Dynamics of Roots of Polynomials

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Three problems

I will discuss three separate problems regarding polynomials, roots and dynamics. All three lead to a lot of open problems.

1. If the roots of the random polynomial p_n are i.i.d. in $\mathbb{C}$, where are the solutions of $p_n(z) - p_n(0)$?
2. Variations on the (deterministic) Erdős-Turan theorem
3. (**Main part**:) Dynamics of Roots under Differentiation

A Motivating Theorem

Theorem (Kalbuchko 2012, conjectured by Rivin & Pemantle)

Let μ be a probability distribution in $\mathbb{C}$, let $z_1, \ldots, z_n$ i.i.d. random variables and consider the random polynomial

$$p_n(z) = \prod_{k=1}^{n} (z - z_k)$$

Then the roots of $p_n'(z)$ are also distributed according to μ.

This has inspired quite a bit of research (O'Rourke and Williams, Hanin, ...).

Random roots of random equations

Problem

Let μ be a probability distribution in $\mathbb{C}$, let $z_1, \ldots, z_n$ i.i.d. random variables and consider the random polynomial

$$p_n(z) = \prod_{k=1}^{n} (z - z_k)$$

Where are the n roots of $p_n(z) - p_n(0)$?

(This question is motivated by some convergence questions in complex analysis about nonlinear systems using Blaschke products – if interested, ask me later.)

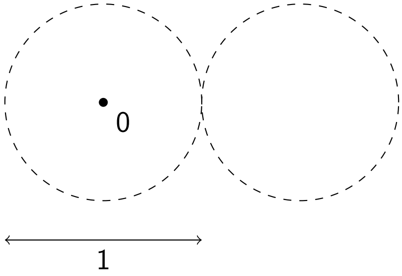
Example 1: μ is standard Gaussian in $\mathbb{C}$

roots of $p_n(z) - p_n(0)$

They appear to also be distributed according to a Gaussian.

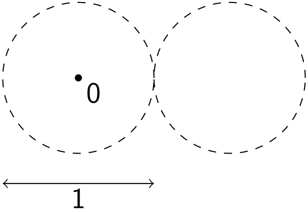
Example 2: μ is the union of two circles

Pick roots uniformly at random from



Where are the roots of $p_n(z) - p_n(0)$?

Example 2: μ is the union of two circles



roots of $p_n(z) - p_n(0)$

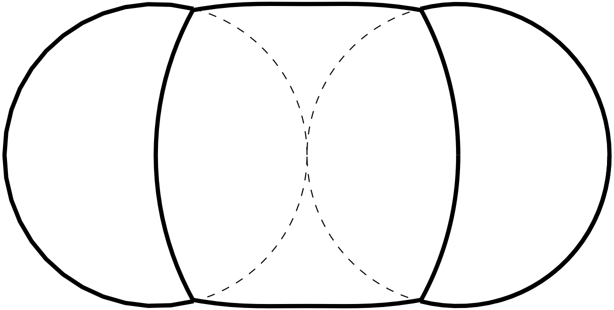
Theorem

Theorem (Hau-tieng Wu and S., 2018)

In some regions of $\mathbb{C}$, the solutions of $p_n(z) - p_n(0) = 0$ are distributed exactly as μ (see Example 1). In other regions, the solutions jump to fixed curves that one can compute.

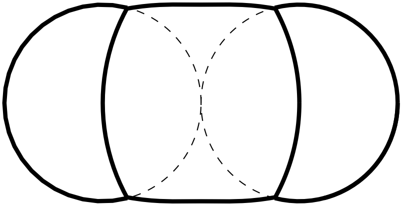
Remark. If μ is radial around 0, then only the first case appears. This is Example 1 (the Gaussian).

Example 2: μ is the union of two circles



If we pick the roots uniformly from the boundary of two circles, then the solutions of $p_n(z)$ land on the bold lines.

