Curvature on Combinatorial Graphs

Stefan Steinerberger

ICERM, June 2023

UNIVERSITY of WASHINGTON

Curvature on Graphs

Curvature is a local infinitesimal property in differential geometry. Clearly, there is no hope of defining something similar on graphs.

Curvature on Graphs

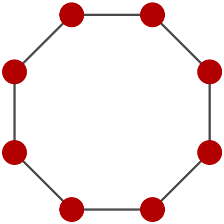
Curvature is a local infinitesimal property in differential geometry. Clearly, there is no hope of defining something similar on graphs.

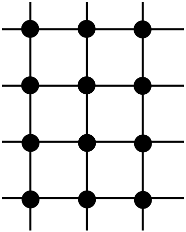
However,....

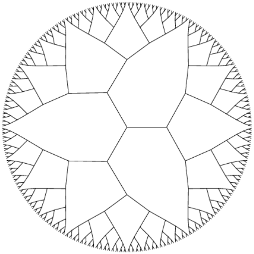
Curvature on Graphs

Curvature is a local infinitesimal property in differential geometry. Clearly, there is no hope of defining something similar on graphs.

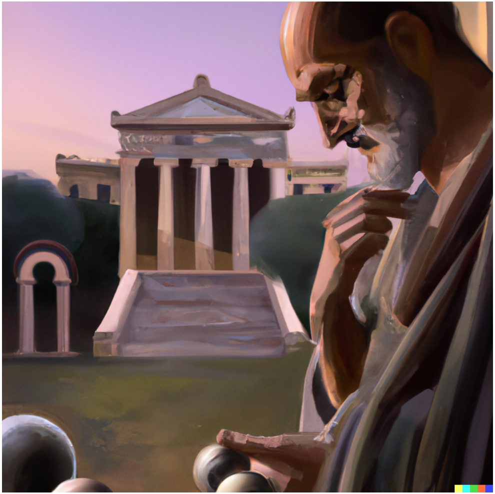
However,....



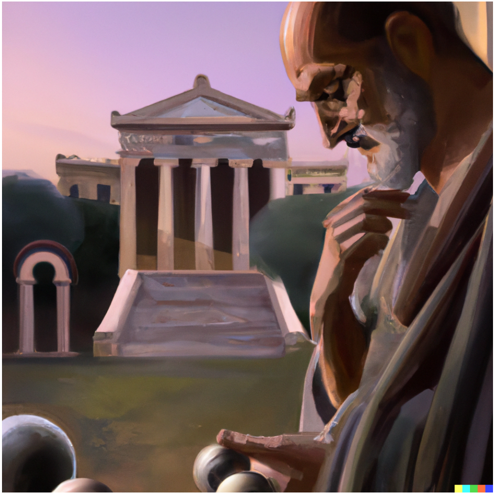




Euclid and curvature (DALL-E)

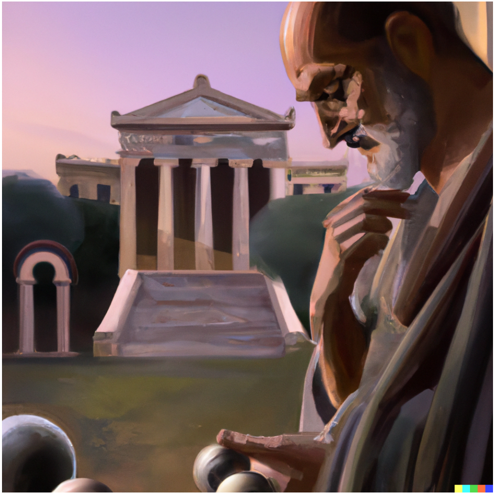


Euclid and curvature (DALL-E)



Which properties are essential and which properties can we live without?

