Tax Mechanisms and Gradient Flows

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The main idea

- Classic Mirrlees static nonlinear income tax problem
  - we still do not have a full solution for the static problem
  - even in the simplest case (iso-elastic, Pareto) Diamond-Saez formula is still a coupled system of differential equations
- Main idea: Study a static optimal tax from the dynamic point of view.
  - Starting from a given tax (either optimal or suboptimal): associate with the tax a dynamical system the analysis
- Key object:
  - a heat equation and the associated heat kernel
  - heat equation is intimately connected to the optimal tax problem
  - allow to derive new properties of tax mechanisms
Main result: optimal tax

- Associate with the first order conditions for the optimal tax a heat kernel:
  - a family of averaging functions at different times (or scales)

- **Main results – a fairness principle:**
  - **Optimal tax at a given income** is equal to the **weighted average of the optimal taxes** at other incomes **at any scale** and of a weighted mechanical effect.

- Weighting is given by the heat kernel
  - show that it behaves as a **Gaussian**
  - The weights at different scales are tightly linked – **one unified weighting scheme at any scale**
Fairness principle

- Not externally imposed
  - It is revenue maximizing to treat agents similarly: a tax at a given income equal to the weighted average of taxes around it
- It is a smoothing result
  - A way to smooth deadweight loss and raised revenues across incomes
- Intuition about connection to the heat equation:
  - Nature wants to distribute heat evenly on the background
Main results: Gradient flows

- Construct a gradient flow for any initial (optimal or suboptimal) tax function:
  - changes the underlying tax function in the direction of maximal revenue increase
  - steepest descent in the space of functions
  - The optimal tax is a stationary point of the gradient flow

- Show that a version of the fairness principle holds along the trajectory of the gradient flow, for short time asymptotics

- Gradient flow gives rise to the heat equation
  - and all of its nice properties, e.g., smoothing
Main contributions

▶ A new way to look at the classical problem:
   ▶ static system from a dynamic point of view
▶ Fairness principle – a new property of the tax systems
   ▶ at the optimum
   ▶ and of the tax reform
▶ Unifying a variational (tax reform) and an optimal tax point of view
▶ Mathematical contribution:
   ▶ heat equation arises in a completely new context
Literature

- Tirole and Guesnerie (1981): tax reform from a gradient descent point of view for linear taxes - ODE
  - our work: nonlinear taxes, PDE
- Sonnenschein (1981, 1982) and Artzner, Simon, and Sonnenschein (1986): derive a heat equation as a gradient process of the firms adjusting the commodity they produce by maximizing the rate of change in profit
- Optimal transport:
  - gradient flow is one of the main objects: Villani (2003), Jordan, Kinderlehrer, and Otto (1998)
  - in (macro)economics, optimal transport is starting to be used: Chiappori, Eeckhout-Kircher, Fajgelbaum-Schaal, Galichon, Lindenlaub, Salanie. A great book by Galichon (2016).
Classic static nonlinear income tax setting

- **Productivity**: $\theta \in \Theta \subset \mathbb{R}_+$, c.d.f. $H(\theta)$, density $h(\theta)\text{\footnote{Assume zero density at the boundaries.}}$

- **Preferences**: $U(c, l) = c - v(l)$

- **Tax**: $T : \mathbb{R}_+ \to \mathbb{R}$, non-linear function of income $y = \theta \times l$

- **Agent**: 
  \[
  \max_{y \geq 0} y - T(y) - v\left(\frac{y}{\theta}\right)
  \]

  $y(\theta, T)$ is the argmax, c.d.f. $\Phi(y)$, density $\phi(y)\text{\footnote{Assume: (1) a unique global maximum; (2) $\phi$ is continuous and bounded away from zero.}}$

- **Government revenue**: $R(T) = \int_{\Theta} T(y(\theta, T))dH(\theta)$
Variations of taxes

- Gateaux differential of revenue $R(T) = \int_{\Theta} T(y(\theta, T))dH(\theta)$:

$$
\delta R(T, \hat{T}) = \int_{\mathcal{Y}} \hat{T}(y) \phi(y) dy + \int_{\mathcal{Y}} T'(y) \delta y(\theta) \phi(y) dy
$$

