The Hot Spots Conjecture on Graphs

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UNIVERSITY *of* WASHINGTON

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You have an insulated room and some non-constant initial distribution of heat $u(0, x)$. The heat equation runs for a long time: where are the hottest and the coldest spots?

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$$\begin{aligned} -\Delta u = \lambda u & \qquad \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \qquad \text{on } \partial\Omega. \end{aligned}$$

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for which there exists a solution ϕ_k. If $\lambda_1 < \lambda_2$ and $\langle u(0,x), \phi_1 \rangle \neq 0$, then

$$u(t,x) = e^{-\lambda_1 t} \langle u(0,x), \phi_1 \rangle + \text{lower order terms.}$$

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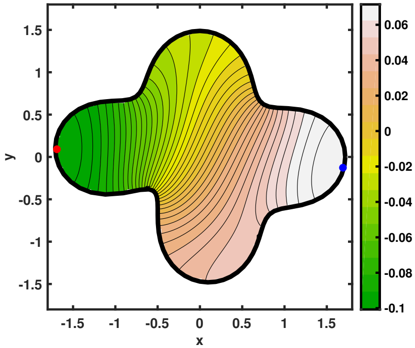

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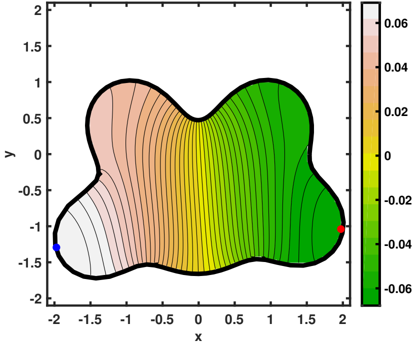

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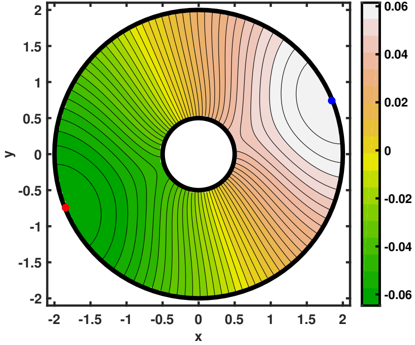

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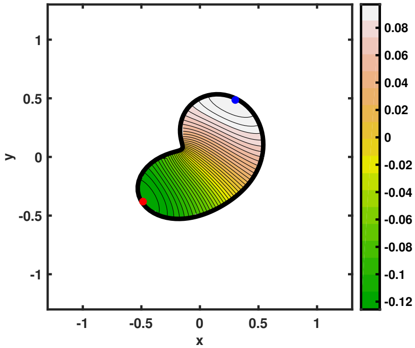

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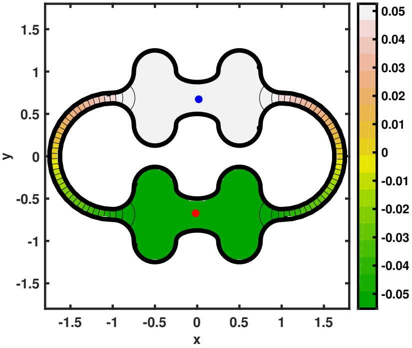

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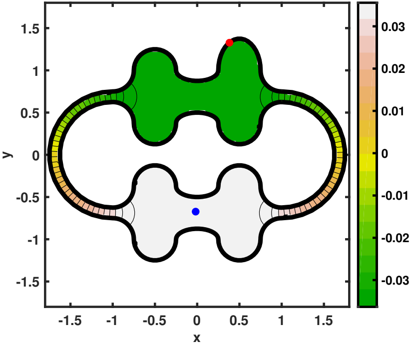

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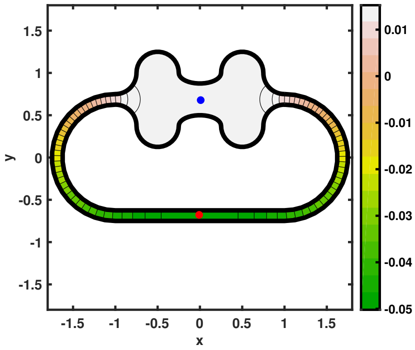

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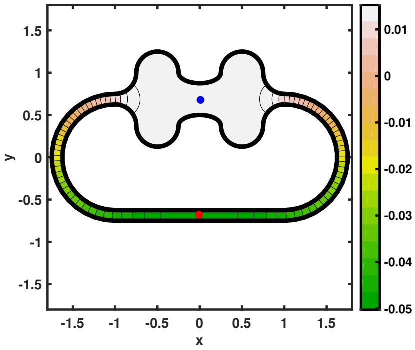

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These examples also lead to the first accurate guess for

$$\sup_{\Omega} \frac{\max_{x\in\Omega} u(x)}{\max_{x\in\partial\Omega} u(x)} \geq 1 + \varepsilon$$

I refer to Andreas' paper for details.