Example 2: μ is the union of two circles



Theorem

Define

$$A = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log|x - z| d\mu(x) > \int_{\mathbb{C}} \log|x| d\mu(x) \right\}$$

and

$$B = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log|x - z| d\mu(x) = \int_{\mathbb{C}} \log|x| d\mu(x) \right\}.$$

Theorem (Hau-tieng Wu and S.)

The distribution of $\{z \in \mathbb{C} : p_n(z) = p_n(0)\}$ converges to ν in distribution, where $\nu = \mu$ on A and ν has the remaining measure on B.

Sketch of Proof

$$A = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log|x-z| d\mu(x) > \int_{\mathbb{C}} \log|x| d\mu(x) \right\}$$

$$B = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log|x-z| d\mu(x) = \int_{\mathbb{C}} \log|x| d\mu(x) \right\}.$$

Main Idea is twofold: (1)

$$\mathbb{E} \log|p_n(z)| = n \int_{\mathbb{C}} \log(z-x) d\mu(x)$$

with very good concentration properties. This means that, for generic $z \in A$,

$$\mathbb{E}|p_n(z)| \gg \mathbb{E}|p_n(0)|$$

but then we can use every single root to find a solution (Rouché's theorem).

Sketch of Proof

$$A = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log|x - z| d\mu(x) > \int_{\mathbb{C}} \log|x| d\mu(x) \right\}$$

$$B = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log|x - z| d\mu(x) = \int_{\mathbb{C}} \log|x| d\mu(x) \right\}.$$

Main Idea is twofold: (2)

$$\mathbb{E} \log|p_n(z)| = n \int_{\mathbb{C}} \log(z - x) d\mu(x)$$

The rest has to concentrate on the lines (counting argument) in an absolutely continuous fashion. Used many times: there are exactly n solutions.

Fun Problems

Theorem (Hau-tieng Wu and S., 2018)

Then the distribution of $\{z \in \mathbb{C} : p_n(z) = p_n(0)\}$ converges to ν in distribution, where $\nu = \mu$ on A and ν has measure $1 - \mu(A)$ supported on B.

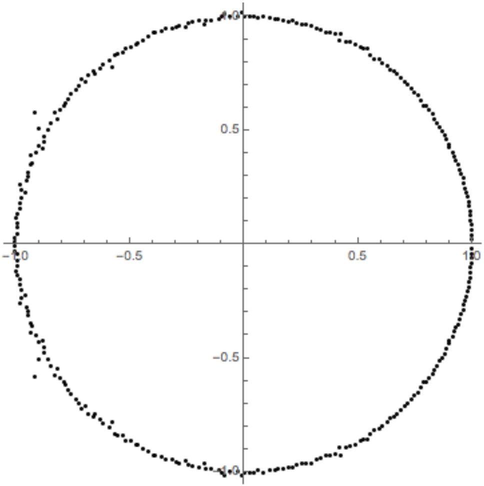
Problems: what does B generically look like? Lines? How many can intersect? What about

$$p_n(z) - p_n(\text{random point})?$$

What if one were to iterate this? (A good understanding of these kinds of questions would actually have implications in signal processing).

The Erdős-Turan Theorem

The Erdős-Turan Theorem: degree 300, i.i.d. coefficients



The Erdős-Turan Theorem

Let $p_n : \mathbb{C} \to \mathbb{C}$ be a (monic) polynomial

$$p_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Main idea: if the coefficients aren't terribly large, then the roots are all very close $|z| = 1$.

Careful: $p(z) = z^n$ has all roots in 0. But somehow this is the only obstruction. (The theorem requires a_0 to not be too small.) Also: the roots are equally distributed in angle.

Erdős-Turan Theorem: equidistribution at scale $n^{-1/2}$

$$h(p) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left(\frac{|p(e^{i\theta})|}{\sqrt{|a_0|}}\right) d\theta,$$

where $\log^+(x) = \max(0, \log x)$.

