How far away is the positive part of a function from its negative part? Interpolation Inequalities for the Wasserstein Distance and Applications

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A Basic Question

Suppose we have the periodic function $f : \mathbb{T} \to \mathbb{R}$ and know that

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(kx) + b_k \sin(kx).$$

What can you say about the number of roots of f?

There are many open problems that are connected to such questions but it's also interesting in itself. Many high frequencies should induce many roots.

A Problem of Littlewood

Let $A \subset \mathbb{N}$ and consider

$$f(x) = \sum_{k \in A} \cos(kx).$$

Conjecture (Littlewood)

The function has at least $|A| - 1$ roots (or not much less).

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Theorem (Sahasrabudhe)

Such a function has at least $(\log \log \log |A|)^{1/2-}$ roots.

A Simpler Question

Suppose

$$f(x) = \sum_{k=n}^{\infty} a_k \cos(kx) + b_k \sin(kx).$$

Then surely, such a function has to have at least $\sim n$ roots. This is true and known as the Sturm-Hurwitz Theorem.

Sturm-Liouville Theory: Some History

Sturm-Liouville theory dates from 1836.

> *In 1833 both* **Sturm** *and* **Liouville** *and their common friend* **Duhamel** *applied for the seat vacated by the death of* **Legendre**. *A fourth applicant was* **Libri-Carucci** *[...] (Lützen, 1984)*

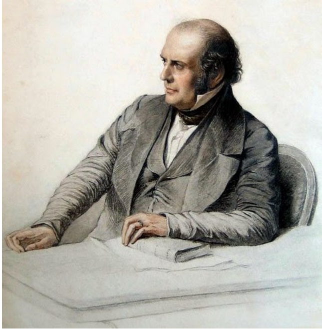
French Academy, 1833

The outcome of the elections is

1. Sturm: 0
2. Liouville: 1
3. Duhamel: 16
4. Libri-Carucci: 37

Who is this mysterious Libri-Carucci?

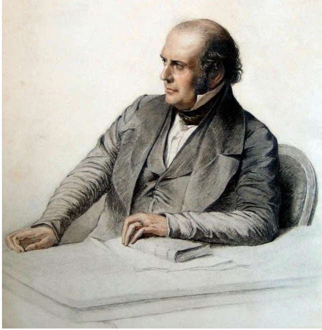
Guglielmo Libri Carucci dalla Sommaja (1803–1869)



Libri-Carucci

In 1841, Libri obtained an appointment as Chief Inspector of French Libraries through his friendship with influential French Chief of Police Francois Guizot. This job, involving in part the cataloguing of valuable books and precious manuscripts allowed Count Libri to indulge his collecting passion **by stealing them.**

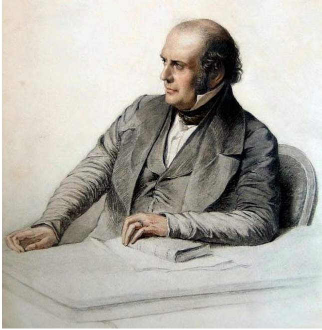
Guglielmo Libri Carucci dalla Sommaja (1803–1869)



Libri-Carucci

In 1848, as France was involved in a liberal revolution and the government fell, a warrant was issued for Libri's arrest. [...] However he received a tip-off and fled to London, shipping 18 large trunks of books and manuscripts, about 30,000 items, before doing so.

Guglielmo Libri Carucci dalla Sommaja (1803–1869)



Libri-Carucci

In June 2010, one of the stolen items, a letter from Descartes to Father Marin Mersenne, dated May 27, 1641 concerning the publication of Meditations on First Philosophy, was discovered in the library of Haverford College. The college returned the letter to the Institut de France on June 8, 2010.

Thee year laters: French Academy, 1836

After the death of Ampere, there are again elections. Sturm, Liouville and Duhamel compete once more for the seat (together with several others).

> *Three weeks before the election of* **Ampere***'s successor,* **Liouville** *presented a paper to the Academy [1837a] in which he praised* **Sturm***'s two memoires on the Sturm-Liouville theory as ranking with the best works of* **Lagrange***. Supporting a rival in this way was rather unusual in the competitive Parisian academic circles [...] (Lützen, 1984)*

Thee year laters: French Academy, 1836

> *[...] and it must have been shocking when on the day of the election, December 5-th,* **Liouville** *and* **Duhamel** *withdrew their candidacies to secure the seat for their friend.* **Sturm** *was elected with an overwhelming majority. (Lützen, 1984)*

Now Sturm is one of the 72 names in the Eiffel tower.