A Related Result

Suppose you have a long convex domain $\Omega \subset \mathbb{R}^2$. Let us fix $N = \text{diam}(\Omega)$ and $\text{inrad}(\Omega) = 1$.

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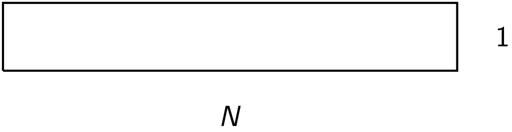

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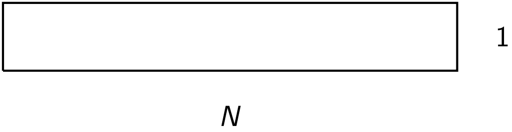

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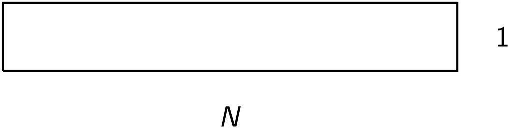

Where do we expect maxima and minima to be? On the boundary, certainly, but also at opposite ends!

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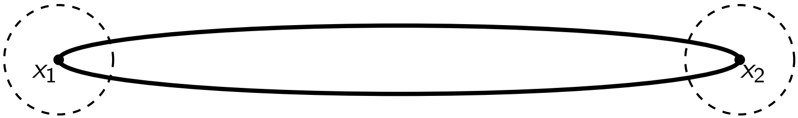

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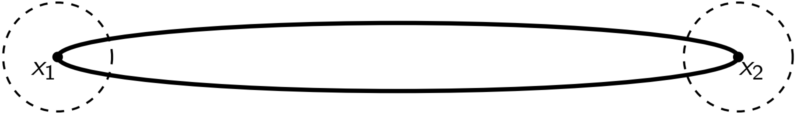

Theorem (S, 2019)

There exists a universal $c > 0$ such that for all bounded, convex $\Omega \subset \mathbb{R}^2$: if $x_1, x_2 \in \Omega$ are at maximal distance,

$$\|x_1 - x_2\| = \text{diam}(\Omega),$$

then ϕ_1 assumes every global maximum and minimum at distance at most $c \cdot \text{inrad}(\Omega)$ from $\{x_1, x_2\}$, where $\text{inrad}(\Omega)$ denotes the inradius of Ω.

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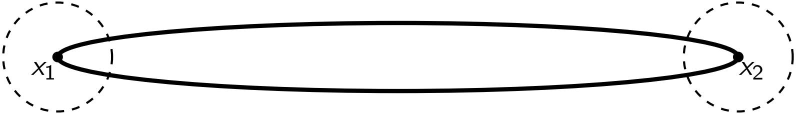

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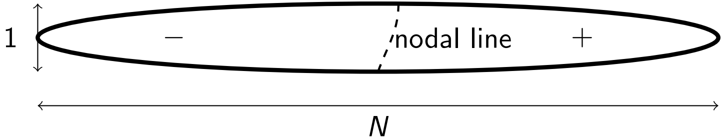
The proof is interesting – if you don't like Brownian motion, feel free to ignore, I will explain things on Graphs later!

