Growth of Laplacian Eigenfunctions

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UNIVERSITY of WASHINGTON

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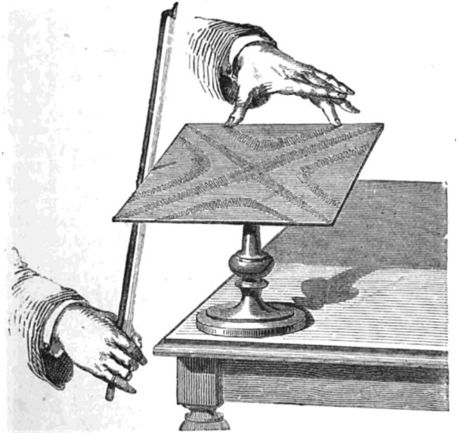
What do these eigenfunctions do? Specifically: $\|\phi_k\|_{L^\infty}$?


H. Adlard, Sc.
Chladni.
London: Longmans & Co



Chladni meets Napoleon

The late blind Justice Fielding walked for the first time into my room, when he once visited me, and after speaking a few words said, This room is about 22 feet long, 18 wide, and 12 high; all which he guessed by the ear with great accuracy.

(Erasmus Darwin, ZOONOMIA*)*

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D. Oliveira e Silva (IST Lisboa)

Felipe Goncalves (IMPA)

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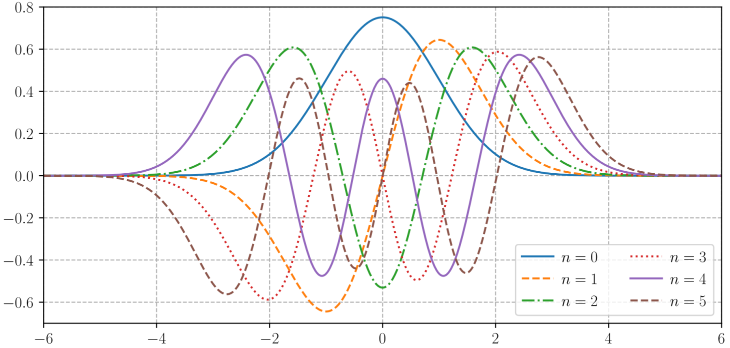

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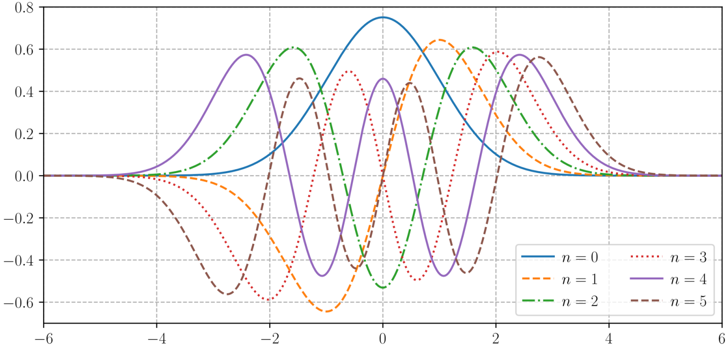

These are also eigenfunctions of the Fourier transform and

$$\mathcal{F}\phi_n = i^n \phi_n.$$

Lemma (G, OeS, S)

For any $\{a_1, \ldots, a_m\} \subset \mathbb{R}$ there are infinitely many $n \in \mathbb{N}$ so that

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This should be contrasted with the following

Fact (G, OeS, S)

There are only finitely many $n \in \mathbb{N}$ such that

$$\mathrm{sgn}(H_{4n}(1), H_{4n}(2), H_{4n}(3), H_{4n}(4)) = (+, +, -, +)$$

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Sketch of Proof. On compact intervals

$$\frac{\Gamma(2n+1)}{\Gamma(4n+1)} e^{\frac{x^2}{2}} H_{4n} = \cos\left(\sqrt{8n}x\right) + \mathcal{O}(1/\sqrt{n}).$$

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The linear flow $\gamma(t) = (a_1 t, a_2 t, \ldots, a_m t)$ gets arbitrarily close to the origin again and again (Poincare Recurrence Theorem) and the cosines are all positive there. □

Suppose now we have a ‘reasonably’ nice potential $V : \mathbb{R} \to \mathbb{R}$ (satisfying $|V(x)| \to \infty$ as $|x| \to \infty$) and suppose we have the eigenfunctions

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First gut instinct is that $\alpha_N \sim N/2$ since these quantities should be sort of unconnected. Second gut instinct: well, maybe they are connected a little because waves in 1D can only go left or right.

