Dynamics of Roots of Polynomials

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I will discuss three separate problems regarding polynomials, roots and dynamics. All three lead to a lot of open problems.

1. If the roots of the random polynomial $p_n$ are i.i.d. in $\mathbb{C}$, where are the solutions of $p_n(z) - p_n(0)$?

2. Variations on the (deterministic) Erdős-Turan theorem

3. (Main part:) Dynamics of Roots under Differentiation
A Motivating Theorem

Theorem (Kalbuckho 2012, conjectured by Rivin & Pemantle)
Let $\mu$ be a probability distribution in $\mathbb{C}$, let $z_1, \ldots, z_n$ i.i.d. random variables and consider the random polynomial

$$p_n(z) = \prod_{k=1}^{n} (z - z_k)$$

Then the roots of $p_n'(z)$ are also distributed according to $\mu$.

This has inspired quite a bit of research (O’Rourke and Williams, Hanin, ...).
Random roots of random equations

Problem
Let $\mu$ be a probability distribution in $\mathbb{C}$, let $z_1, \ldots, z_n$ i.i.d. random variables and consider the random polynomial

$$p_n(z) = \prod_{k=1}^{n} (z - z_k)$$

Where are the $n$ roots of $p_n(z) - p_n(0)$?

(This question is motivated by some convergence questions in complex analysis about nonlinear systems using Blaschke products – if interested, ask me later.)
Example 1: $\mu$ is standard Gaussian in $\mathbb{C}$

roots of $p_n(z) - p_n(0)$

They appear to also be distributed according to a Gaussian.
Example 2: $\mu$ is the union of two circles

Pick roots uniformly at random from

Where are the roots of $p_n(z) - p_n(0)$?
Example 2: \( \mu \) is the union of two circles

\[ \text{roots of } p_n(z) - p_n(0) \]
Theorem (Hau-tieng Wu and S., 2018)

In some regions of $\mathbb{C}$, the solutions of $p_n(z) - p_n(0) = 0$ are distributed exactly as $\mu$ (see Example 1). In other regions, the solutions jump to fixed curves that one can compute.

**Remark.** If $\mu$ is radial around 0, then only the first case appears. This is Example 1 (the Gaussian).
Example 2: $\mu$ is the union of two circles

If we pick the roots uniformly from the boundary of two circles, then the solutions of $p_n(z)$ land on the bold lines.
Example 2: $\mu$ is the union of two circles
Theorem

Define

\[ A = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log |x - z| d\mu(x) > \int_{\mathbb{C}} \log |x| d\mu(x) \right\} \]

and

\[ B = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log |x - z| d\mu(x) = \int_{\mathbb{C}} \log |x| d\mu(x) \right\}. \]

Theorem (Hau-tieng Wu and S.)

The distribution of \( \{ z \in \mathbb{C} : p_n(z) = p_n(0) \} \) converges to \( \nu \) in distribution, where \( \nu = \mu \) on \( A \) and \( \nu \) has the remaining measure on \( B \).
Sketch of Proof

\[ A = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log |x - z| d\mu(x) > \int_{\mathbb{C}} \log |x| d\mu(x) \right\} \]

\[ B = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log |x - z| d\mu(x) = \int_{\mathbb{C}} \log |x| d\mu(x) \right\} . \]

Main Idea is twofold: (1)

\[ \mathbb{E} \log |p_n(z)| = n \int_{\mathbb{C}} \log (z - x) d\mu(x) \]

with very good concentration properties. This means that, for generic \( z \in A \),

\[ \mathbb{E}|p_n(z)| \gg \mathbb{E}|p_n(0)| \]

but then we can use every single root to find a solution (Rouché’s theorem).
Sketch of Proof

\[ A = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log |x - z| d\mu(x) > \int_{\mathbb{C}} \log |x| d\mu(x) \right\} \]

\[ B = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log |x - z| d\mu(x) = \int_{\mathbb{C}} \log |x| d\mu(x) \right\}. \]

Main Idea is twofold: (2)

\[ \mathbb{E} \log |p_n(z)| = n \int_{\mathbb{C}} \log (z - x) d\mu(x) \]

The rest has to concentrate on the lines (counting argument) in an absolutely continuous fashion. Used many times: there are exactly \( n \) solutions.
Fun Problems

Theorem (Hau-tieng Wu and S., 2018)
Then the distribution of \( \{ z \in \mathbb{C} : p_n(z) = p_n(0) \} \) converges to \( \nu \) in distribution, where \( \nu = \mu \) on \( A \) and \( \nu \) has measure \( 1 - \mu(A) \) supported on \( B \).

