse Function and its Properties)

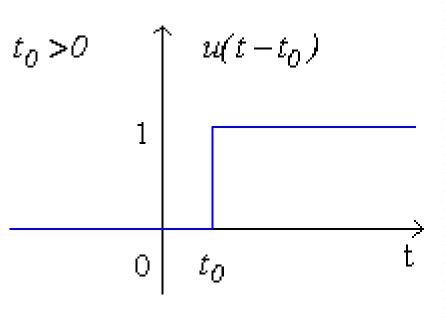
The Unit Step Function

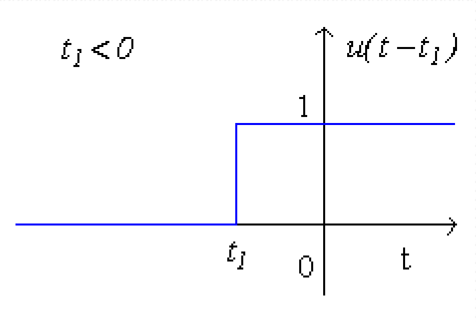
We already defined the unit step function $u(t)$ as:

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \\ \text{undefined}, & t = 0 \end{cases}$$



Ex: Find and plot $u(t - t_o)$ and $u(t - t_1)$



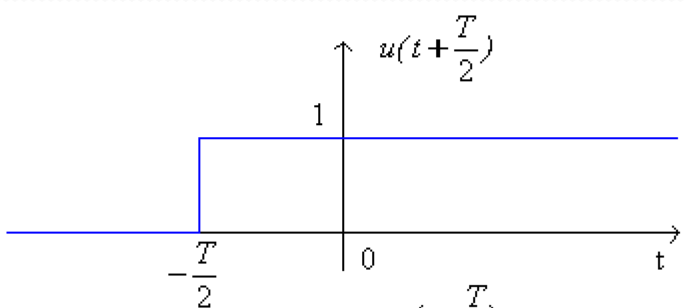


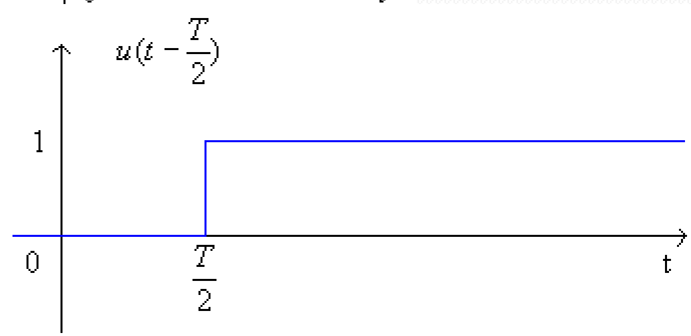
Ex. Define a block function (window) as

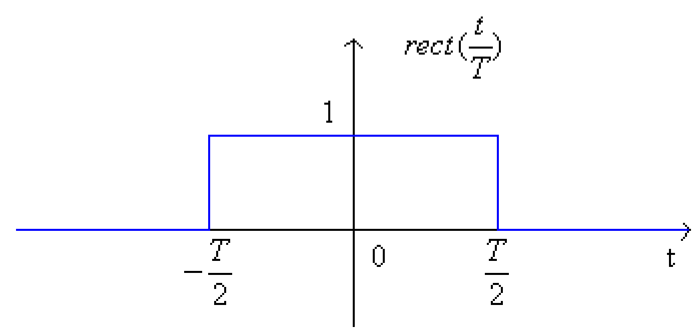
$$rect(\frac{t}{T}) = \begin{cases} 1, & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Then $rect(\frac{t}{T}) = u(t+\frac{T}{2}) - u(t-\frac{T}{2})$ This is an ideal low pass filter

plot $rect(\frac{t}{T})$



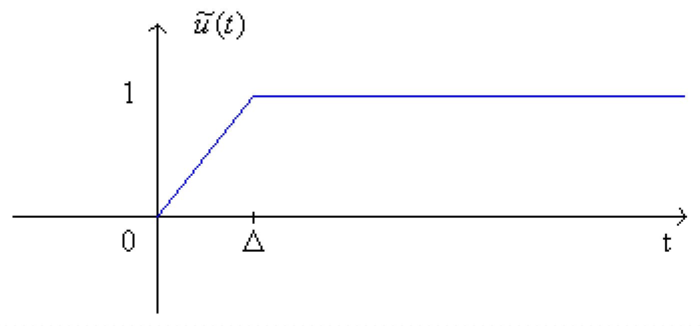




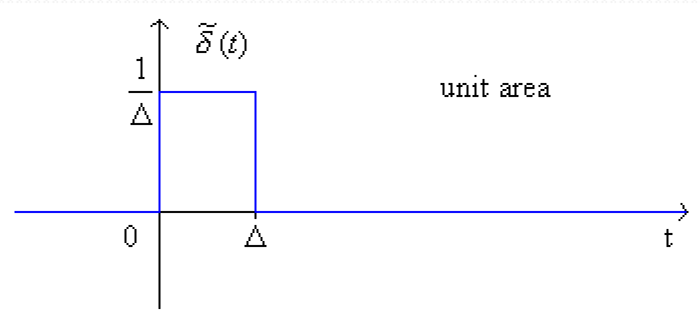
$\tilde{u}(t)$

The Unit Impulse Function

Now let's look at a signal: $\tilde{u}(t)$

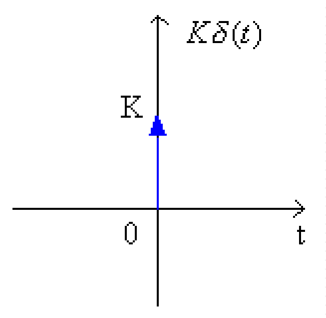


What is its derivative? Define it as: $\tilde{\delta}(t) = \frac{d}{dt}\tilde{u}(t)$ which has unit area.



Now, $\lim_{\Delta \to 0} \tilde{u}(t) = u(t)$

so what if we take $\lim_{\Delta \to 0} \tilde{\delta}(t)$?