- Gateaux differential of income$^3$:

$$
\delta y(\theta) = -\varepsilon(y(\theta)) \hat{T}'(y(\theta))
$$

Lemma

The Gateaux differential of revenue in the direction $\hat{T}$ is:

$$
\delta R(T, \hat{T}) = \int_{\mathcal{Y}} \hat{T}(y) \phi(y) dy - \int_{\mathcal{Y}} T'(y) \varepsilon(y(\theta)) \hat{T}'(y(\theta)) \phi(y) dy
$$

$^3 \varepsilon(\theta) = \frac{y(\theta)}{1 - T'(y(\theta))} \frac{e(\theta)}{1 + p(y(\theta))e(\theta)}$ is the elasticity along the nonlinear budget constraint.
Optimal tax

\[ \delta R(T, \hat{T}) = \int_Y \hat{T}(y) \phi(y) \, dy - \int_Y T'(y) \varepsilon(y(\theta)) \hat{T}'(y(\theta)) \phi(y) \, dy \]

- Integrate by parts

\[ \delta R(T, \hat{T}) = \int_Y \hat{T}(y) \phi(y) \, dy + \int_Y \hat{T}(y) \frac{d}{dy} \left( T'(y) \varepsilon(y) \phi(y) \right) \, dy \]

- Optimal tax \( T_* \) satisfies (Mirrlees 1971, Diamond 1998, Saez 2001):

\[ 0 = \phi_*(y) + \frac{d}{dy} \left( T'_*(y) \varepsilon_*(y) \phi_*(y) \right) \]
Optimal tax is not in a closed form

\[ 0 = \phi_*(y) + \frac{d}{dy} \left( T'_*(y) \varepsilon_*(y) \phi_*(y) \right) \]

▶ Saez:

\[ T'_*(y) = \frac{1}{\varepsilon_*(y)} \frac{1 - \Phi_*(y)}{\phi_*(y)} \]

▶ \( \varepsilon_*(y) \), \( \phi_*(y) \) are endogenous objects
Optimal tax is not in a closed form

\[ 0 = \phi_*(y) + \frac{d}{dy} \left( T'_*(y) \varepsilon_*(y) \phi_*(y) \right) \]

- **Diamond** (assume isoelastic):

\[
\frac{T'_*(y(\theta))}{1 - T'_*(y(\theta))} = \left( 1 + \frac{1}{\varepsilon} \right) \frac{1 - F(\theta)}{\theta f(\theta)}
\]

- **rhs** is a closed form as a function of \( \theta \)
- **lhs** is the marginal tax rate paid faced a type \( \theta \)
- **lhs** is not a closed form expression for the **income tax schedule** \( T(\cdot) \) as we do not know \( y \) as a function of \( \theta \)
- **this is the essence of the Mirrlees problem**
Plan: optimal tax

- Consider the static problem from the dynamic point of view
  - Associate a dynamic object: a heat kernel
- Derive a new characterization of the optimum:
  - A fairness principle (an invariance relationship for the optimum)
Heat kernel and the Operator

\[ 0 = \phi_*(y) + \frac{d}{dy} \left( T'_*(y) \varepsilon_*(y) \phi_*(y) \right) \]

- Second order differential operator \( L = \frac{\partial}{\partial y} \left( \varepsilon_*(y) \phi_*(y) \frac{\partial}{\partial y} \right) \):

\[ LT_* = -\phi_* \]

- Associate the heat kernel \( q_t (x, y) \) given by Kolmogorov forward equation:

\[ \frac{\partial}{\partial t} q_t (x, y) = \frac{\partial}{\partial y} \left( \varepsilon_*(y) \phi_*(y) \frac{\partial}{\partial y} q_t (x, y) \right) \]

\[ \lim_{t \to 0} q_t (x, y) = \delta (x - y) \]
Invariance relationship: proof

\[ \frac{\partial}{\partial t} \int q_t (x, y) T_*(y) \, dy = \int \frac{\partial}{\partial t} q_t (x, y) T_*(y) \, dy \]

