Mean-Value Inequalities for Harmonic Functions

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Goal of the Talk

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The Mean Value Theorem

Let $B_r(0) \subset \mathbb{R}^d$ and let $\Delta f = 0$ for some nice $f : B_r(0) \to \mathbb{R}$. Then

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Proof.

Mean-Value Theorem and Maximum Principle. □

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General Mean-Value Inequalities?

What if the domain is not a ball?

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Let's start with something 'simpler': convex functions.

My interest in this arose when seeing a fun paper on the arXiv.

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JENSEN-TYPE GEOMETRIC SHAPES

PAWEŁ PASTECZKA

ABSTRACT. We present both necessary and sufficient conditions to the convex closed shape X such that the inequality

$$\frac{1}{|X|}\int_X f(x)\,dx \le \frac{1}{|\partial X|}\int_{\partial X} f(x)\,dx$$

is valid for every convex function $f\colon X \to \mathbb{R}$ (∂X stands for the boundary of X).

It is proved that this inequality holds if X is (i) an n-dimensional parallelotope, (ii) an n-dimensional ball, (iii) a convex polytope having an inscribed sphere (tangent to all its facets) with center in the center of mass of ∂X.

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or, for $\Omega = [0,1]$,

$$\frac{1}{|\Omega|}\int_\Omega f(x)dx \leq \frac{1}{|\partial\Omega|}\int_{\partial\Omega} f(x)dx.$$

Hermite-Hadamard Inequalities

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- when Ω is a ball in $\mathbb{R}^d$ (Dragomir, Pearce)

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- all Platonic solids (Pasteczka)

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Proof.

Plug in $f(x) = \langle a, x \rangle + b$. Both f and $-f$ are convex, therefore

$$\frac{1}{|\Omega|} \int_{\Omega} f(x)dx = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} f(x)dx$$

for all functions of this type. □

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Conjecture (Pasteczka)

'Our conjecture is that every convex shape which satisfies this condition is of Jensen-type'.

I think this would be really nice if it were true (maybe too nice?)

Hermite-Hadamard Inequalities

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If, for all convex $f : \Omega \to \mathbb{R}$

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then Ω and $\partial\Omega$ have the same center of mass.

In particular, if Ω and $\partial\Omega$ have different centers of mass, then the optimal constant c_Ω

$$\frac{1}{|\Omega|} \int_{\Omega} f(x) dx \leq \frac{c_\Omega}{|\partial\Omega|} \int_{\partial\Omega} f(x) dx$$

satisfies $c_\Omega > 1$.

Hermite-Hadamard Inequalities

$f : \Omega \to \mathbb{R}^d$ convex and

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Theorem (S. 2018)

$c_{\Omega} \leq c_n$ for all convex domains $\Omega \subset \mathbb{R}^n$.

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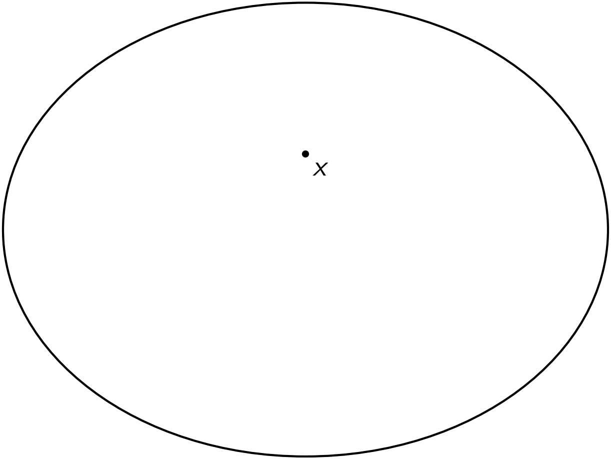
I will later show much better results. But what is interesting here is that there is a fun transport problem hiding here. I always thought that this was independently interesting.