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Example. For $(x, y) = (0.5, 2.5)$ and $V(x) = x^2$ (Hermite functions), we have $\lim_{N \to \infty} \alpha_N = 3/5$. For example, $\alpha_{1000} = 603$.

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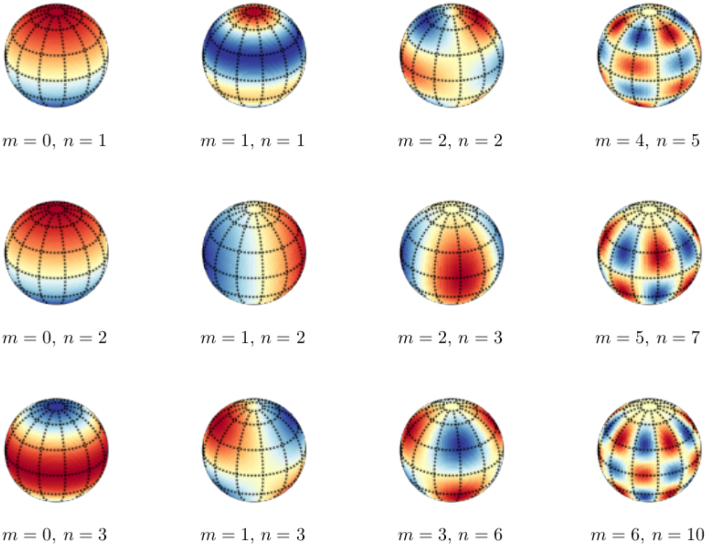
Main Takeaway

Growth of $\|\phi_k\|_{L^\infty}$ (beyond log) and sign correlations are intertwined.

Theorem (Levitan, 1952), (Avakumovic, 1956)

$$\|\phi_k\|_{L^\infty} \lesssim \lambda_k^{\frac{d-1}{4}}.$$

This is sharp on $\mathbb{S}^d$ for spherical harmonics.


$m = 0,\ n = 1$
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$m = 2,\ n = 2$
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$m = 1,\ n = 2$
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$m = 5,\ n = 7$
$m = 0,\ n = 3$
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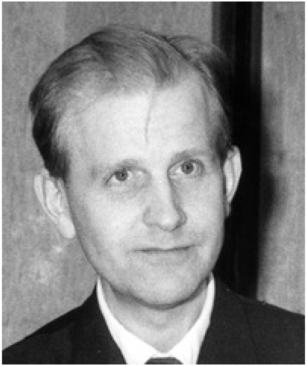
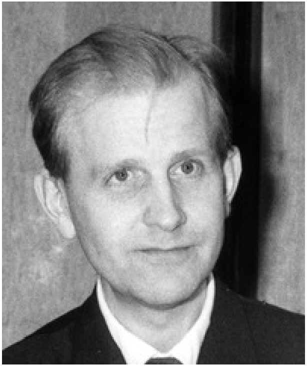
$m = 3,\ n = 6$
$m = 6,\ n = 10$



Local Weyl Law (Hörmander, 1966)

If normalized $\mathrm{vol}(M) = 1$, then

$$\sum_{k=1}^{n} \phi_k(x)^2 = n + \mathcal{O}(n^{\frac{d-1}{d}}).$$



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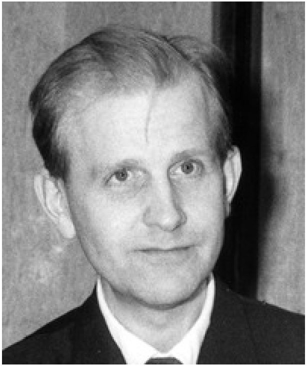

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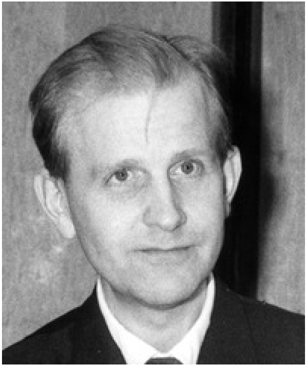

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The same kind of improvement is known for negatively curved compact manifolds, compact hyperbolic manifolds, ... On generic negatively curved manifolds it is expected that one has

$$\|\phi_n\|_{L^\infty} \lesssim \lambda_n^{\varepsilon}.$$

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Prediction (Michael Berry, Hejhal-Rackner,...)