The pulse height gets higher and higher and its width goes to zero, but its area is still 1! So define $\delta(t)$ as unit impulse:

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{undefined}, & t = 0 \end{cases}$$

And

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

or equivalently,

$$\delta(t - t_0) = \begin{cases} 0, & t \neq t_0 \\ \text{undefined}, & t = t_0 \end{cases}$$

And

$$\int_{-\infty}^{\infty} \delta(t - t_0)\,dt = 1$$

Also

$$\int_{t_1}^{t_2} \delta(t)\,dt = \begin{cases} 1, & \text{if } t_1 < 0 < t_2 \\ 0, & \text{otherwise} \end{cases}$$

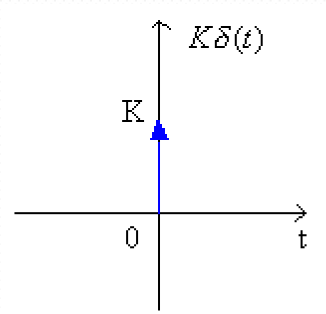
$\delta(t)$ can be considered to be the derivative of $u(t)$ but only in a restricted sense since $u(t)$ is a discontinuous function.

Note that the impulse function is not a true function since it is not defined for all values of *t*. It's a "generalized function." But its idealization will allow us to derive many interesting results.

$\tilde{u}(t)$

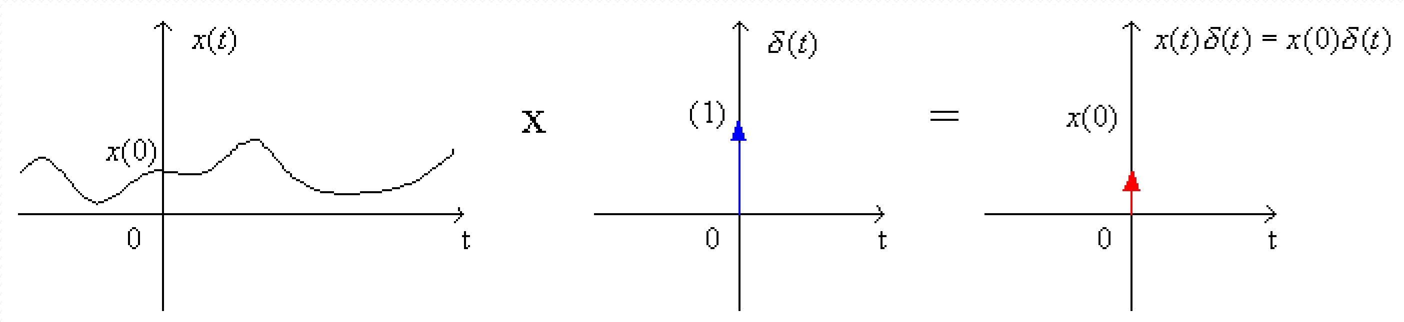
Unit Impulse Properties

1. Scaling $K\delta(t)$ is an impulse with weight or area K:



2. Multiplication of a function x(t) (that is continuous at 0) by an impulse δ(t):

We get an impulse with area or weight $x(0)$.

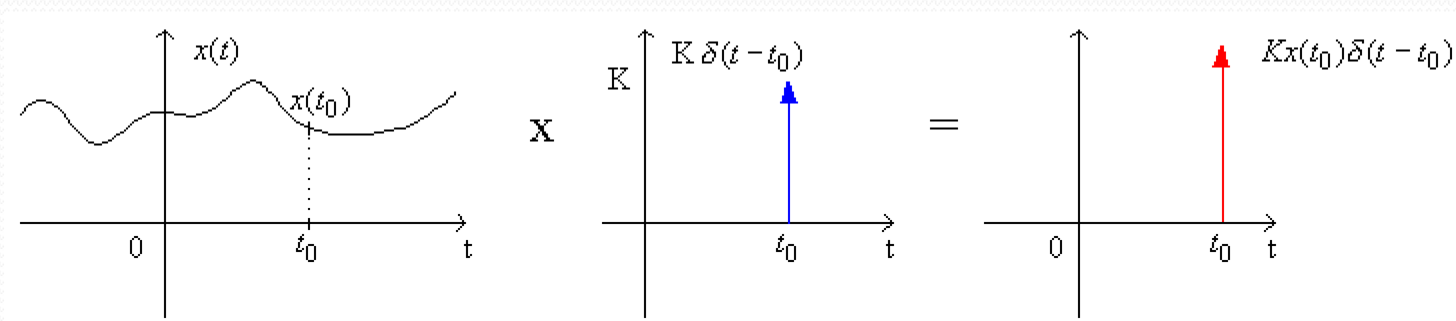


3. Time Shift of an impulse
$y(t) = x(t)\delta(t-t_o)$

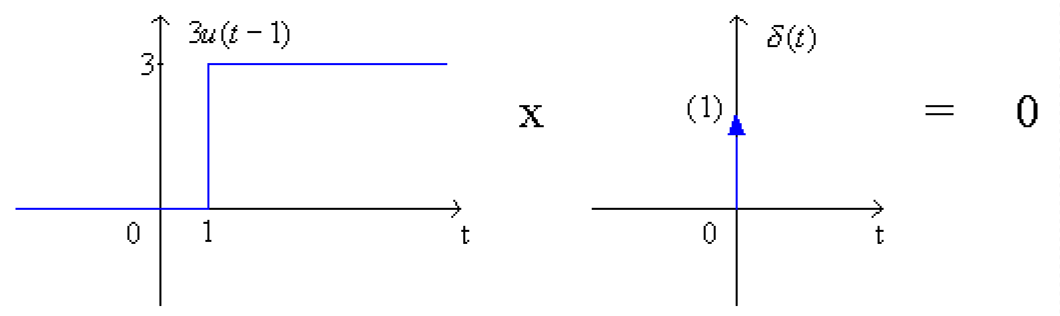
$$\delta(t - t_0) = \begin{cases} 0, & t \neq t_0 \\ \text{undefined}, & t = t_0 \end{cases}$$

So we get an impulse with weig ... re the impulse is located: $y(t) = x(t_o)\delta(t-t_o)$

Example What is $x(t) * K\delta(t-t_o)$



Example What is $3u(t\text{-}1)\delta(t)$?



SHIFTING PROPERTY

What if we multiply a function by an impulse and then integrate?

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = ?$$

$$= \int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt$$

$$= x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)dt = x(t_0)$$

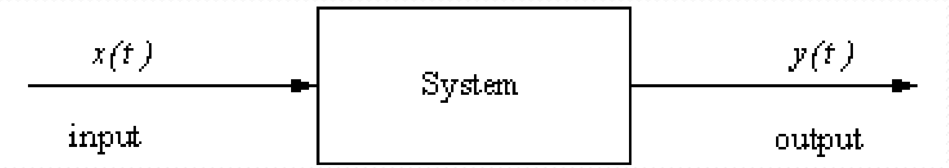
We integrate out the time variable so the integral is just equal to a number (or later on, a function). We'll see this many times. In this case, the impulse $\delta(t\text{-}t_o)$ is defined by the integral (as long as the function $x(t)$ is continuous at t_o).