Sketch of the Proof

We know roughly how the first nontrivial eigenfunction behaves in a long skinny domain (see



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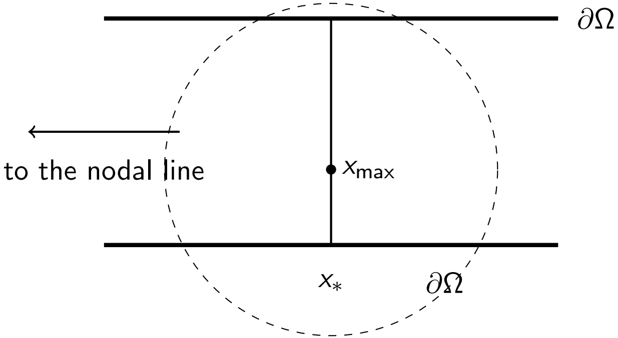

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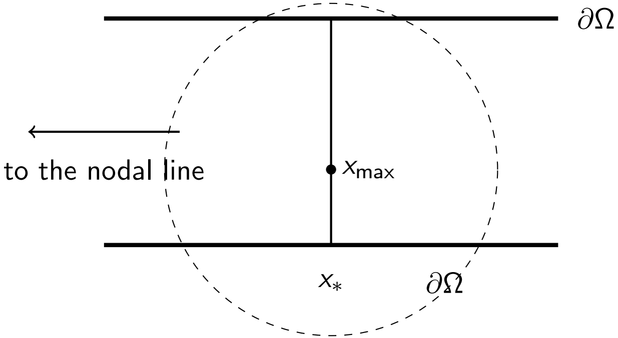

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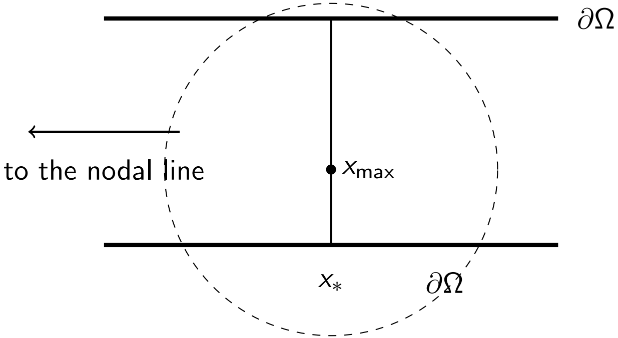

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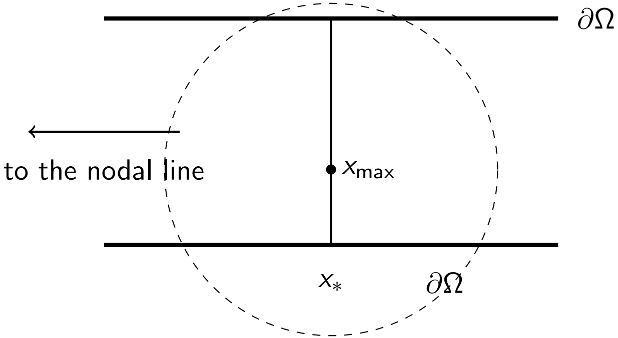

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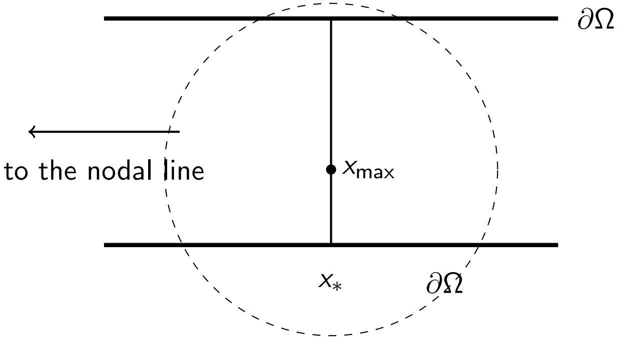

Suppose the maximum is in $x_{\max}$. Then $\nabla u(x_{max}) = 0$ and the local behavior is given by the second derivative. The Laplacian is $\Delta = \mathrm{tr} D^2$, so we get some local estimates. We get that the function is locally flat.

Sketch of the Proof

Let $\omega_x(t)$ denote a Brownian motion started in x and running up to time t (being reflected on the boundary of Ω), then

$$e^{\mu_1 t}\mathbb{E}\left(\phi_1(\omega_x(t))\right) = \phi_1(x).$$

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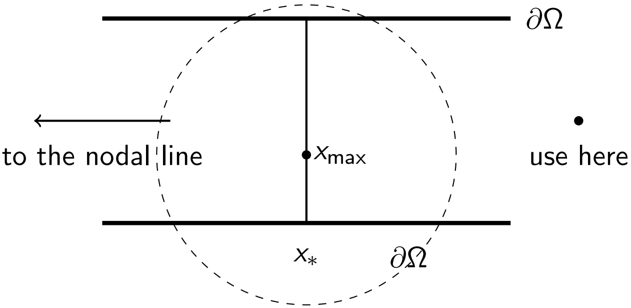

Markov property!