A Transport Problem



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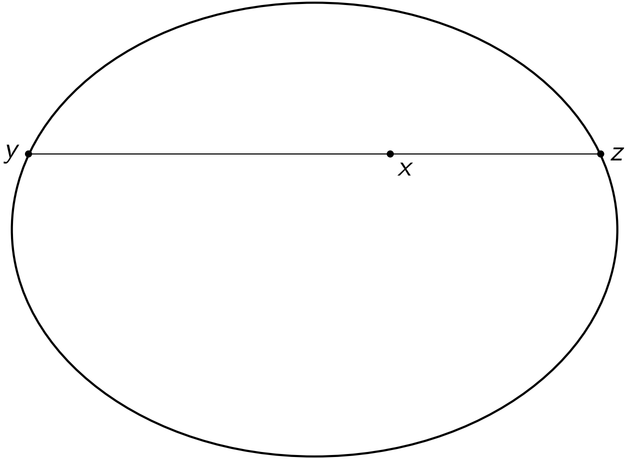

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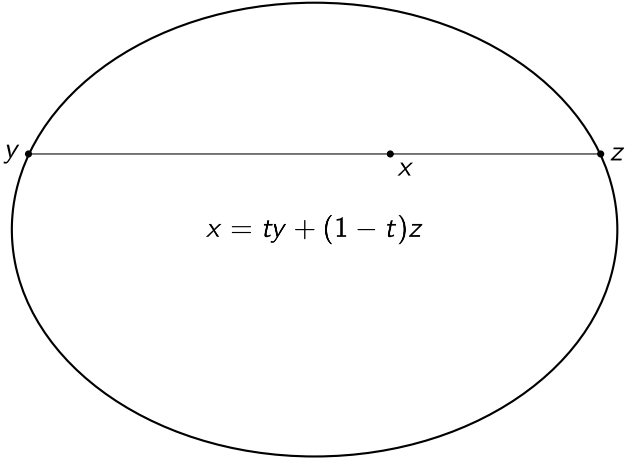

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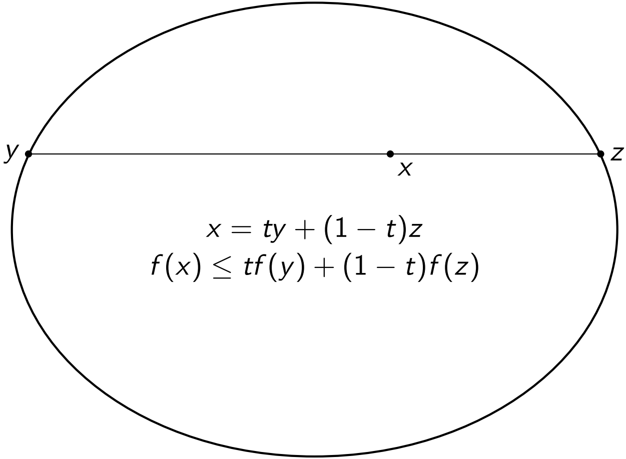

$$x = ty + (1 - t)z$$
$$f(x) \leq tf(y) + (1 - t)f(z)$$

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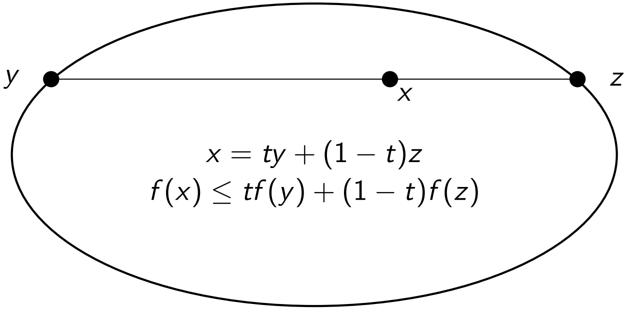

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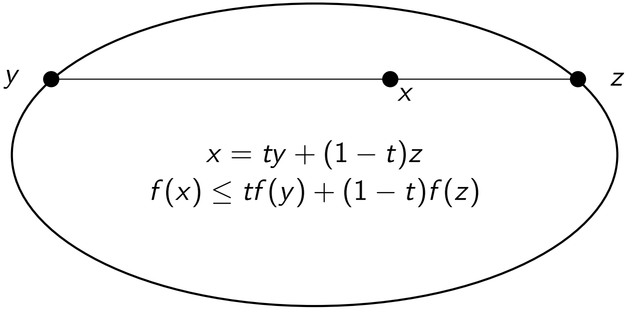