ALSO:

- $\int_{-\infty}^{\infty} f(t-t_0)\delta(t-t_1)dt = f(t_1-t_0)$, if $f(t)$ is continuous at t_1-t_0
- $\int_{-\infty}^{t} \delta(\tau-t_0)d\tau = u(t-t_0)$
- $\delta(t) = \delta(-t)$
- $\delta(at) = \frac{1}{|a|}\delta(t)$

Continuous-Time System

A SYSTEM is an operation for which cause-and-effect relations exist.



Properties of Continuous-Time Systems

Here, we discuss some of the properties that a continuous-time system could have. We will use $x(t)$ for the input to the system, $y(t)$ as its output, and use the notation:

$$y(t) = \mathrm{T}[x(t)]$$

Or

$$y(t) = \mathrm{S}[x(t)]$$

Or

$$x(t) \rightarrow y(t)$$

Systems with memory

Systems whose output depend on values of the input other than just at the time of the output have memory.
A system $y(t_o)$ has memory if its output at time t_o depends on the input $x(t)$ for $t > t_o$ or $t < t_o$, i.e., it depends on values of the input other than $x(t_o)$.

Otherwise, the system is MEMORYLESS

Example of a memoryless system:

Resistor $v(t_o) = R\, i(t_o)$; the voltage depends only on current at time t_o.

Example of System with Memory:

Capacitor $v(t_0) = \frac{1}{C}\int_{-\infty}^{t_0} i(t)\,dt$

the voltage depends on past values of current so a capacitor has memory.

Ex. Does $y(t) = x(t) + 5$ have memory?

y(t) is memoryless.

Ex. Does $z(t) = x(t+5)$ have memory?

z(t) has memory

Ex. Does $y(t) = (t+5)x(t)$ have memory?

y(t) is memoryless

Ex. Does $z(t) = [x(t+5)]^2$ have memory?

Z(t) has memory

Ex. Does $a(t) = x(5)$ have memory?

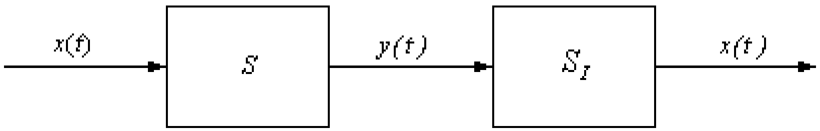
a(t) has memory

Ex. Does $v(t) = x(2t)$ have memory?

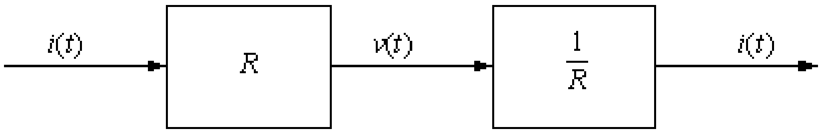
V(t) has memory

Inverse of a System

A system is invertible if you can determine the input uniquely from the output, i.e., there is a one-to-one relationship between the input and output.



Resistor is Invertible, $x(t) = i(t)$, $y(t) = v(t)$, $x(t) = y(t)/R$.



$y(t) = x^5(t)$ is an invertible system.

Noninvertible:

$y(t) = x(t)u(t) \rightarrow$ zeros out much of the input

$y(t) = x^2(t) \rightarrow$ don't know sign

$y(t) = \cos[x(t)] \rightarrow$ add 2π to $x(t)$

Causality

Output $y(t)$ depends only on past and present inputs and **not on the future.**

All physical real-time systems are causal because we can not anticipate the future.
Image processing-Non-causal filters like blurring masks.
Music processing - record and process later - noncausal but not real-time

Ex. Resistor, Capacitor, and stock market are causal,

- $v(t_0) = i(t_0)R \rightarrow$ memoryless $\Rightarrow$ Causal
- $v(t_0) = \frac{1}{C}\int_{-\infty}^{t_0} i(t)dt \rightarrow$ Causal since only depends on past and present

Non causal systems need off-line processing
Intuition: system doesn't laugh until it's tickled

Ex. $y(t_0) = \int_{-\infty}^{t_0+a} x(t)dt$ Is this Causal? You fill in.

It is non causal. Since the value of 'a' is not specified, the value of 'a' can be positive (a >o). In that case, it is non causal.

FACT: Memoryless $\rightarrow$ Causal but not vice versa. In fact, most causal systems have memory.

Ex. Let $y(t) = x(-t)$.

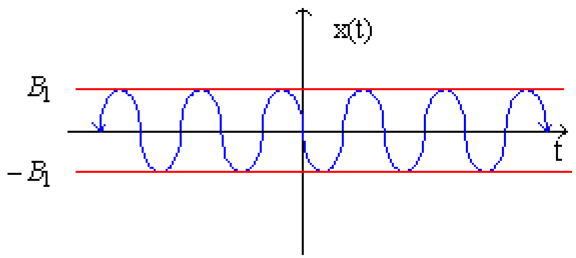
Is this causal? Try letting t be a negative number.

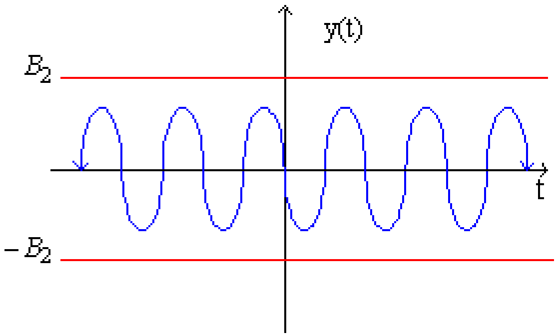
This is non causal. For example $y(-1) = x(1)$

Stability

Bounded Input - Bounded Output (BIBO) Stability

Input $x(t)$ bounded produces bounded output.
If $|\,x(t)\,| \leq B_1 \rightarrow |\,y(t)\,| \leq B_2$, where $y(t)$ is output.





Ex. Resistor is stable $V=iR$, $|\,i(t)\,| \leq B_1 \rightarrow |\,v(t)\,| \leq RB_1$

Example: Capacitor: $i(t) = C \frac{dV_c(t)}{dt}$

Is this stable?