(Roy Lederman, Yale Statistics)

Chatting after lunch in front of the Stats Department: “What happens if you try it on trees?”

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In terms of linear algebra, $L = D - A$, where D is the diagonal matrix recording the degree of vertices and A is the adjacency matrix. (There are other notions of the Laplacian on Graphs but we will focus on this one henceforth). So, in particular, there is something like the first 'nontrivial' eigenvector.

$$\lambda_2(G) = \min_{x \perp \mathbf{1}} \frac{\sum_{v_i \sim_E v_j} (x_i - x_j)^2}{\sum_{i=1}^n x_i^2}.$$

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In particular, if you can prove something nice on Graphs, it may just translate to the continuous setting (graphs are harder but also change your perspective).

Theorem (Fiedler)

The induced subgraph on $\{v \in V : \phi_2(v) \geq 0\}$ is connected.

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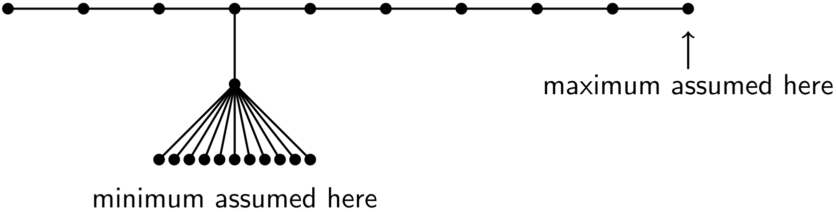

Figure: The ‘Fiedler rose’ counterexample of Evans (2011).

A Representation Formula

Let us fix $G = (V, E)$ to be a Graph on n vertices. Let v_1, v_2 be two arbitrary vertices. We introduce a game that results in a representation formula for eigenvector ϕ_2 associated to the eigenvalue λ_2. (It also works for other eigenvectors.)

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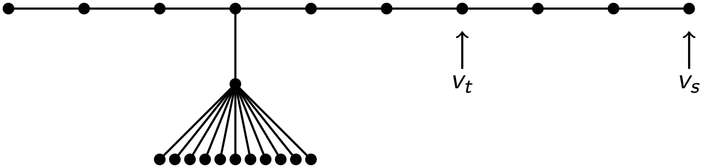

Pick v_t such that $\phi_2(v_t) > 0$. Then, by Fiedler's theorem, ϕ_2 is positive on the right half. The game is thus positive and we have monotonicity.

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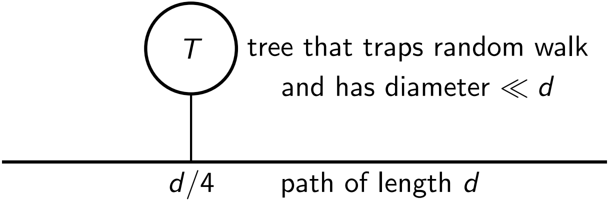

Figure: A generic counterexample to the conjecture that things happen at the endpoints of the longest path.

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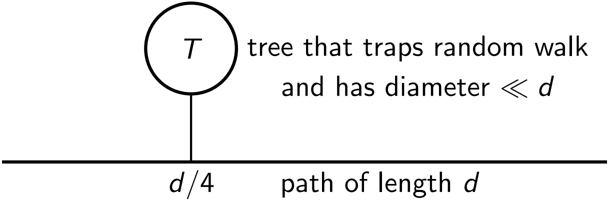
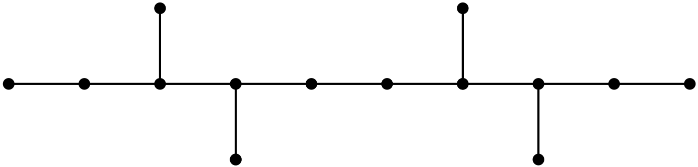


Figure: A generic counterexample to the conjecture that things happen at the endpoints of the longest path.

What's important is not length, it's number of steps in the game.

Caterpillar Graphs are trees where, if you remove the vertices of degree 1, you have a path.