Theorem (Soundararajan 2018, Erdős-Turán 1948)

We have, assuming $|a_n| = 1$,

$$\max_{J\subset\mathbb{T}} \left| \frac{\#\left\{1 \le k \le n : \arg\ z_k \in J\right\}}{n} - \frac{|J|}{2\pi} \right| \le \frac{8}{\pi}\frac{\sqrt{h(p)}}{\sqrt{n}},$$

where the maximum runs over all intervals $J \subset \mathbb{T}$, where $\mathbb{T}$ is the one-dimensional torus scaled to lentgh 2π and identified with the boundary of the unit disk.

An Accidental Discovery

Very important in Analytic Number Theory: sums of cosines that are positive. For example

$$\sum_{|k|\leq n}\left(1-\frac{|j|}{n}\right)e^{ijx}=1+2\sum_{1\leq k\leq n}\left(1-\frac{j}{n}\right)\cos\left(jx\right)\geq 0$$

which is the Fejér kernel. Another nice example is Fejér-Gronwall-Jackson (1910, 1911)

$$\sum_{k=1}^{n}\frac{\sin kx}{k}>0 \qquad \text{for } 0<x<\pi$$

and the Young inequality (1913)

$$1+\sum_{k=1}^{n}\frac{\cos kx}{k}>0 \qquad \text{for } 0<x<\pi.$$

An Accidental Discovery

Theorem (Vietoris, 1958)

If a_k is a decreasing sequence of positive real numbers such that

$$2k \cdot a_{2k} \leq (2k-1) \cdot a_{2k-1},$$

then, for $0 < x < \pi$,

$$\sum_{k=1}^{n} a_k \sin kx > 0 \quad \text{and} \quad \sum_{k=0}^{n} a_k \cos kx > 0.$$

This inspired *a lot* of subsequent research.
Leopold Vietoris (1891 – 2002)

- became famous in the 1920s for work in topology
- wrote his last paper at age 103 on trigonometric sums
- died at 110 years, 309 days. His wife died at age 100, they are the second oldest couple (combined: 211 years) in history.

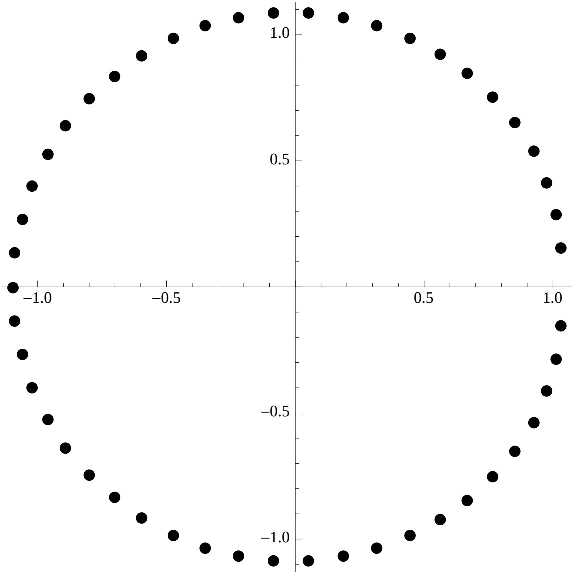
An Accidental Discovery

$$\sum_{k=0}^{n} a_k \cos kx \quad \text{corresponds to} \quad \sum_{k=0}^{n} a_k z^k.$$

If the trigonometric sum has few or no roots, then what about the polynomial?

An Accidental Discovery

Roots of the Fejer polynomial: slightly *too* regular.



An Accidental Discovery

Theorem (S. 2019)

If

$$\sum_{k=0}^{n} a_k \cos kx$$

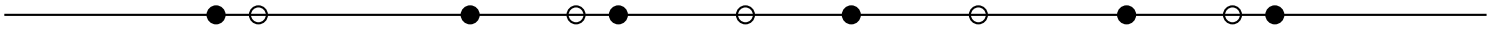
has $\lesssim n^{\delta}$ real roots, then the roots of $\sum_{k=0}^{n} a_k z^k$ are equidistributed at scale $\lesssim n^{\delta-1}$ in angle.