Let $i(t) = B_1 u(t)$, where $B_1 \neq 0$

$$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

$$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} B_1 u(\tau)d\tau$$

$$V_c(t) = \frac{1}{C}\int_{0}^{t} B_1 d\tau$$

$$V_c(t) = \frac{B_1}{C}t$$

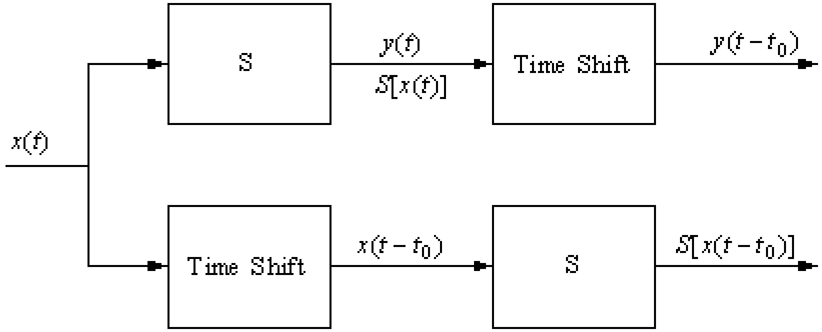
$\therefore$ BIBO : unstable

$V_c(t) = \frac{B_1}{C}t$ grows linearly with t and as $t \to \infty,\ V_c(t) \to \infty.$

So capacitor is not BIBO stable.

Time-Invariance

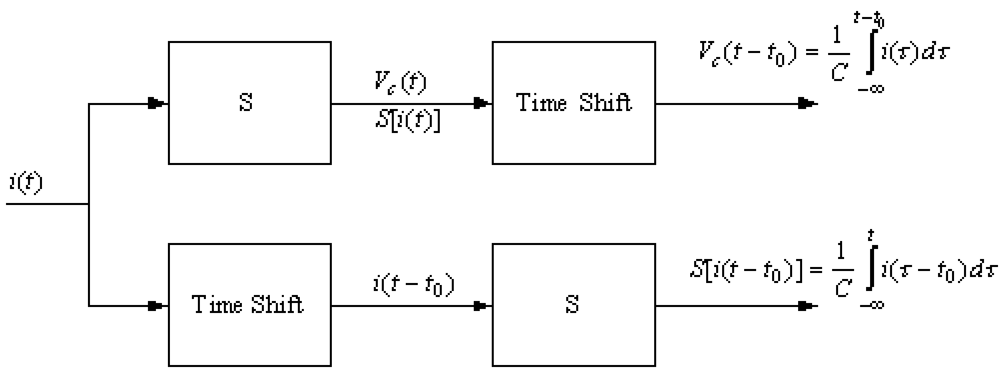
Given a system that is *time-invariant*, if the input signal is shifted in time, all that will happen is the output signal will be shifted by the same amount in time. It will not change in any other way. An alternative way to state this is that the system does not change over time. It will perform the same today as it would next week.



- If $y(t - t_o) = S[x(t - t_o]$ (i.e. the outputs of both branches of the above block diagram are equal), then the system is **TIME-INVARIANT**. This means that the system is not changing with time.
- If $y(t - t_o) \neq S[x(t - t_o]$ (the outputs of both branches of the above block diagram are different), the system is **TIME-VARYING.** This means that the system will perform differently depending on when you use it.

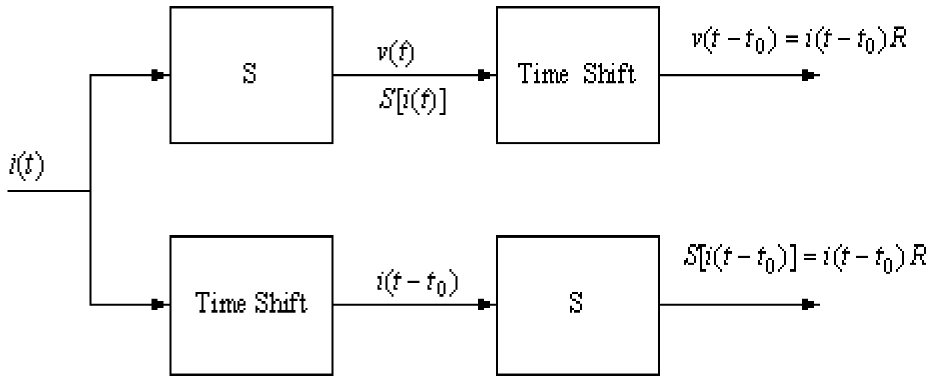
<u>Ex.</u> Is a capacitor time-invariant? $V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$

compare $V_c(t - t_o)$ with $S[i(t - t_o)]$:

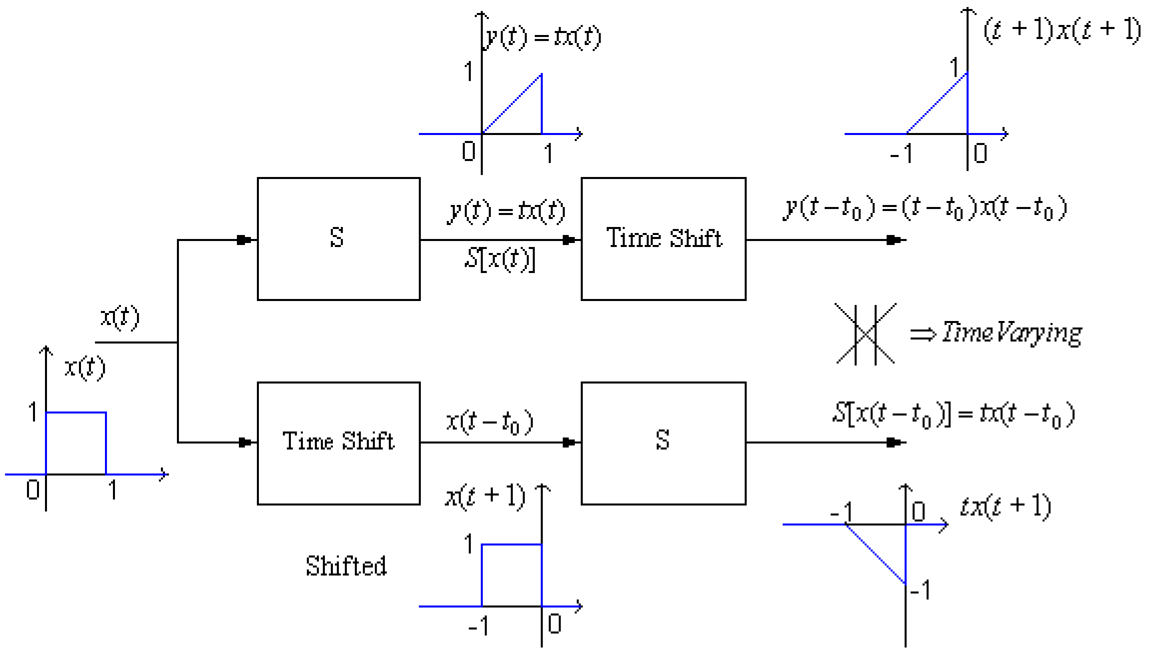


Let $\alpha = \tau - t_0 \quad \Rightarrow S[i(t - t_0)] = \frac{1}{C}\int_{-\infty}^{t-t_0} i(\alpha)d\alpha = V_c(t - t_0)$