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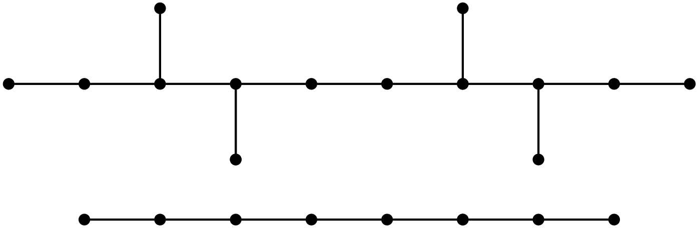

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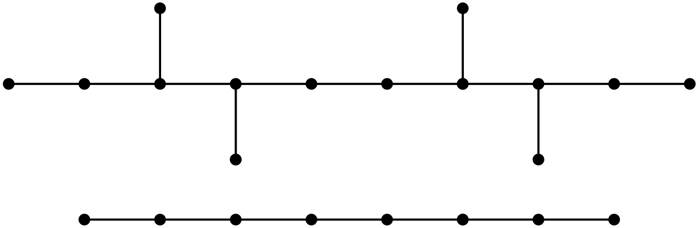

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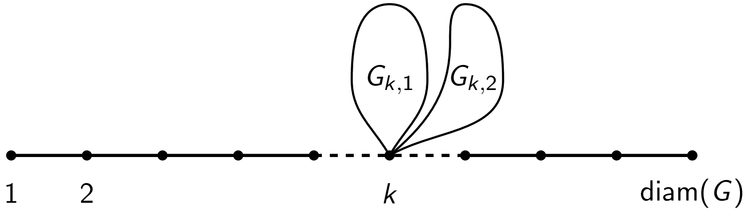
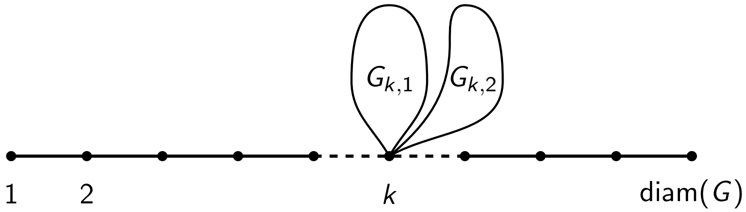


Figure: The class of admissible graphs: a long path whose attached Graphs are connected to exactly one vertex on the path and do not have any connections between them.



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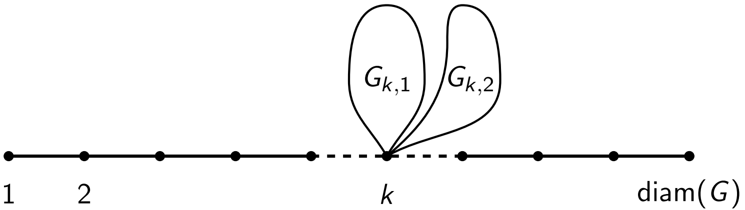
We also define hit($G_{k,i}$) as the largest expected number of steps necessary until you hit the path. The idea is that this should be small.



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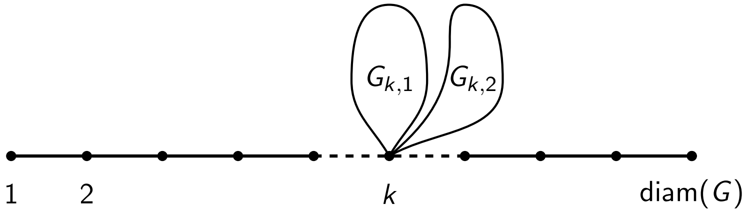
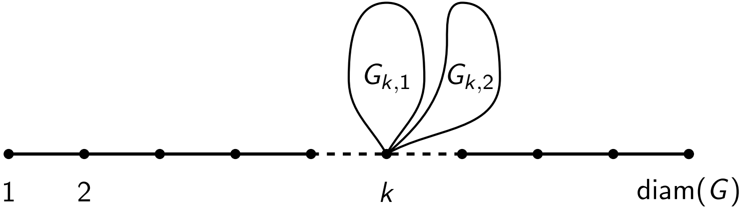


Theorem (Lederman and S)