Sturm-Liouville Theory

Sturm-Liouville Theory covers general second order linear operators. For simplicity, we will always discuss the Laplacian

$$-\frac{d^2}{dx^2}u_k(x) = \lambda_k u_k \qquad \text{on } [a, b]$$

with Dirichlet boundary conditions. This corresponds

Theorem (Sturm Oscillation Theorem)

u_k has $k - 1$ roots.

Theorem (Sturm Oscillation Theorem, 1836)

u_k has $k-1$ roots.

In our special case of the Laplacian, this is not all that surprising: just count the roots of the sign.

In fact, stronger results are possible.

Theorem (Sturm-Hurwitz Theorem, 1903)

The function

$$f(x) = \sum_{k=n}^{\infty} a_k \sin(kx)$$

has at least $2n$ distinct roots in $[0, 2\pi)$.

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Proof.

$f(x)$ is the imaginary part of the holomorphic function

$$g(z) = \sum_{k=n}^{\infty} a_k z^k$$

when evaluated on the boundary of the unit circle $g(e^{it})$. It has at least n roots in the origin and thus at least n roots inside the unit disk. By the argument principle, $g(e^{it})$ winds around the origin at least n times creating at least $2n$ roots. □

Sturm-Hurwitz

Theorem (Sturm-Hurwitz Theorem, 1903)

The function

$$f(x) = \sum_{k=n}^{\infty} a_k \sin(kx)$$

has at least $2n$ distinct roots in $\mathbb{T}$.

So, unsurprisingly, we can strenghten the Sturm Oscillation Theorem in the case of the Laplacian: surely that is because the trigonometric system is special.....

The REAL Sturm Oscillation Theorem

Theorem (Sturm Oscillation Theorem, 1836)

u_k has $k-1$ roots.

Theorem (**Sturm Oscillation Theorem**, 1836)

Unless all coefficients vanish, the function

$$\sum_{k=m}^{n} a_k u_k$$

has at least $m-1$ roots and at most $n-1$ roots.

This is a remarkable statement, even for sines and cosines alone. A sum of oscillating functions is still oscillating.

Berard & Helffer, 2017

Although well known in the nineteenth century, this theorem seems to have been ignored or forgotten by some of the specialists in spectral theory since the second half of the twentieth-century.

Timeline from Berard & Helffer, 2017

- **1833.** *Sturms Memoir presented to the Paris Academy of sciences in September.*
- **1836.** *Sturms papers published.*
- **1877.** *Lord Rayleigh writes a beautiful theorem has been discovered by Sturm*
- **1891.** *F. Pockels [30, pp. 68-73] gives a summary of Sturms results [...] provided by Sturm, Liouville and Rayleigh.*
- **1903.** *Hurwitz gives a lower bound for the number of zeros of the sum of a trigonometric series with a spectral gap and refers, somewhat inaccurately, to Sturms Theorems. This result, known as the Sturm-Hurwitz theorem, already appears in a more general framework in Liouvilles paper.*

Timeline from Berard & Helffer, 2017

- **1931.** *Courant & Hilbert extensively mention the Sturm-Liouville problem. They do not refer to the original papers of Sturm, but to Bochers book [8] which does not include Theorem 1.4 [the full result].*
- **1956.** *Pleijel mentions Sturms Theorem 1.4, somewhat inaccurately [...]*
- **...**

Back to our little problem

Suppose

$$f(x) = \sum_{k=n}^{\infty} a_k \cos(kx) + b_k \sin(kx).$$

Then surely, such a function has to have at least $\sim n$ roots. This is true and known as the Sturm-Hurwitz Theorem.

But what if we add some **small** low-frequency functions? Are there stable results?

Theorem

Let $f : [0, 2\pi] \to \mathbb{R}$ be a function with mean value 0. Then

$$\# \{x : f(x) = 0\} \cdot \sum_{k \neq 0} \frac{|\widehat{f}(k)|}{|k|} \gtrsim \|f\|_{L^1}.$$

Proof.

Let us integrate the function. Between any two roots, we have

$$\int_{r_k}^{r_{k+1}} f(x) dx \leq \sum_{k=1}^{\infty} \frac{\widehat{f}(k)}{k}.$$

Summing over all the roots gives us at least the L^1-norm. □

Theorem (S. 2018)

Let $f : [0, 2\pi] \rightarrow \mathbb{R}$ be a function with mean value 0. Then

$$\#\left\{x : f(x) = 0\right\} \cdot \left(\sum_{k=1}^{\infty} \frac{|\widehat{f}(k)|^2}{|k|^2} \right)^{\frac{1}{2}} \gtrsim \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}}.$$

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Reformulation

Let $f : \mathbb{T} \to \mathbb{R}$ be $C^1(\mathbb{T})$ with mean value 0. Then

$$(\text{number of critical points of } f) \cdot \|f\|_{L^2(\mathbb{T})} \gtrsim \frac{\|f'\|_{L^1(\mathbb{T})}^2}{\|f'\|_{L^\infty(\mathbb{T})}}.$$

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Conjecture. If a function is wiggly, it has many critical points.