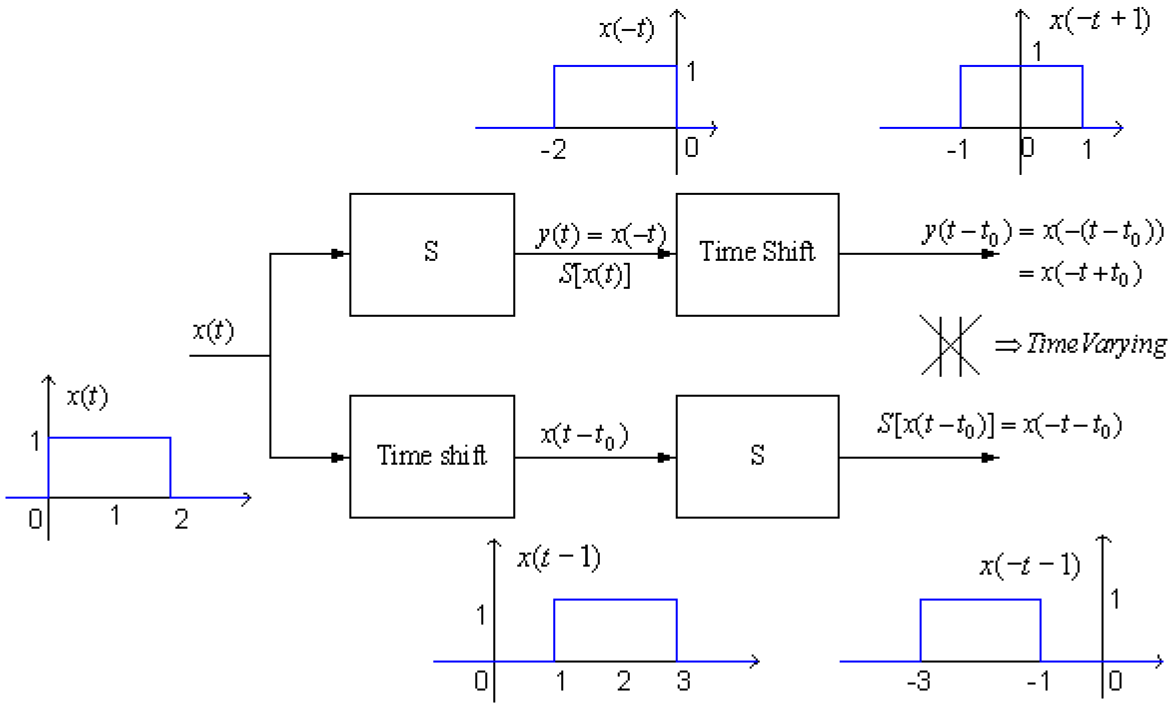
Ex. Resistor $v(t) = i(t)R$. Is this Time-Invariant?



Ex. $y(t) = t\,x(t)$ Is this Time-Invariant?

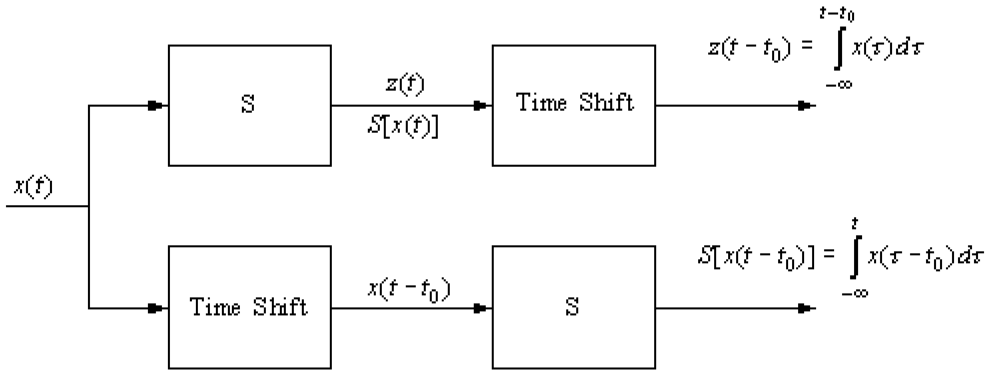


Ex. Time Reversal $y(t) = x(-t)$ Is this Time-Invariant?



Ex. Test the following systems for time-invariance:

1. $z(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$



2. $y(t) = \int_0^t x(\tau)d\tau$

This is time varying. Because the length of the window over which you integrate changes.

3. $a(t) = \sin[x(t)]$

This is time invariant.

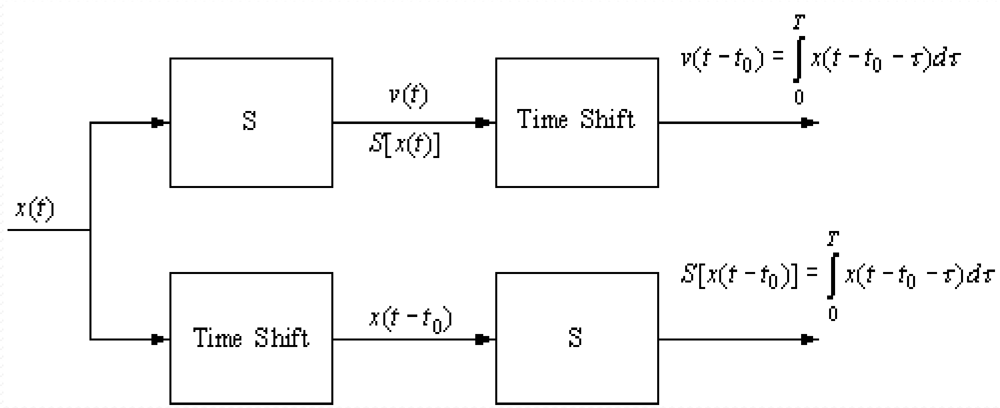
4. $b(t) = \sin(t)x(t)$

This is time varying.