Suppose that each graph $G_{k,i}$ attached to vertex k satisfies



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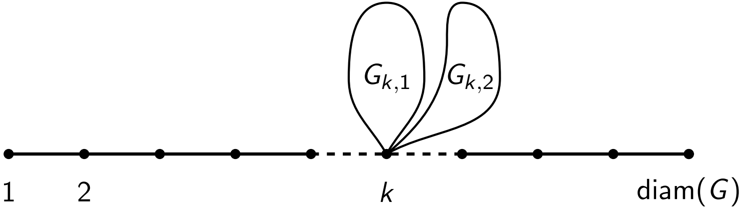
1. the attached Graph $G_{k,i}$ does not have too many vertices

$$|G_{k,i}| \leq \frac{\text{diam}(G)}{32}.$$



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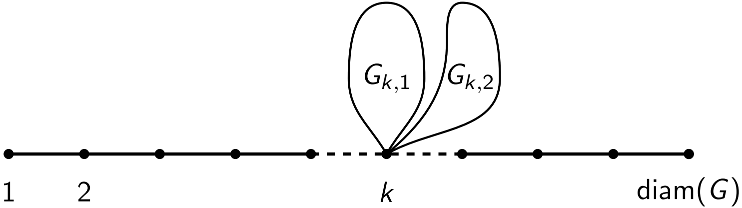

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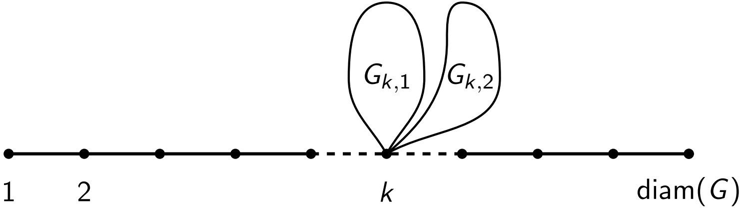
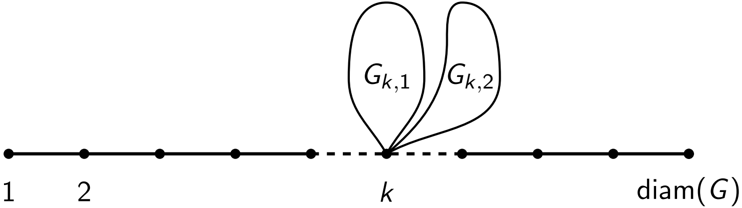
Then the second eigenvector of the Graph Laplacian assumes its extrema at the endpoints of the graph.



Corollary (Lederman and S)

If $G_{k,i}$ is a path graph, then maxima and minima of ϕ_2 are assumed at the end of the longest path if

$$\text{length}(G_{k,i}) < c \cdot \min\{k, n-k\}.$$



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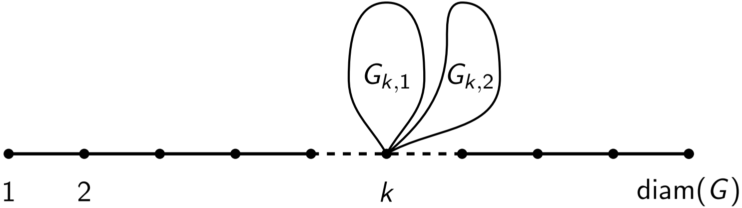

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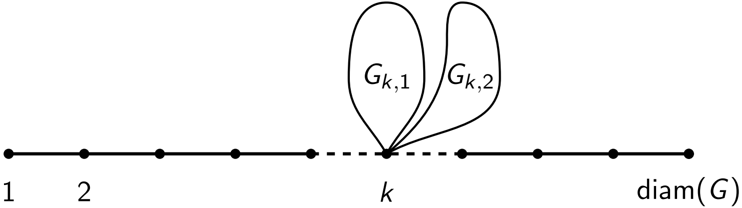

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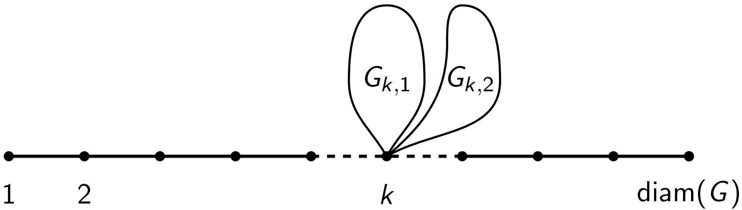

In summary, the Hot Spots conjecture is interesting and there should be interesting versions of it on Graphs.