\[ = [\text{use KFE}] \int \frac{\partial}{\partial y} \left( \varepsilon_*(y) \phi_*(y) \frac{\partial}{\partial y} q_t (x, y) \right) T_*(y) \, dy \]

\[ = [\text{integrate twice by parts}] \int q_t (x, y) \frac{\partial}{\partial y} \left( \varepsilon_*(y) \phi_*(y) \frac{\partial}{\partial y} T_*(y) \right) \]

\[ = [\text{use } LT_* = -\phi_*] - \int q_t (x, y) \phi_*(y) \, dy. \]

\[ \int q_t (x, y) T_*(y) \, dy = T_*(x) - \int_0^t \int q_s (x, y) \phi_*(y) \, dy \, ds \]
Proposition

**Fairness principle for the optimum.** The optimal tax $T_\ast(y)$ is invariant under the heat kernel $q_t(x, y)$ given by, for any $x \in Y$ and any $t > 0$:

\[
T_\ast(x) = \int_0^t \int q_s(x, y) \phi_\ast(y) \, dy \, ds + \int q_t(x, y) T_\ast(y) \, dy
\]

- $q_t(x, y)$ is an averaging function at scale $t$ (very similar to Gaussian)
- An optimal tax $T_\ast(x)$ at any income $x$ is equal to a (weighted by $q_t$) average of the taxes at all other incomes $T_\ast(y)$ at any (!) scale $t$
  - plus a mechanical effect
Why fairness?

A planner wants to raise revenue:
- no notion of treating people equally in any way

The most efficient way to do it is such that
- an agent at a given income $x$ is paying roughly the average of the amount of taxes paid by people working just more or less than him

More precisely: smoothing of the behavioral effect of taxes (distortions) and raised revenues at any scale $t$
What is $q_t(x, y)$?

**Proposition**

The heat kernel satisfies a Gaussian upper bound, **for any** $t > 0$:

$$q_t(x, y) \leq \frac{c_1}{\sqrt{t}} \exp(c_2 t) \exp\left(-c_3 \frac{(x - y)^2}{t}\right),$$

for some positive constants $c_1, c_2, c_3$.

**For** $t \to 0$,

$$q_t(x, y) \sim \frac{1}{\sqrt{4\pi \sigma_*(x) t}} \exp\left(-\frac{(y - x - \sigma'_*(x) t)^2}{4\sigma_*(x) t}\right),$$

where $\sigma_*(x) = \varepsilon_*(x) \phi_*(x)$.
Proposition

For all points $x, y$ and all times $t, s > 0$

$$q_{t+s}(x, y) = \int q_t(x, z)q_s(z, y)dz.$$ 

▶ A fact of crucial importance:

▶ the behavior of $q_t$ is tightly linked to both past and future behavior of the heat kernel and is thus far from arbitrary

▶ One weighting scheme at any scale!
Marginal taxes

\[ T_\star (x) = \int_0^t \int q_s (x, y) \phi_\star (y) \, dy \, ds + \int q_t (x, y) T_\star (y) \, dy \]

Corollary

The optimal marginal tax satisfies, for all \( t > 0 \)

\[ \frac{\partial}{\partial x} T_\star (x) = \int_0^t \int \left( \frac{\partial}{\partial x} q_s (x, y) \right) \phi_\star (y) \, dy \, ds + \int \left( \frac{\partial}{\partial x} q_t (x, y) \right) T_\star (y) \, dy \]

- The marginal tax is determined by the \textbf{global} behavior of the optimal tax code
  - a priori not obvious: understanding of the level does not necessarily translate to understanding the derivative
  - trivial for \( t \to 0 \) (def of a derivative), but we show \textbf{for all} \( t \)
What is $\frac{\partial}{\partial x} q_s (x, y)$?