5. $w(t) = x(2t)$

This is time varying.

6. $$y(t) = \int_0^T x(t-\tau)\,d\tau$$



Linearity

For a system to be linear, it must satisfy both the additivity and homogeneity properties:

1. Additivity

If $S[x_1(t)] = y_1(t)$ and $S[x_2(t)] = y_2(t) \rightarrow S[x_1(t) + x_2(t)] = y_1(t) + y_2(t)$ means that a system satisfies the additivity property.

2. Homogeneity or Scaling

$S[x(t)] = y(t) \rightarrow S[ax(t)] = ay(t)$ means that a system satisfies the scaling or homogeneity property.

Combine Additivity and Homogeneity to get the SUPERPOSITION CONDITION:

If $S[x_1(t)] = y_1(t)$ and $S[x_2(t)] = y_2(t)$
then $S[ax_1(t) + bx_2(t)] = ay_1(t) + by_2(t)$

Examples of Linear systems

Multiplication by a constant:

$$S[x(t)] = cx(t)$$

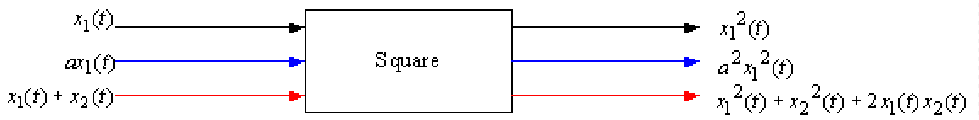
Try: $S[ax_1(t) + bx_2\ (t)\ ]$:

$$\begin{aligned} S[x_1(t)] &= cx_1(t) = y_1(t) \\ S[x_2(t)] &= cx_2(t) = y_2(t) \\ S[ax_1(t) + bx_2\ (t)\ ] &= acx_1(t) + bcx_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Therefore, linearly combined input produces linearly combined output and the system is linear.

Examples of Nonlinear systems

1) Squaring



$$S[x(t)] = y(t) = x^2(t)$$

Violates homogeneity,

$$S[x(t)] = x^2(t)$$

$$S[ax(t)] = a^2x^2(t) \neq ax^2(t)$$

(It also violates Additivity due to the cross-terms.)

2) Affine or Incrementally linear system

$$S[x(t)] = y(t) = x(t) + a$$

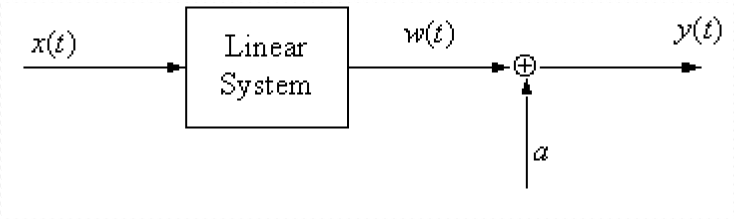
This system violates homogeneity:

$$S[x(t)] = x(t) + \mathrm{a}$$
$$S[cx(t)] = cx(t) + a \neq c[x(t) + a]$$

It also violates additivity:

$$S[x_1(t)] = x_1(t) + a$$
$$S[x_2(t)] = x_2(t) + a$$
$$S[x_1(t) + x_2(t)] = x_1(t) + x_2(t) + a \neq S[x_1(t)] + S[x_2(t)]$$

But we can think of this as a system that is "incrementally linear" or affine (note that the first part of the system is linear):



Note: If the input to a linear system is zero, the output will also be zero. Use the scaling property to show this:

$$S[x(t)] = y(t) \rightarrow S[ax(t)] = ay(t)$$

If we let $a = 0$, then we get that $S[0] = 0$.
For an affine (nonlinear) system such as

$$S[x(t)] = x(t) + 3$$

A zero input produces non-zero output, i.e. $S[0]=3$. This violates the requirement that a linear system produce a zero output to a zero input.

Superposition:

We can generalize superposition to more than 2 functions, i.e. given a set of inputs $x_k(t)$ with a set of corresponding outputs $y_k(t)$, we can take a linear combination of any number of the inputs and get the same linear combination of the corresponding outputs:

$$x(t) = \sum_k a_k x_k(t) \text{ produces output}$$

$$y(t) = \sum_k a_k y_k(t)$$

You will find this very useful in doing some convolutions.

Are the following systems linear?

1. $y(t) = tx(2t)$

This is Linear.

2. $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

This is Linear.

3. $y(t) = cos[x(t)]$

This is non linear

4. $y(t) = \begin{cases} x(t) & t < 0 \\ -x(t) & t \geq 0 \end{cases}$

This is Linear.

5. $y(t) = |x(t)|$

This is non linear.

Continuous-Time Linear Time-Invariant Systems

In this lesson, we will discuss linear time-invariant (LTI) systems - these are systems that are both linear and time-invariant. We will see that an LTI system has an input-output relationship described by a convolution.

Impulse Representation of Continuous-Time Signals

Using the sifting property, we can write a signal $x(t)$ as:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

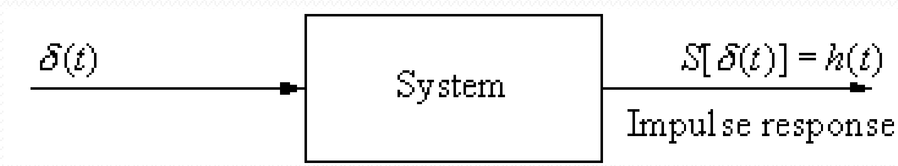
which is writing a general signal $x(t)$ as a function of an impulse function. This expresses the input $x(t)$ as an integral (continuum sum) of shifted impulses that are weighted by weights $x(\tau)$. Another way to put this is that you can build a CT signal out of impulses.