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$$\text{1 unit of Lebesgue at } x \to \begin{cases} t & \text{at } y \\ 1-t & \text{at } z. \end{cases}$$

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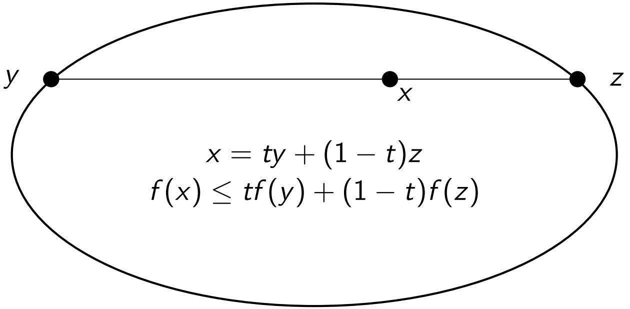

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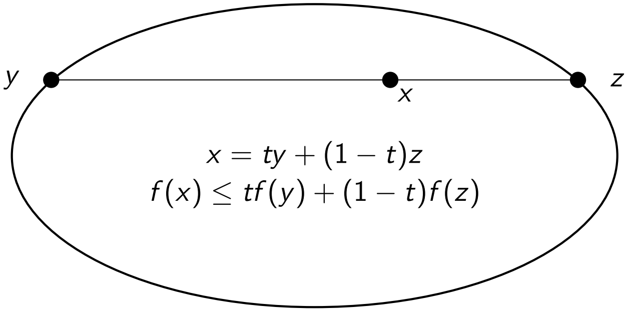
Mechanism to send $\mathcal{H}^d(\Omega)$ to $\mathcal{H}^{d-1}(\partial\Omega)$.

A Transport Problem



Main Question

How do you send the $\mathcal{H}^d$ to the boundary so that the final measure on the boundary is as flat as possible?

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$$\nu(\partial\Omega) = |\Omega|.$$

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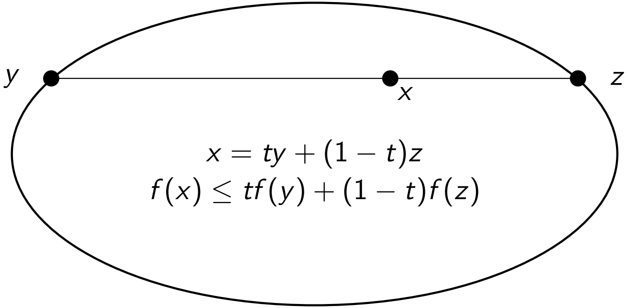
Moreover, for all convex $f : \Omega \to \mathbb{R}$, we have

$$\int_\Omega f(x)dx \leq \int_{\partial\Omega} f \ d\nu$$

and in particular, in terms of the Radon-Nikodym derivative,

$$\int_\Omega f \ dx \leq \left\| \frac{d\nu}{d\sigma} \right\|_{L^\infty} \cdot \int_{\partial\Omega} f \ d\sigma$$

A Transport Problem



If Ω and $\partial\Omega$ have a different center of mass, then the final measure satisfies

$$\left\| \frac{d\mu}{d\sigma} \right\| > \frac{|\Omega|}{|\partial\Omega|},$$

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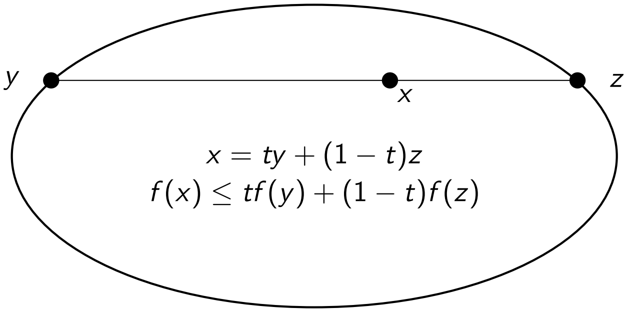

If Ω and $\partial\Omega$ have a different center of mass, then the final measure satisfies

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so it cannot be too evenly distributed. How evenly distributed can it be?