- Has a positive part and a negative part with the same total mass
- **Marginal tax** = weighted taxes paid by individuals with income below **minus** weighted taxes paid by individuals with income above
- Similar formulas for the higher derivatives
Key to the results

- View the static optimal tax from the dynamic point of view
  - by associating the heat kernel \( q_t(x, y) \) with the operator \( L \) determining the optimal tax

- The optimal tax \( T_*(x) \) is of course time-independent

- Yet, can think of it as being invariant under a dynamic system that starts from this tax and applies the heat kernel \( q_t(x, y) \) to it
  - \( t \) can be time or scale
Summary: optimal tax

- Static optimal tax can be studied from the dynamic point of view
  - key object: a heat kernel
- New characterization: optimal tax satisfies an invariance relationship, the fairness principle
  - optimal tax is determined by the weighted average at all scales of optimal taxes at all other incomes
  - weighting behaves like a Gaussian and is tightly linked at all scales
  - behavioral effect and raised revenues are smoothed at all scales
  - applies to the higher derivatives
- Not a closed form solution but a very different view on the classic problem
Gradient flows

- Construct a dynamic system, a **gradient flow**,
  - starts at any (optimal or suboptimal) tax function
  - changes the tax system in the direction of the steepest increase in revenues.
  - The optimal tax is a stationary point of this system.

- Show that a version of the fairness principle holds for the gradient flow

- Gradient flow is a heat equation
  - all kinds of nice properties
Variations of taxes

- Start with any initial tax system \( T_t \)
- Change in revenue from a perturbation

\[
\delta R(T_t, \hat{T}_t) = \int_Y \left\{ \phi_t + \frac{\partial}{\partial y} [T'_t \varepsilon_t \phi_t] \right\} \hat{T}_t (y) \, d \]

- The gradient flow

\[
\frac{\partial T_t (y)}{\partial t} = \phi_t (y) + \frac{\partial}{\partial y} [T'_t (y) \varepsilon_t (y) \phi_t (y)]
\]

- maximizes the change in revenue in \( L^2 \) norm over infinitesimally short period of time
- a continuous version of the steepest descent in a functional space
- evaluated at the current tax system: \( T_t, \varepsilon_t, \phi_t \) are given.
Fairness principle for the gradient flow

Proposition

(Fairness principle for the gradient flow). For small $\tilde{t}$, the tax $T_{\tilde{t}}(y)$, generated by the gradient flow, is given by a weighted Gaussian average of the incomes:

$$T_{\tilde{t}}(x) \sim (\tilde{t} - t) \phi_t(x) + \int_y^\infty \frac{1}{\sqrt{4\pi \sigma_t(x)(\tilde{t} - t)}} e^{-\frac{(y-x-\sigma'_t(x)(\tilde{t}-t))^2}{4\sigma_t(x)(\tilde{t}-t)}} T_t(y) dy$$

- Key idea of the proof: for the short $\tilde{t}$, the coefficients $\varepsilon_t(y)$ and $\phi_t(y)$ are essentially “frozen”: get the short time asymptotics of the Brownian motion.
- Extends the notion of fairness to the trajectory of the gradient flow of taxes
- Everything in the closed form but unlike the optimum holds for short $\tilde{t}$.
  - of course, also holds for the optimum
Operator Splitting

- Gradient flow:
  - changes the tax schedule in favor of increasing government revenue,
  - letting $\phi_t$ and $\varepsilon_t$ being endogenously driven by $T_t$
  - density and elasticity change in response to the evolution of taxes.

- Propose to evolve the system separately:
  - fix the distribution of incomes $\phi_t$ and the elasticity $\varepsilon_t$ for a short period of time $\delta t$,
  - evolve $T_t$,
  - then re-compute $\phi_{t+\delta t}$ and $\varepsilon_{t+\delta t}$ based on the new tax function $T_{t+\delta t}$

- Government evaluates the changes in revenues under the current information given by the **exogenous sufficient statistics** evaluated at a given initial time.
Fixing $\phi_t$ and $\varepsilon_t$, the gradient flow

$$\frac{\partial T_t(y)}{\partial t} = \phi_t(y) + \frac{\partial}{\partial y} [T'_t(y) \varepsilon_t(y) \phi_t(y)]$$

is a heat equation, i.e., a second order PDE of the form

$$\frac{\partial T}{\partial t} = \phi(y) + \frac{\partial}{\partial y} [\sigma(y) \frac{\partial T(y, t)}{\partial y}]$$