Continuous Time Convolution

We can write:

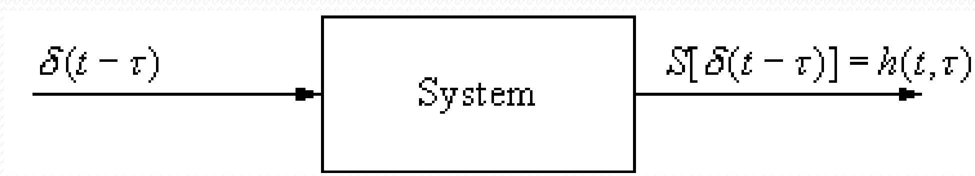
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad \text{using the sifting property}$$

This expresses the input $x(t)$ as an integral (continuum sum) of shifted impulses that are weighted by weights $x(\tau)$.
Now take a system and define the impulse response of the system as



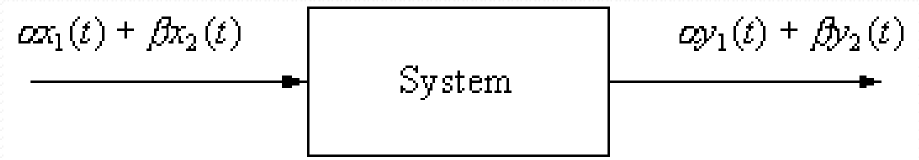
$$h(t) = S[\delta(t)]$$

and the response of the system to a shifted impulse as:



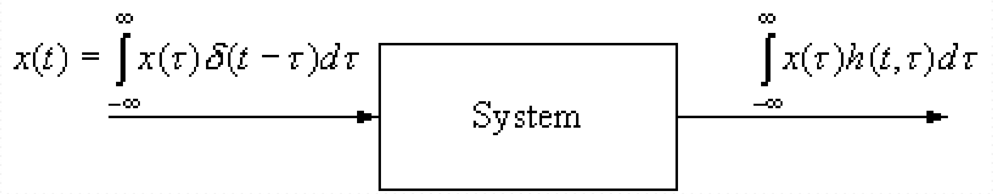
$$h(t,\tau) = S[\delta(t - \tau)]$$

If the system is linear, then



$$S[\alpha x_1(t) + \beta x_2(t)] = \alpha y_1(t) + \beta y_2(t)$$

Let



$$y(t) = S[x(t)] = S[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau]$$

$$= \int_{-\infty}^{\infty} x(\tau)S[\delta(t-\tau)]d\tau \quad \text{due to linearity}$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t,\tau)d\tau$$

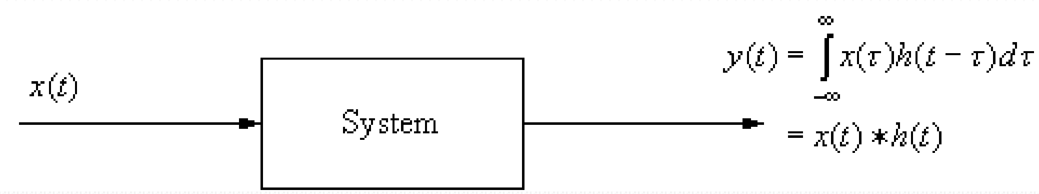
But what if the system is also Time-Invariant?

Then $S[\delta(t - \tau)] = h(t, \tau) = h(t - \tau)$, since we had $S[\delta(t)] = h(t)$. Therefore,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

We have seen that if we have a linear time-invariant system, then the output is the input convolved with the system's impulse response $h(t)$. **In other words, we can completely characterize an LTI system by its impulse response.**

This is a very important result!



Convolution Integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Here, $h(\tau)$ is flipped and shifted across $x(\tau)$.

Convolution is a tough concept to get at first. I have 2 rules that will greatly improve the quality of your life:

- 1.DRAW A PICTURE of $x(\tau)$ and $h(t - \tau)$
- 2.FLIP THE "EASY" FUNCTION

Why can we pick which function to flip?
Because convolution is commutative:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Change variables: $\lambda = t - \tau \rightarrow \tau = t - \lambda$, $d\tau = -d\lambda$.

$$y(t) = -\int_{\infty}^{-\infty} x(t-\lambda)h(\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda = h(t) * x(t) = x(t) * h(t)$$

(minus signs cancel)

Convolution is a tough concept to get at first. I have 2 rules that will greatly improve the quality of your life:

- 1.DRAW A PICTURE of $x(\tau)$ and $h(t - \tau)$
- 2.FLIP THE "EASY" FUNCTION

Why can we pick which function to flip?
Because convolution is commutative:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Change variables: $\lambda = t - \tau \rightarrow \tau = t - \lambda$, $d\tau = -d\lambda$.

$$y(t) = -\int_{\infty}^{-\infty} x(t-\lambda)h(\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda = h(t) * x(t) = x(t) * h(t)$$

(minus signs cancel)

Let's examine convolution formula:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

1. Flip $h(\tau)$ and shift it t

Note: $h(t - \tau)$ is a function of τ, not t!

t is the shift parameter.

2. Fix t and multiply $x(\tau)$ with $h(t - \tau)$ for all values of τ.

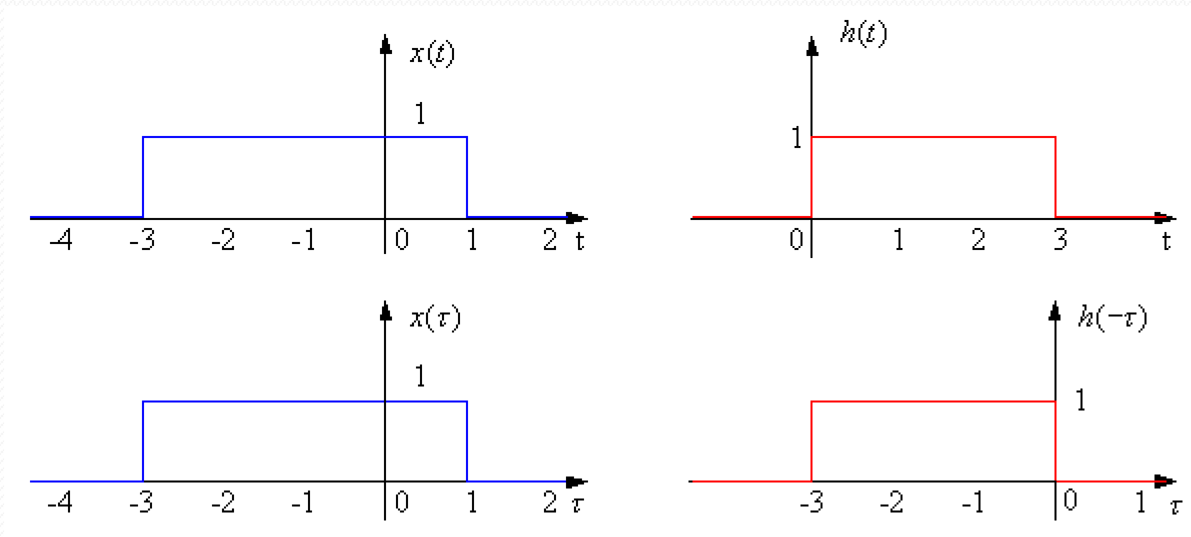
3. Integrate $x(\tau)h(t - \tau)$ over all τ to get $y(t)$ which is a single value that depends on t. Remember that τ is the integration variable and that t is treated like a constant when doing the integral.