This improves Erdős-Turán for $\delta < 1/2$. It also raises many questions: is $\delta < 1/2$ necessary? Somehow one would expect that $\delta < 1$ might be enough because 'typical' trigonometric polynomials have $\sim n$ roots.

Dynamics of Roots of Derivatives

Problem (Polya, Riesz)

Let $p_n : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n. What can you say about the roots of p_n'? Are they more regular?



Basic counting argument: between any two roots, there is a maximum or a minimum. That corresponds to a root of p_n'. Since there are n roots, there are $n - 1$ gaps between the roots and therefore the roots of p_n and p_n' interlace.

Dynamics of Roots of Derivatives

Problem (Polya, Riesz)

Let $p_n : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n. What can you say about the roots of p_n'? Are they more regular?

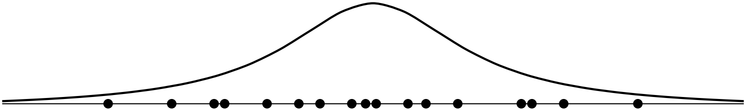
'Gaps become bigger Theorem' (Riesz)

Let $p_n : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n. Then the largest gap between roots of $p_n'(x)$ is at least as big as that of p_n.

There are is a vague folklore conjecture that if you keep differentiating, the roots even out (Polya, Farmer-Rhoades). Nothing is known.

The Big Question

Let $p_n : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n all of whose roots are on the real line (and, say, distributed according to a nice function). What does it look like when I differentiate it *a lot*?



A Motivating Proposition

Let $p_n : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n whose roots are randomly chosen on $[0, 1]$. Then the roots of $p_n^{(k)}$ are also uniformly distributed on $[0, 1]$ for all

$$k \lesssim n/(\log n)^2$$

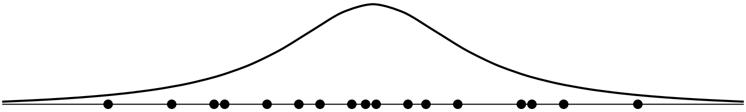
as $n \to \infty$.

Proof: the roots can only move at most the distance of the largest gap, which is size $\sim \log n/n$ or something like that.

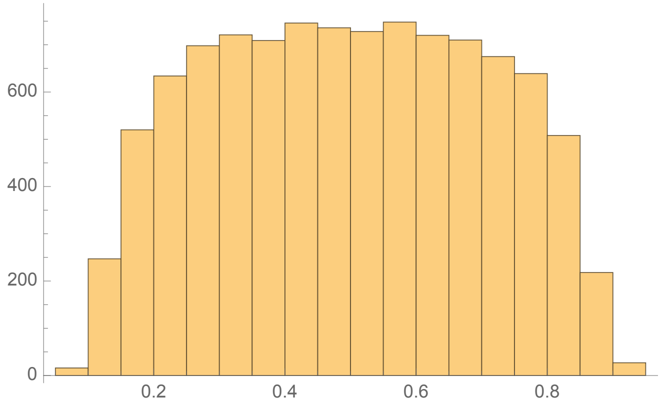
The Big Question made precise

Let $p_n : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n all of whose roots are on the real line. Where are the roots of

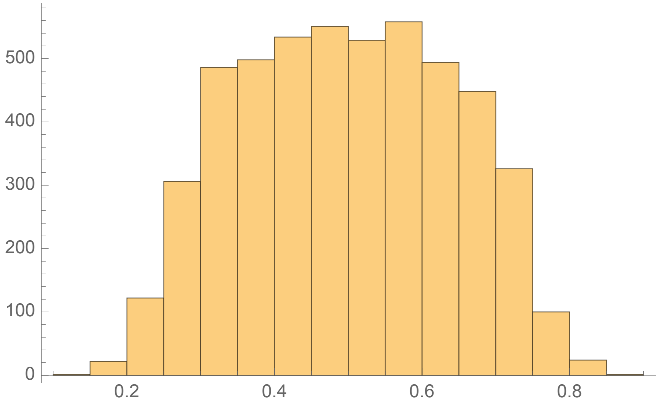
$$p_n^{(t \cdot n)} \qquad \text{for } 0 < t < 1?$$