Generically, we should have something like

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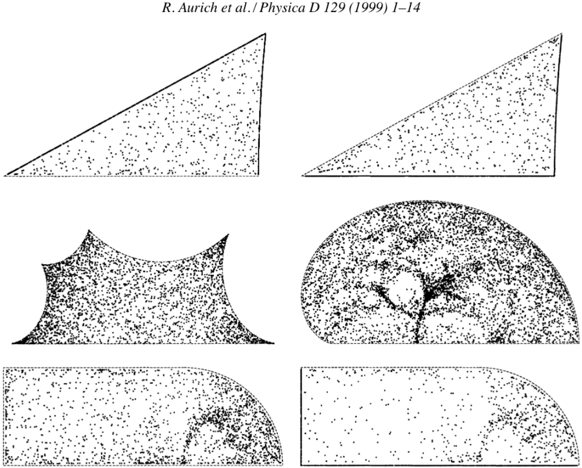

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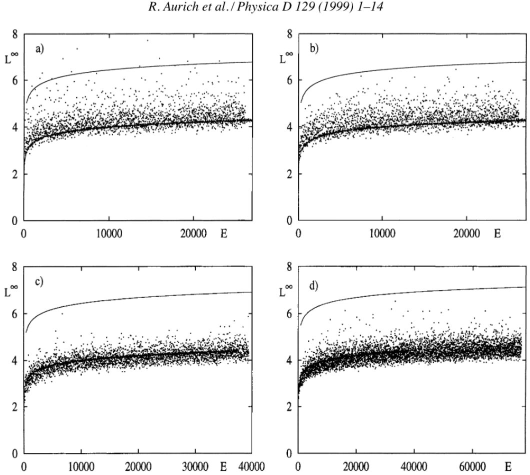

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many such local waves. Maximum of m Gaussians is $\sim \sqrt{\log m}$.

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In particular, for any choice of signs

$$\int_M (\pm\phi_1 \pm \phi_2 \pm \cdots \pm \phi_n)\phi_{n+1} dx = 0.$$

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$$\pm\phi_1, \pm\phi_2, \pm\phi_3, \dots$$

is an orthonormal basis of L^2 (eigen**spaces** not eigen**vectors**)

In particular, for any choice of signs

$$\int_M (\pm\phi_1 \pm \phi_2 \pm \cdots \pm \phi_n)\phi_{n+1} dx = 0.$$

Is there a particularly smart choice of signs?

New Idea

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We could flip all signs of ϕ_i in such a way that the eigenfunction ϕ_i is positive in x_0. Do we learn anything from

$$\int_M \left(\sum_{i=1}^{n} \text{sign}(\phi_i(x_0))\phi_i(x) \right) \phi_{n+1}(x) dx = 0?$$

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Basic Facts

We have

$$\int_M \int_M \amalg^{(n)}(x,y)^2 dxdy = n.$$

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$$\int_M \int_M \amalg^{(n)}(x,y)^2 dxdy = n.$$

On the diagonal $x = y$, we have $\amalg^{(n)}(x,x) = \sum_{i=1}^{n} |\phi_i(x)| dx$ and

$$n^{\frac{d+1}{2d}} \lesssim_{(M,g)} \sum_{k=1}^{n} |\phi_k(x)| \leq n + \mathcal{O}(n^{\frac{d-1}{d}}).$$

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Question. Is there a universal estimate

$$\int_M |\, \amalg^{(n)}(x,x)| dx = \sum_{k=1}^{n} \|\phi_k\|_{L^1} \gtrsim \frac{n}{(\log n)^{\alpha}}$$

for some $\alpha \geq 0$?

Prediction (Michael Berry, Hejhal-Rackner,...)

Generically, we should have something like

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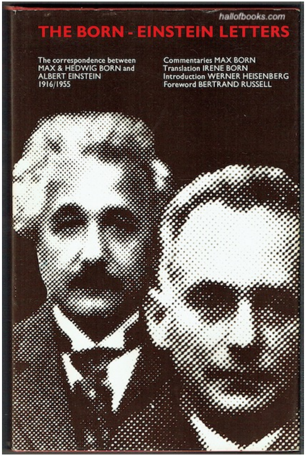
I will argue that these two things are highly intertwined. If this heuristic is violated, then we will speak of *spooky action at a distance* (picture of what that looks like in 2 slides).

Letter from Einstein to Born, March 3 1947

[...] die Physik eine Wirklichkeit in Zeit und Raum darstellen soll, ohne spukhafte Fernwirkungen.