Theorem (S. 2018)

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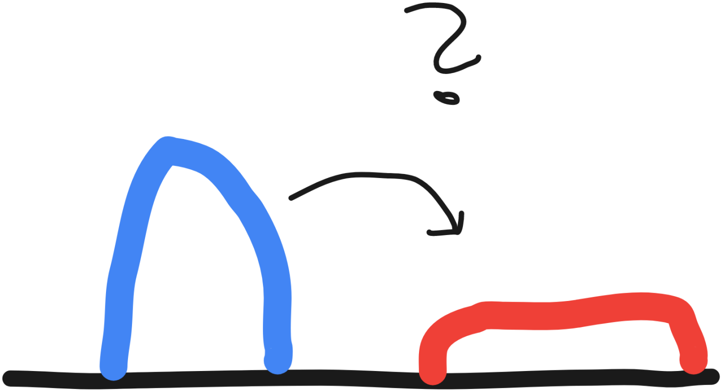
- A function that has many large coefficients has the second term (which is merely the Sobolev norm $\|\cdot\|_{\dot{H}^{-1}}$) small: thus the first term has to be large.
- One of very few *lower* bounds on Fourier coefficients.
- I would be interested in any related estimates of this flavor.
- The proof is **Optimal Transport**.

Optimal Transport

A very, very basic introduction.

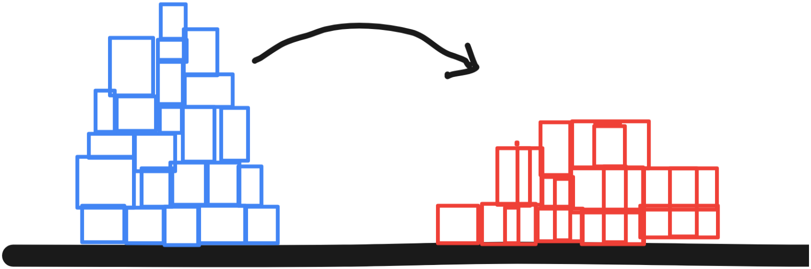
Optimal Transport

Suppose we are given two measure μ and ν having same total mass and want to transport one to the other.



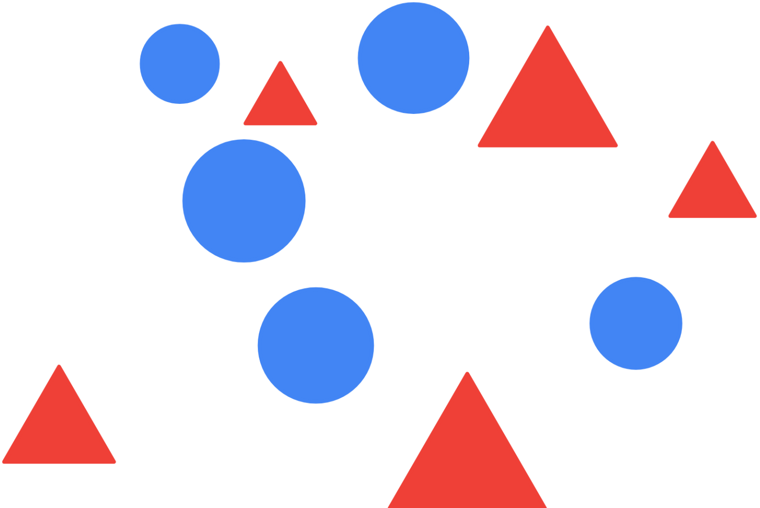
Optimal Transport

Think of both measures as being a collection of little boxes. Suppose it costs $\delta \cdot \varepsilon$ to move a box of weight ε distance δ. What is the cheapest way to move the boxes to the desired goal?



Optimal Transport

As it turns out, this problem is not really an issue in one dimension but it is already quite tricky in $d = 2$. We call the answer the 1-Wasserstein distance of μ, ν, denoted by $W_1(\mu, \nu)$.