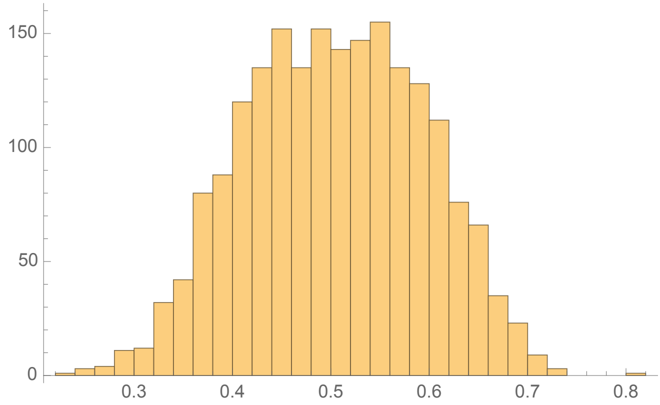
Example: take p_{20} to be given by 20 roots from $U[0,1]$. Where are the roots of $p_{20}^{(10)}$?



Example: take p_{20} to be given by 20 roots from $U[0,1]$. Where are the roots of $p_{20}^{(15)}$?



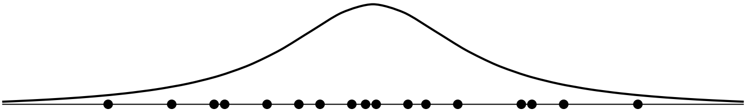
Example: take p_{20} to be given by 20 roots from $U[0,1]$. Where are the roots of $p_{20}^{(18)}$?



The Big Question made precise

Let $p_n : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n all of whose roots are on the real line. Where are the roots of

$$p_n^{(t \cdot n)} \qquad \text{for } 0 < t < 1?$$



If the roots of $p_n^{(t \cdot n)}$ are distributed according to $u(t, x)$, then what sort of equation does $u(t, x)$ satisfy?

The BIG equation

$$\text{roots of } p_n^{(t\cdot n)} \sim u(t,x)$$

Educated Guess (S. 2018)

$u(t,x)$ satisfies a nonlinear, nonlocal transport equation

$$\boxed{\frac{\partial u}{\partial t} + \frac{1}{\pi}\frac{\partial}{\partial x}\left(\arctan\left(\frac{Hu}{u}\right)\right) = 0}$$

where

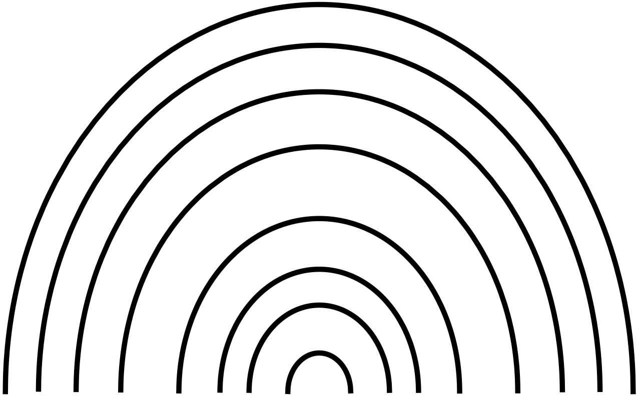
$$Hf(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy \qquad \text{is the Hilbert transform.}$$