[...] that physics should represent reality in time and space, without spooky action at a distance.



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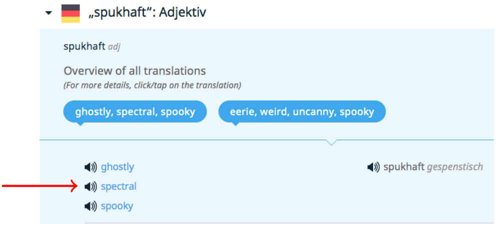

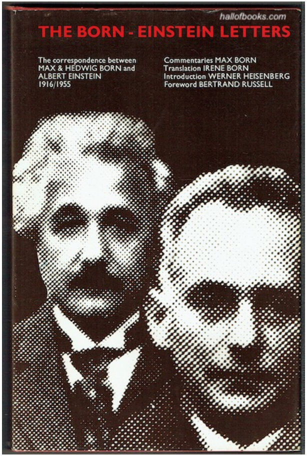

Spooky Action at a Distance

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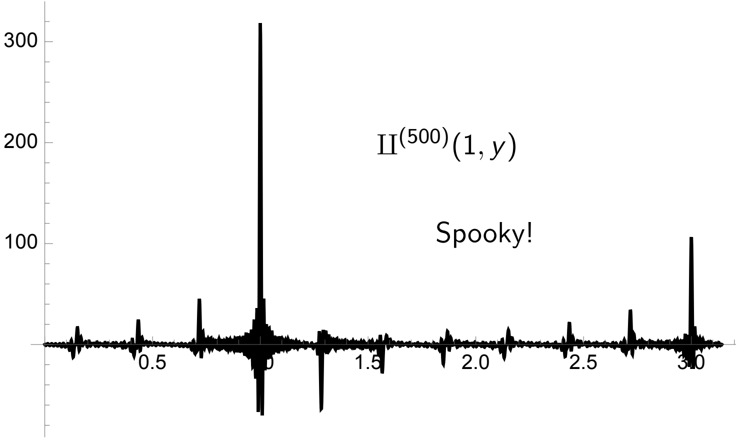

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Square $[0, \pi]^2$ (Dirichlet boundary): $\amalg^{(675)}((1.3, 1.3), (x, y))$.

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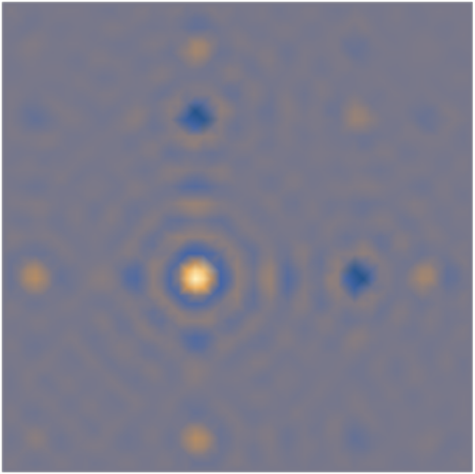

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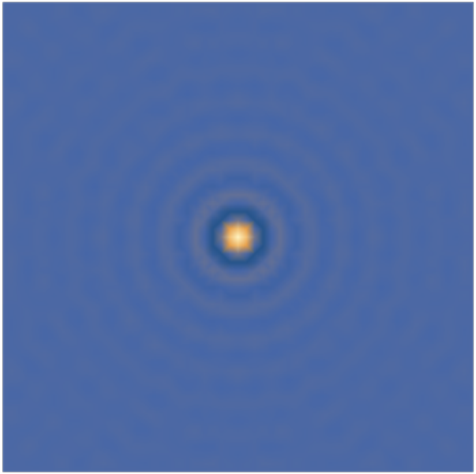

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Theorem (informal)

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The usual problem: eigenfunctions we can explicitly study are those with closed form expression. They have a closed form expression because their manifold is structured which is *not* generic. It is hard to get your hand on ‘generic’ eigenfunctions. Standard trick: take manifolds where eigenspaces have large multiplicity and take a random linear combination.

An Example: $\mathbb{S}^1$

Theorem

The canonical basis of eigenfunctions on $\mathbb{S}^1$ (sines and cosines) exhibits spooky action at a distance.

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One would expect such results to be true at a rather great level of generality (with guarantees to be formulated in terms of the multiplicity of the eigenspace).