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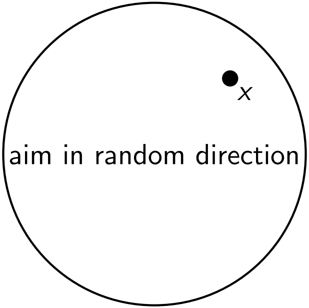

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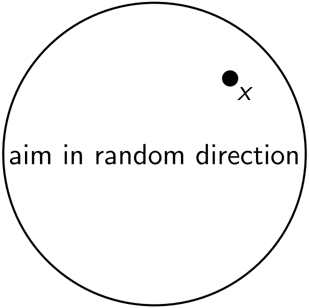

If we distribute the mass randomly in all directions, then the final distribution on the boundary has to be uniform. Thus the constant is 1 for the ball is admissible (and clearly optimal).

Back to subharmonic

Let $\Omega \subset \mathbb{R}^d$, let $f : \Omega \to \mathbb{R}$ satisfy $\Delta f \geq 0$ and suppose $f\big|_{\partial\Omega} \geq 0$.

$$\int_\Omega f(x)dx \leq c_\Omega \int_{\partial\Omega} f(x)dx$$

and how does the constant c_Ω depend on Ω?

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and how does the constant c_Ω depend on Ω?

We start by recalling some arguments from Niculescu-Persson. To this end, we introduce the function $\phi : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta\phi &= 1 \qquad \text{in } \Omega \\ \phi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Integration by Parts

$$\begin{aligned}\int_\Omega f(x)dx &= \int_\Omega f(x)(-\Delta\phi(x))dx \\ &= \int_\Omega (-\Delta f(x))\phi(x)dx + \int_{\partial\Omega} f(x)\frac{\partial\phi}{\partial n}d\sigma \\ &\leq \int_{\partial\Omega} f(x)\frac{\partial\phi}{\partial n}d\sigma\end{aligned}$$

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Find the harmonic function corresponding to boundary data given by a characteristic function in the neighborhood where that gradient is large.

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We have

$$c_\Omega \leq \left\| \frac{\partial \phi}{\partial n} \right\|_{L^\infty}$$

where $\phi : \Omega \to \mathbb{R}$ is such that

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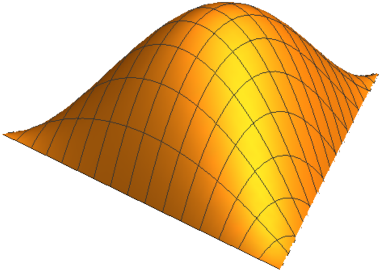

(solution on an equilateral triangle)

This turns out to be a classical problem and there are lots of estimates that are known.

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Theorem (Beck, Brandolini, Burdzy, Henrot, Langford, Larson, Smits, S, 2019)

Let $f : \Omega \to \mathbb{R}$ be positive, $\Delta f \geq 0$ and let Ω be convex. Then

$$\frac{1}{|\Omega|} \int_{\Omega} f dx \leq \frac{c_n}{|\partial\Omega|} \int_{\partial\Omega} f d\sigma,$$

where $n \lesssim c_n \lesssim n^{3/2}$.

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Theorem (Simon Larson, 2020)

Let $f : \Omega \to \mathbb{R}$ be positive, $\Delta f \geq 0$ and let Ω be convex. Then

$$\frac{1}{|\Omega|} \int_\Omega f dx < \frac{n}{|\partial\Omega|} \int_{\partial\Omega} f d\sigma$$

and n is the sharp constant. (No extremizers!)

$$\frac{1}{|\Omega|}\int_{\Omega} f dx \leq \frac{n}{|\partial\Omega|}\int_{\partial\Omega} f d\sigma$$

is the sharp result in terms of $|\Omega|$ and $|\partial\Omega|$. However, one could also invoke other (or fewer/other) geometric quantities.

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Let $f : \Omega \to \mathbb{R}$ be positive, $\Delta f \geq 0$ and let Ω be convex. Then

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What's nice about this is the nice universal constant 1 in front of everything.

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What's nice about this is the nice universal constant 1 in front of everything. **Open Problem:** the sharp constant, how it scales with n, whether there is an extremal domain and how the extremal domain looks like, that is less clear.