Testcase 1: the Hermite polynomials

1. the roots of the Hermite polynomial H_n are approximately (in the sense of weak convergence after rescaling) given by the measure
$$\mu = \frac{1}{\pi}\sqrt{2n - x^2}dx$$
2. the derivatives of Hermite polynomials are again Hermite polynomials
$$\frac{d^m}{dx^m}H_n(x) = \frac{2^n n!}{(n-m)!}H_{n-m}(x).$$

Testcase 1: the Hermite polynomials

for Hermite polynomials, we expect a family of shrinking semicircles



$$u(t,x) = \frac{2}{\pi}\sqrt{1-t-x^2} \quad \text{solves the equation}$$

Testcase 2: the Laguerre polynomials

The family of associated Laguerre polynomials $L_n^{(\alpha)}$ satisfies

$$\frac{d^k}{dx^k} L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x).$$

The roots are given by the Marchenko-Pastur distribution

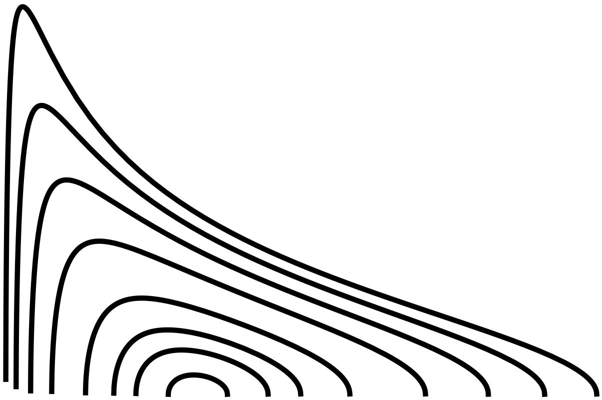
$$v(c,x) = \frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi x} \chi_{(x_-,x_+)} dx$$

where

$$x_\pm = (\sqrt{c+1} \pm 1)^2.$$

Testcase 2: Orthogonal Polynomials

for Laguerre, we expect a one-parameter family within the Marchenko-Pastur family



$$u_c(t,x) = v\left(\frac{c+t}{1-t}, \frac{x}{1-t}\right)$$ solves the equation!

Testcase 3: Integration!

$p_n^{(t \cdot n)}$ has $(1-t)n$ roots. There is a constant loss of roots.

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}} u(t,x)dx &= \int_{\mathbb{R}} \frac{\partial u}{\partial t}(t,x)dx \\ &= -\int_{\mathbb{R}} \frac{1}{\pi}\frac{\partial}{\partial x}\left(\arctan\left(\frac{Hu}{u}\right)\right)dx \\ &= -1.\end{aligned}$$

The equation has a constant loss of mass. If we start with a probability distribution, then there is a constant loss of mass and finite time blow-up at time $t = 1$ (when we have differentiated the polynomial n times).

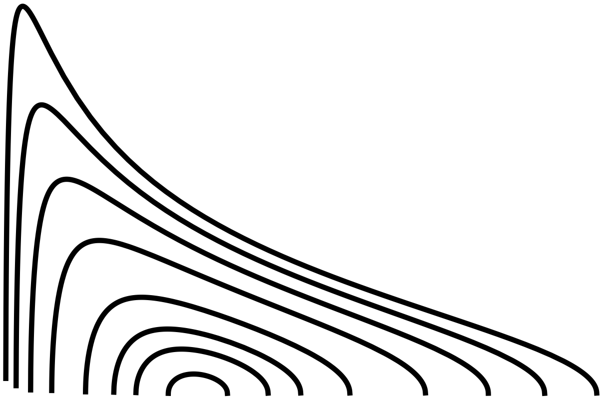
Science Fiction!

Magic!

There seems to be a nonlinear, nonlocal partial differential equation that describes the roots of polynomials under differentiation.

More Magic!

The equation looks a bit like those one-dimensional model equations for Euler/Navier stokes (cf. Rafael Granero Belinchon). Do roots of polynomials flow like water?



Thank you!