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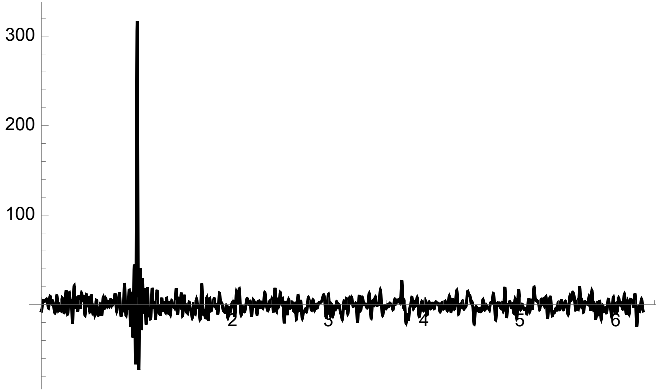

Figure: $\text{Ш}^{(250)}(1, y)$ for a fixed randomization of the Fourier basis.

An Example: $\mathbb{S}^1$ with randomly rotated basis

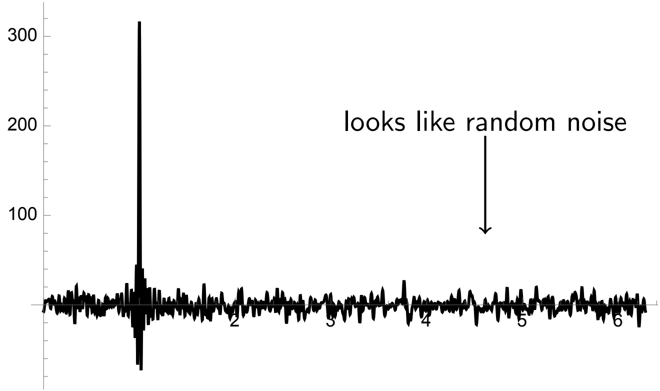

Figure: $\amalg^{(250)}(1, y)$ for a fixed randomization of the Fourier basis.

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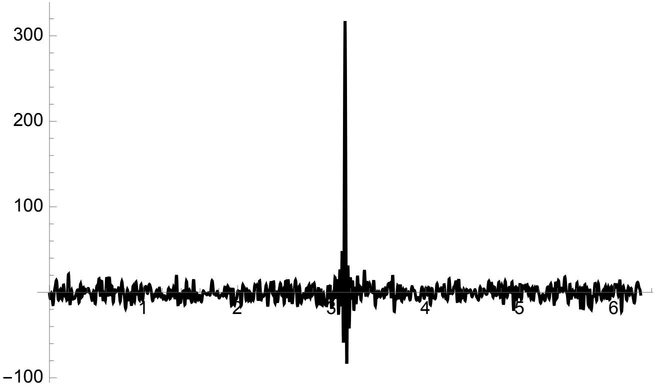
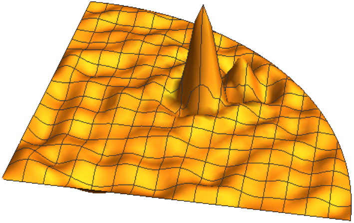
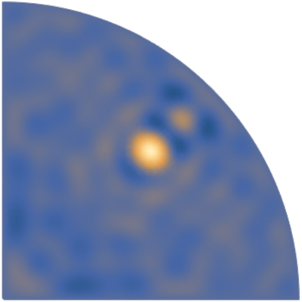


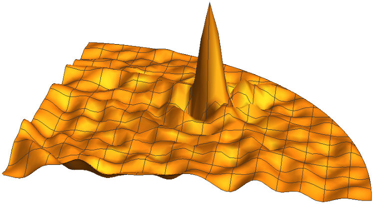
Figure: $\Pi^{(250)}(1, y)$ for a fixed randomization of the Fourier basis.

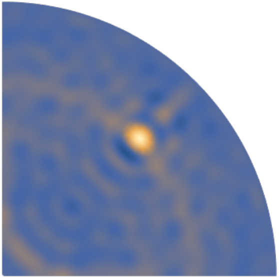
1/4−disk with Dirichlet boundary conditions





1/4−Disk with Neumann boundary conditions





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Lemma

On manifolds normalized to $\mathrm{vol}(M) = 1$, we have

$$\min_{x \in M} \amalg^{(n)}(x,x) \geq \frac{n - \mathcal{O}(n^{\frac{d-1}{d}})}{\max_{1 \leq k \leq n} \|\phi_k\|_{L^\infty}}.$$

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The only way for this term to be small is if previous terms were large. So this term by itself cannot be the (sole) reason for growth.