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The optimal constant has to satisfy $1/(2\sqrt{\pi e}) \leq c_n \leq 1$. (Lower bound given by ellipsoids, example by Thomas Beck.)

$$\frac{1}{|\Omega|}\int_{\Omega} f dx \leq \frac{n}{|\partial\Omega|}\int_{\partial\Omega} f d\sigma$$

is the sharp result in terms of $|\Omega|$ and $|\partial\Omega|$. However, one could also invoke other (or fewer/other) geometric quantities.

Theorem (Jianfeng Lu and S, 2019)

Let $f : \Omega \to \mathbb{R}$ be positive, $\Delta f \geq 0$ and let Ω be convex. Then

$$\int_{\Omega} f dx \leq |\Omega|^{1/n} \int_{\partial\Omega} f d\sigma.$$

The optimal constant has to satisfy $1/(2\sqrt{\pi e}) \leq c_n \leq 1$. (Lower bound given by ellipsoids, example by Thomas Beck.) It's not entirely clear how extremal domain has to look.

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As it turns out, this question is 165 years old!

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> *Les points dangereux sont donc, comme dans l'ellipse et le rectangle, les points du contour les plus rapproches de l'axe de torsion, ou les extremites des petits diametre. (Saint Venant, 1856)*

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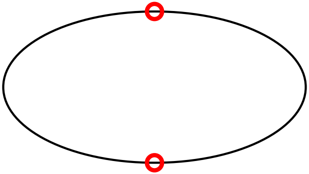

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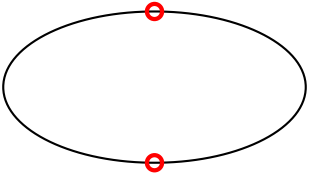

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> *M. de St. Venant also calls attention to a conclusion from his solutions which to many may be startling, that in the simpler cases the places of greatest distortion are those points of the boundary which are nearest to the axis [...] and the places of least distortion those farthest from it. (Thomson & Tait, Treatise on Natural Philosophy, 1867)*

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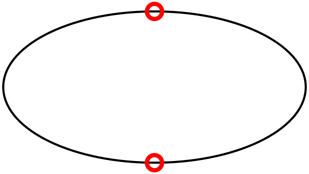

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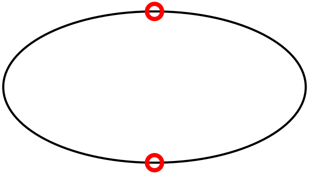

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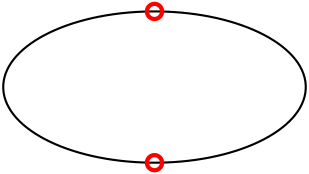

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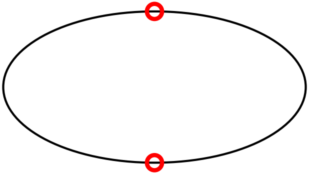

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- 1930: Polya proves the maximum is on the boundary.

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What we therefore looking for

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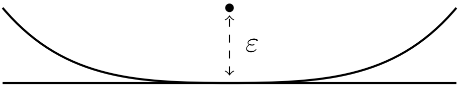

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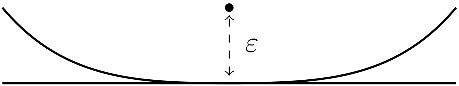

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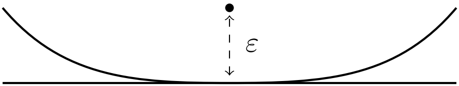

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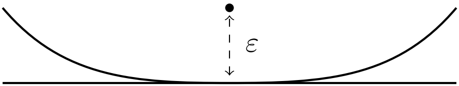

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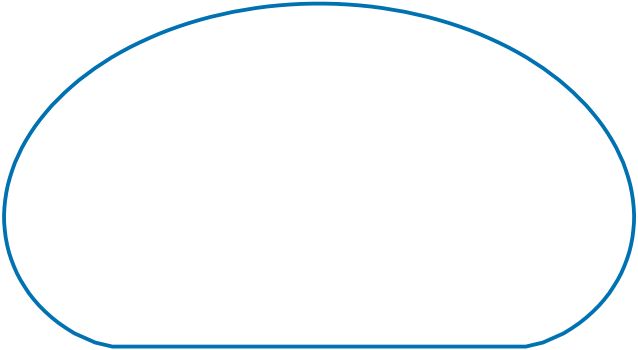
Here's the result of some high precision numerics.