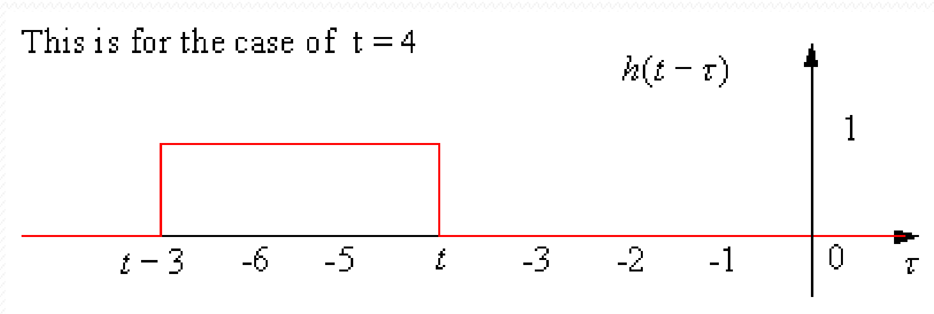
4. Repeat for all values of t.

Fortunately, it usually falls out that there are only several regions of interest and the rest of $y(t)$ is zero.

Ex. Find $y(t) = x(t)*h(t)$.

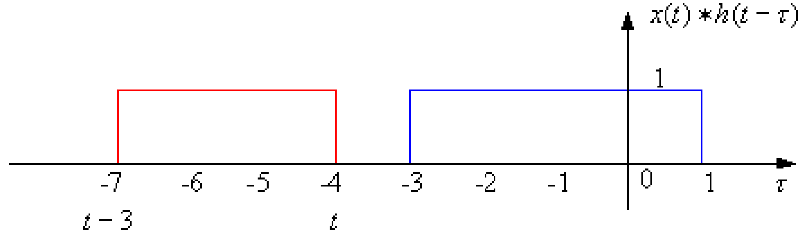
Form $x(\tau)$ and $h(t - \tau)$ (to shift by $h(-\tau)$ by t, just add t to all points) and continue from the





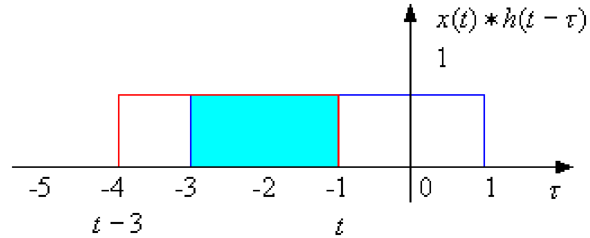
(1) $t < -3$, no overlap

$y(t) = 0$



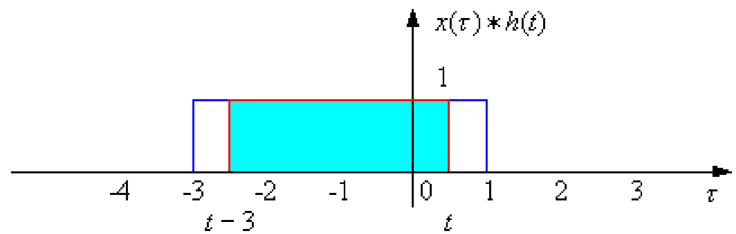
(2) $-3 \le t \le 0$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-3}^{t} d\tau = t+3$$



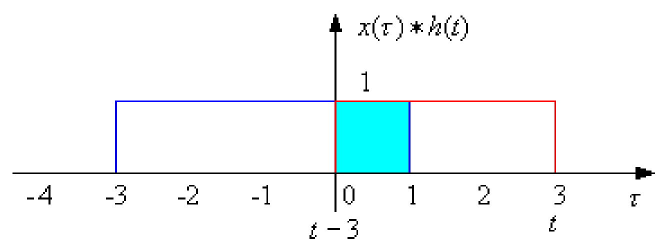
(3) $0 \le t \le 1$

$y(t) = 3$



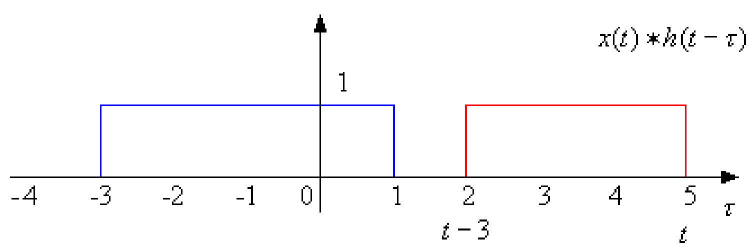
(4) $t > 1$ & $(t-3) < 1$

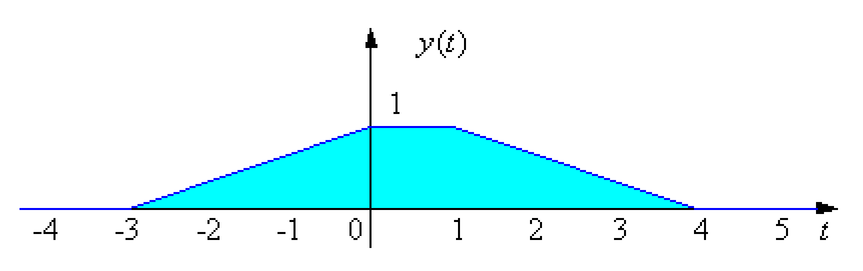
$$y(t) = \int_{t-3}^{1} d\tau = 4 - t$$



(5) $t > 4$

$y(t) = 0$





When you finish notice:

1. (a) nonzero "width" of $x(t) = 3$
   (b) nonzero "width" of $h(t) = 4$
   (c) nonzero "width" of $y(t) = 7$
2. $y(t)$ is "smoother" than $x(t)$ or $h(t)$