Corollary

Eigenfunction growth can only happen when

$$\left| \int_{M \backslash B(z,1/\sqrt{\lambda})} \amalg^{(n)}(z,y)\phi_{n+1}(y)dy \right| \qquad \textit{is large.}$$

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This, however, is tremendously interesting: there is no reason why an eigenfunction should correlate strongly with a particular linear combination of eigenfunctions.

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$$\left| \int_{M \setminus B(z, 1/\sqrt{\lambda})} \amalg^{(n)}(z,y)\phi_{n+1}(y)dy \right| \leq \sqrt{n} \sim \lambda_n^{d/4}$$

which is a factor $1/4$ worse than the $L^{\infty}-$bound. This being sharp would require $\amalg^{(n)}(z,y)$ and $\phi_{n+1}(y)$ to be proportional!

So how large do we expect

$$X = \int_{M \setminus B(z, 1/\sqrt{\lambda})} \amalg^{(n)}(z,y)\phi_{n+1}(y)dy \qquad \text{to be?}$$

Using the random wave model, we expect ϕ_{n+1} to behave like a random Gaussian at scale $1/\sqrt{\lambda_{n+1}} \sim n^{-1/d}$ and

$$\begin{aligned} X &\sim \amalg^{(n)}(x,y_1) \cdot \frac{\pm 1}{n} + \cdots + \amalg^{(n)}(x,y_n) \cdot \frac{\pm 1}{n} \\ &\sim \pm \sqrt{\frac{\amalg^{(n)}(x,y_1)^2}{n^2} + \cdots + \frac{\amalg^{(n)}(x,y_n)^2}{n^2}} \sim \mathcal{N}(0,1). \end{aligned}$$

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The maxima of n Gaussians is $\sim \sqrt{\log n}$ and we recover the random wave heuristic assuming an **integrated** version of the random wave model.

What I am not talking about: special function identities

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Spooky Action implies interesting special function correlations. On $\mathbb{T}$, $\sum_{k=1}^{n} \cos(ky)$ is orthogonal to $\cos{(n+1)y}$ but has nontrivial negative correlation outside the origin. On $\mathbb{S}^2$, we deduce that a suitable linear combination of Legendre Polynomials

$$\sum_{k=1}^{n} \sqrt{k + \frac{1}{2}} \cdot P_k(\cos\theta)$$

has some curious behavior with respect to $P_{k+1}(\cos\theta)$.

What I am not talking about II: other sign flips

We looked at one particular choice of sign flips

$$\int_M (\pm\phi_1 \pm \phi_2 \pm \cdots \pm \phi_n)\phi_{n+1} dx = 0.$$

But one could look at others!

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$$\text{Gaussian Free Field} = \sum_{k=1}^{\infty} \gamma_k \frac{\phi_k(x)}{\sqrt{\lambda_k}} \qquad \text{where } \gamma_k \sim \mathcal{N}(0,1).$$

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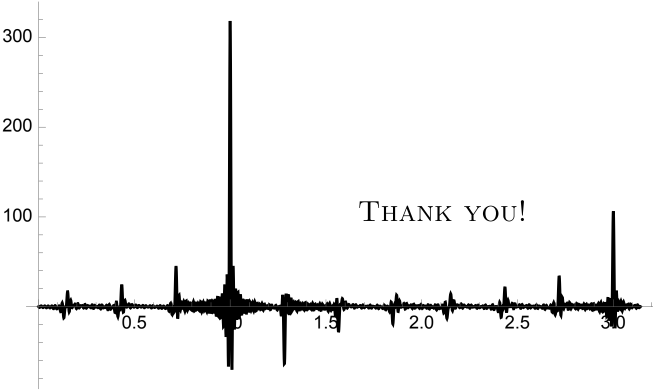
5. Special Function Identities as byproduct.

Long story short

1. Growth of eigenfunctions requires manifold structure but unclear what.
2. Random wave model: $\|\phi_n\|_{L^\infty} \lesssim \sqrt{\log \lambda_n}$
3. New ingredient: linear combinations $\amalg^{(n)}$. Growth of eigenfunctions happens iff there is strong global correlation between ϕ_{n+1} and **many** suitable linear combination of the first n eigenfunctions.
4. Not 'generically' expected, **integrated** random wave heuristic.
5. Special Function Identities as byproduct.
6. Counterexamples present Spooky Action at a Distance: **eigenfunctions synchronize across large scales.**



Thank you!