One of the most well-known and well-behaved partial differential equation

obtain a number of textbook properties
Why heat equation?: loose intuition

- Heat tries to escape the larger is the gradient of temperature around the point

- Agents:
  - if the tax changes by \( \frac{\partial T_t(y_t(\theta))}{\partial t} \)
  - the agents move their income by \( \frac{\partial y_t(\theta)}{\partial t} = -\varepsilon_t(\theta) \frac{\partial T'_t(y_t(\theta))}{\partial t} \).
  - individual \( \theta \) adjusts his income in the opposite direction and proportionally to the change in the marginal tax rate that he faces

- If the tax is high on me compared to agents nearby, I will “escape” to a lower income
The initial tax system can be rather arbitrary.

For $\phi(y)$ and $\sigma(y)$ fixed, the solution will converge to unique fixed point $\tau(y)$ as $t \to \infty$:

$$0 = \phi(y) + \frac{\partial}{\partial y} \left[ \sigma(y) \frac{\partial \tau(y)}{\partial y} \right]$$

**Important:** we are not interested in letting the gradient flow for all $t \to \infty$ and thus finding $\tau$ but only evolving it for a very small time $t$ (so as to respect the operator splitting).

But, $\tau$ serves as a proper reference point to describe various smoothing properties of the gradient flow.
Proposition

The gradient flow of the revenue functional coincides with the gradient flow of the functional \( J \) given by the weighted Sobolev-type seminorm \( H^1 \)

\[
J(T) = \frac{1}{2} \int \sigma(y) (T'(y) - \tau'(y))^2 dy
\]

- This gradient flow is trying to smooth out rough irregularities in the difference between \( T \) and \( \tau \)
  - a large value of \(|T'(y) - \tau'(y)|\) implies to existence of a large value of \((T'(y) - \tau'(y))^2\)
  - the flow is trying to decrease this as quickly as possible.
- \( \sigma(y) \) serves as a natural weighting measure:
  - if \( \sigma(y) \) is large, then irregularities in that region count even more severely and are dampened quicker.
Proposition

Let $T(t, y)$ denote the solution of the heat equation associated with the gradient flow of the tax revenue functional. If $\sigma(y)$ is bounded away from 0 and $T(0, y)$ is bounded, then $T(t, y)$ is infinitely differentiable for any $t > 0$.

- Tax reform viewed as the gradient flow leads to the continuous tax systems.
- Moreover, the gradient flow has the effect of mollifying any tax scheme instantaneously.
  - If $T(y)$ has large amounts of strong oscillations or maybe even discontinuous jumps, then the gradient flow acts strongest on those parts first.
Proposition

Let \( T(t, y) \) denote the solution of the heat equation associated with the gradient flow of the tax revenue functional, \( T(0, y) \) be an arbitrary initial tax schedule, and \( \lambda_1 \) be the first eigenvalue of the associated Sturm-Liouville operator. Then, \( \forall t > 0 : \)

\[
\int_a^b (T(t, y) - \tau(y))^2 \, dy \leq e^{-2\lambda_1 t} \int_a^b (T(0, y) - \tau(y))^2 \, dy.
\]

- The gradient flow smooths a measure of variability of the tax schedule, the squared deviation from the limiting stationary solution.
  - Moreover, such smoothing is exponential.

- This result is not about the convergence to the optimal tax \( T^* \) but rather about the smoothing behavior of the gradient flow at each step of the operator splitting.
Global convergence

- Outside of the scope of this paper:
  - would be a nice Ph.D. in math thesis
- But what we have is a classical operator splitting
  - similar to splitting convection from diffusion
  - which pretty much converges under very broad circumstances
    Glowinski, Osher, and Yin (2016)
- Results on the fairness principle do not need operator splitting.
Conclusion and extensions

- Dynamical systems can provide new insights on the analysis of static optimal taxation problems
  - Fairness principle for the optimum and for the trajectory
  - Evolution of taxes is a heat equation: unifying variational and the optimal tax approach
- Extensions (immediate):
  - Maximizing social welfare
  - Extensive margin
  - Non quasi-linear preferences
- Multidimensional types?
- More broadly: nonlinear pricing, matching, etc.