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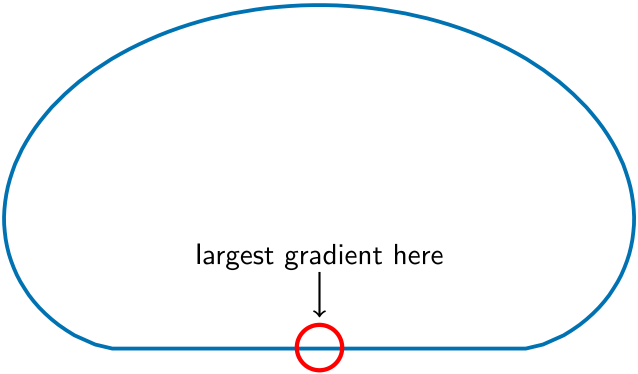

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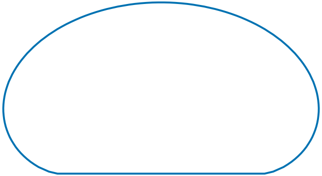

Independent verification by Guido Sweers.

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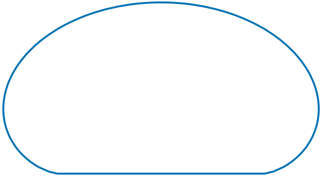

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coming with an explicit construction. Let Ω be a simply connected domain and let $h : E \to \Omega$ be a biconformal map. Then the solutions of

$$\begin{cases} -\Delta w = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta u = \left|h'(\cdot)\right|^2 (f \circ h) & \text{in } E \\ u = 0 & \text{on } \partial E \end{cases}$$

are related via

$$(w \circ h)(x, y) = u(x, y).$$

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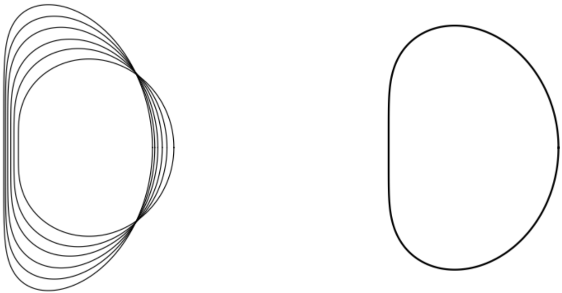
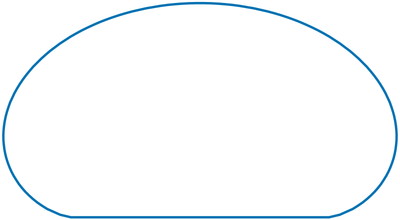


FIGURE 13. Left: $h_q(E_q)$ for $1 \leq q \leq 2$. Right: $h_q(E_q)$ for $q = 1.386$.

Shape Optimization?



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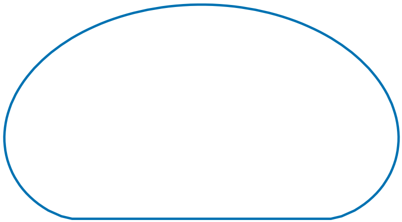

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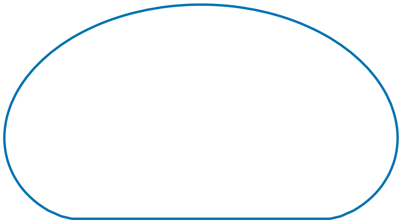

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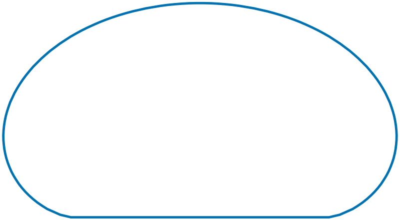

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And what about higher dimensions?

Thank you!