Problems: what does \( B \) generically look like? Lines? How many can intersect? What about \( p_n(z) - p_n(\text{random point})? \)

What if one were to iterate this? (A good understanding of these kinds of questions would actually have implications in signal processing).
The Erdős-Turan Theorem
The Erdős-Turan Theorem: degree 300, i.i.d. coefficients
Let $p_n : \mathbb{C} \to \mathbb{C}$ be a (monic) polynomial

$$p_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0.$$  

Main idea: if the coefficients aren’t terribly large, then the roots are all very close $|z| = 1$.

Careful: $p(z) = z^n$ has all roots in 0. But somehow this is the only obstruction. (The theorem requires $a_0$ to not be too small.) Also: the roots are equally distributed in angle.
Erdős-Turan Theorem: equidistribution at scale $n^{-1/2}$

$$h(p) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left( \frac{|p(e^{i\theta})|}{\sqrt{|a_0|}} \right) d\theta,$$

where $\log^+(x) = \max(0, \log x)$.

**Theorem (Soundararajan 2018, Erdős-Turán 1948)**

We have, assuming $|a_n| = 1$,

$$\max_{J \subset \mathbb{T}} \left| \frac{\# \{1 \leq k \leq n : \arg z_k \in J \}}{n} - \frac{|J|}{2\pi} \right| \leq \frac{8}{\pi} \frac{\sqrt{h(p)}}{\sqrt{n}},$$

where the maximum runs over all intervals $J \subset \mathbb{T}$, where $\mathbb{T}$ is the one-dimensional torus scaled to length $2\pi$ and identified with the boundary of the unit disk.
An Accidental Discovery

Very important in Analytic Number Theory: sums of cosines that are positive. For example

\[
\sum_{|k| \leq n} \left( 1 - \frac{|j|}{n} \right) e^{i j x} = 1 + 2 \sum_{1 \leq k \leq n} \left( 1 - \frac{j}{n} \right) \cos(j x) \geq 0
\]

which is the Fejér kernel. Another nice example is Fejér-Gronwall-Jackson (1910, 1911)

\[
\sum_{k=1}^{n} \frac{\sin(k x)}{k} > 0 \quad \text{for } 0 < x < \pi
\]

and the Young inequality (1913)

\[
1 + \sum_{k=1}^{n} \frac{\cos(k x)}{k} > 0 \quad \text{for } 0 < x < \pi.
\]
An Accidental Discovery

Theorem (Vietoris, 1958)

If \( a_k \) is a decreasing sequence of positive real numbers such that

\[
2k \cdot a_{2k} \leq (2k - 1) \cdot a_{2k-1},
\]

then, for \( 0 < x < \pi \),

\[
\sum_{k=1}^{n} a_k \sin kx > 0 \quad \text{and} \quad \sum_{k=0}^{n} a_k \cos kx > 0.
\]

This inspired a lot of subsequent research.

Leopold Vietoris (1891 – 2002)

- became famous in the 1920s for work in topology
- wrote his last paper at age 103 on trigonometric sums
- died at 110 years, 309 days. His wife died at age 100, they are the second oldest couple (combined: 211 years) in history.
An Accidental Discovery

\[ \sum_{k=0}^{n} a_k \cos kx \] corresponds to \[ \sum_{k=0}^{n} a_k z^k. \]

If the trigonometric sum has few or no roots, then what about the polynomial?
An Accidental Discovery

Roots of the Fejer polynomial: slightly *too* regular.
Theorem (S. 2019)

If

$$\sum_{k=0}^{n} a_k \cos kx$$

has $\lesssim n^\delta$ real roots, then the roots of $\sum_{k=0}^{n} a_k z^k$ are equidistributed at scale $\lesssim n^{\delta-1}$ in angle.

This improves Erdős-Turán for $\delta < 1/2$. It also raises many questions: is $\delta < 1/2$ necessary? Somehow one would expect that $\delta < 1$ might be enough because 'typical' trigonometric polynomials have $\sim n$ roots.
Dynamics of Roots of Derivatives

Problem (Polya, Riesz)

Let $p_n : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree $n$. What can you say about the roots of $p'_n$? Are they more regular?

Basic counting argument: between any two roots, there is a maximum or a minimum. That corresponds to a root of $p'_n$. Since there are $n$ roots, there are $n - 1$ gaps between the roots and therefore the roots of $p_n$ and $p'_n$ interlace.
Problem (Polya, Riesz)

Let $p_n : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree $n$. What can you say about the roots of $p'_n$? Are they more regular?

’Gaps become bigger Theorem’ (Riesz)

Let $p_n : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree $n$. Then the largest gap between roots of $p'_n(x)$ is at least as big as that of $p_n$.