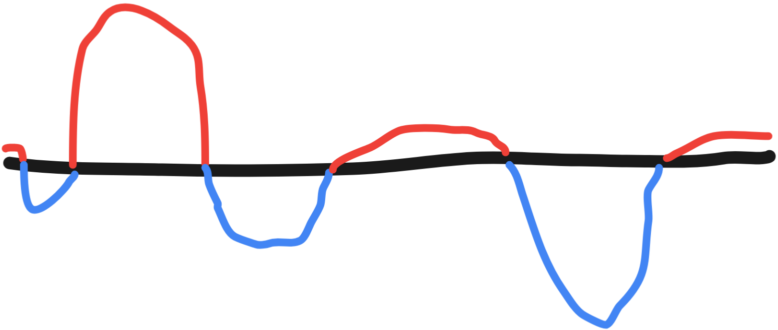


Optimal Transport

Let $f : \mathbb{T} \to \mathbb{R}$ be a function of mean value 0 and set

$$\mu = f^+ dx \qquad \text{and} \qquad \nu = f^- dx.$$

How much does it cost to transport one to the other?



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Theorem (Peyre)

Let $f : [0, 2\pi] \to \mathbb{R}$ be a function with mean value 0. Then

$$W_1(f^+, f^-) \lesssim \|f\|_{\dot{H}^{-1}}.$$

Theorem (S. 2018)

Let $f : [0, 2\pi] \to \mathbb{R}$ be a function with mean value 0. Then

$$W_1(f^+, f^-) \cdot \# \{x : f(x) = 0\} \gtrsim \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}}.$$

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Combining both the Peyre upper bound this lower bounds, we get

$$\# \{x : f(x) = 0\} \cdot \left(\sum_{k=1}^{\infty} \frac{|\widehat{f}(k)|^2}{|k|^2} \right)^{\frac{1}{2}} \gtrsim \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}}.$$

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What about higher-dimensional generalizations?

Main Result

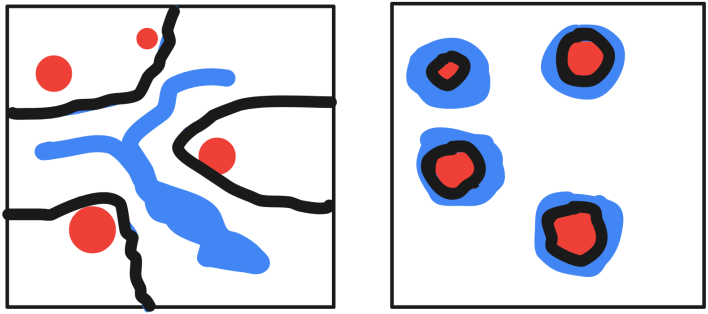
Wasserstein Uncertainty Principle. *If there are very few post offices, some of your letters will have to travel a very long time. If letters arrive quickly, there must be many post offices.*

Theorem (S. 2019)

Let $f : [0,1]^2 \to \mathbb{R}$ be a function with mean value 0. Then

$$W^1(f^+ dx, f^- dx) \cdot \mathcal{H}^1 \left(x : f(x) = 0 \right) \gtrsim \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}}.$$

Main Result



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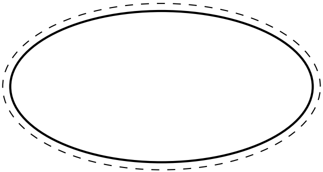
Sketch of the Argument



$$|\varepsilon - \text{neighborhood}(\Omega)| \leq \varepsilon|\partial\Omega|$$

when $\varepsilon \ll \text{diam}(\Omega)$

Further developments

Theorem for $[0,1]^d$ (Amir Sagiv and S. 2019)

$$W^1(f^+dx, f^-dx) \cdot \mathcal{H}^{d-1}(x : f(x) = 0) \gtrsim \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \right)^{4-\frac{1}{d}} \|f\|_{L^1}.$$

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Theorem for $[0,1]^d$ (Carroll, Massaneda, Ortega-Cerda, 2020)

$$W^1(f^+dx, f^-dx) \cdot \mathcal{H}^{d-1}(x : f(x) = 0) \gtrsim \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \right)^{2-\frac{1}{d}} \|f\|_{L^1}.$$

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Conjecture

$$W^1(f^+dx, f^-dx) \cdot \mathcal{H}^{d-1}(x : f(x) = 0) \gtrsim \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \right) \|f\|_{L^1}.$$

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1. Divide into many small boxes. $\|f\|_{L^1}/\|f\|_{L^\infty}$ tells you how many boxes have to have nontrivial supply/demand.
2. If such a box is mainly supply or mainly demand, then there has to be transport.
3. If such a box has evenly matched supply demand, then

Relative Isoperimetric Inequality

Let $\Omega \subset [0,1]^d$ with $|\Omega| \leq 1/2$. Then

$$\partial\Omega \cap (0,1)^d \gtrsim_d |\Omega|^{\frac{d-1}{d}}.$$

Thank you!