There are is a vague folklore conjecture that if you keep differentiating, the roots even out (Polya, Farmer-Rhoades). Nothing is known.
The Big Question

Let \( p_n : \mathbb{R} \to \mathbb{R} \) be a polynomial of degree \( n \) all of whose roots are on the real line (and, say, distributed according to a nice function). What does it look like when I differentiate it \emph{a lot}?
A Motivating Proposition

Let $p_n : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree $n$ whose roots are randomly chosen on $[0, 1]$. Then the roots of $p_n^{(k)}$ are also uniformly distributed on $[0, 1]$ for all $k \ll n/(\log n)^2$ as $n \to \infty$.

Proof: the roots can only move at most the distance of the largest gap, which is size $\sim \log n/n$ or something like that.
The Big Question made precise

Let $p_n : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree $n$ all of whose roots are on the real line. Where are the roots of

$$p_n^{(t \cdot n)}$$

for $0 < t < 1$?
Example: take $p_{20}$ to be given by 20 roots from $U[0, 1]$. Where are the roots of $p_{20}^{(10)}$?
Example: take $p_{20}$ to be given by 20 roots from $U[0, 1]$. Where are the roots of $p_{20}^{(15)}$?
Example: take $p_{20}$ to be given by 20 roots from $U[0, 1]$. Where are the roots of $p^{(18)}_{20}$?
The Big Question made precise

Let \( p_n : \mathbb{R} \to \mathbb{R} \) be a polynomial of degree \( n \) all of whose roots are on the real line. Where are the roots of

\[
p_n^{(t \cdot n)} \quad \text{for } 0 < t < 1?
\]

If the roots of \( p_n^{(t \cdot n)} \) are distributed according to \( u(t, x) \), then what sort of equation does \( u(t, x) \) satisfy?
The BIG equation

roots of $p_{n}^{(t \cdot n)} \sim u(t, x)$

Educated Guess (S. 2018)

$u(t, x)$ satisfies a nonlinear, nonlocal transport equation

$$\frac{\partial u}{\partial t} + \frac{1}{\pi} \frac{\partial}{\partial x} \left( \arctan \left( \frac{H_u}{u} \right) \right) = 0$$

where

$$Hf(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy$$

is the Hilbert transform.
Testcase 1: the Hermite polynomials

1. the roots of the Hermite polynomial $H_n$ are approximately (in the sense of weak convergence after rescaling) given by the measure

$$
\mu = \frac{1}{\pi} \sqrt{2n - x^2} \, dx
$$

2. the derivatives of Hermite polynomials are again Hermite polynomials

$$
\frac{d^m}{dx^m} H_n(x) = \frac{2^n n!}{(n - m)!} H_{n-m}(x).
$$
Testcase 1: the Hermite polynomials

for Hermite polynomials, we expect a family of shrinking semicircles

\[ u(t, x) = \frac{2}{\pi} \sqrt{1 - t - x^2} \]

solves the equation
Testcase 2: the Laguerre polynomials

The family of associated Laguerre polynomials $L_n^{(\alpha)}$ satisfies

$$\frac{d^k}{dx^k} L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x).$$

The roots are given by the Marchenko-Pastur distribution

$$v(c, x) = \frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi x} \chi(x_-, x_+) dx$$

where

$$x_\pm = (\sqrt{c + 1} \pm 1)^2.$$
Testcase 2: Orthogonal Polynomials

for Laguerre, we expect a one-parameter family within the Marchenko-Pastur family

\[ u_c(t, x) = v \left( \frac{c + t}{1 - t}, \frac{x}{1 - t} \right) \] solves the equation!
Testcase 3: Integration!

\[ p_n^{(t \cdot n)} \] has \((1 - t)n\) roots. There is a constant loss of roots.

\[
\frac{\partial}{\partial t} \int_{\mathbb{R}} u(t, x) \, dx = \int_{\mathbb{R}} \frac{\partial u}{\partial t}(t, x) \, dx \\
= -\int_{\mathbb{R}} \frac{1}{\pi} \frac{\partial}{\partial x} \left( \arctan \left( \frac{Hu}{u} \right) \right) \, dx \\
= -1.
\]

The equation has a constant loss of mass. If we start with a probability distribution, then there is a constant loss of mass and finite time blow-up at time \(t = 1\) (when we have differentiated the polynomial \(n\) times).
Science Fiction!

Magic!
There seems to be a nonlinear, nonlocal partial differential equation that describes the roots of polynomials under differentiation.

More Magic!
The equation looks a bit like those one-dimensional model equations for Euler/Navier stokes (cf. Rafael Granero Belinchon). Do roots of polynomials flow like water?
Thank you!