Euclid and curvature (DALL-E)



Which properties are essential and which properties can we live without? *A matter of taste.*

Things that you MAYBE want to be true

Things that you MAYBE want to be true

1. Graphs with curvature bounded below by $K > 0$ satisfy $\operatorname{diam}(G) \leq f(K)$ for some f. (**Bonnet-Myers Theorem**)

Things that you MAYBE want to be true

1. Graphs with curvature bounded below by $K > 0$ satisfy $\text{diam}(G) \leq f(K)$ for some f. (**Bonnet-Myers Theorem**)
2. Equality cases in Bonnet-Myers ($\text{diam}(G) = f(K)$) imply constant curvature (**Cheng Diameter Rigidity Theorem**)

Things that you MAYBE want to be true

1. Graphs with curvature bounded below by $K > 0$ satisfy $\mathrm{diam}(G) \leq f(K)$ for some f. (**Bonnet-Myers Theorem**)
2. Equality cases in Bonnet-Myers ($\mathrm{diam}(G) = f(K)$) imply constant curvature (**Cheng Diameter Rigidity Theorem**)
3. Graphs with curvature bounded from below by $K > 0$ have a spectral gap $\lambda_2 \geq g(K) > 0$ (**Lichnerowicz Theorem**)

Things that you MAYBE want to be true

1. Graphs with curvature bounded below by $K > 0$ satisfy $\mathrm{diam}(G) \leq f(K)$ for some f. (**Bonnet-Myers Theorem**)
2. Equality cases in Bonnet-Myers ($\mathrm{diam}(G) = f(K)$) imply constant curvature (**Cheng Diameter Rigidity Theorem**)
3. Graphs with curvature bounded from below by $K > 0$ have a spectral gap $\lambda_2 \geq g(K) > 0$ (**Lichnerowicz Theorem**)

A matter of taste again.

Things that you MAYBE want to be true

1. Graphs with curvature bounded below by $K > 0$ satisfy $\text{diam}(G) \leq f(K)$ for some f. (**Bonnet-Myers Theorem**)
2. Equality cases in Bonnet-Myers ($\text{diam}(G) = f(K)$) imply constant curvature (**Cheng Diameter Rigidity Theorem**)
3. Graphs with curvature bounded from below by $K > 0$ have a spectral gap $\lambda_2 \geq g(K) > 0$ (**Lichnerowicz Theorem**)

A matter of taste again. I **personally** also want

Things that you MAYBE want to be true

1. Graphs with curvature bounded below by $K > 0$ satisfy $\text{diam}(G) \leq f(K)$ for some f. (**Bonnet-Myers Theorem**)
2. Equality cases in Bonnet-Myers ($\text{diam}(G) = f(K)$) imply constant curvature (**Cheng Diameter Rigidity Theorem**)
3. Graphs with curvature bounded from below by $K > 0$ have a spectral gap $\lambda_2 \geq g(K) > 0$ (**Lichnerowicz Theorem**)

A matter of taste again. I **personally** also want

1. Many nice examples of graphs with positive curvature.

Things that you MAYBE want to be true

1. Graphs with curvature bounded below by $K > 0$ satisfy $\text{diam}(G) \leq f(K)$ for some f. (**Bonnet-Myers Theorem**)
2. Equality cases in Bonnet-Myers ($\text{diam}(G) = f(K)$) imply constant curvature (**Cheng Diameter Rigidity Theorem**)
3. Graphs with curvature bounded from below by $K > 0$ have a spectral gap $\lambda_2 \geq g(K) > 0$ (**Lichnerowicz Theorem**)

A matter of taste again. I **personally** also want

1. Many nice examples of graphs with positive curvature.
2. Everything happens on the graph.

Things that you MAYBE want to be true

1. Graphs with curvature bounded below by $K > 0$ satisfy $\mathrm{diam}(G) \leq f(K)$ for some f. (**Bonnet-Myers Theorem**)
2. Equality cases in Bonnet-Myers ($\mathrm{diam}(G) = f(K)$) imply constant curvature (**Cheng Diameter Rigidity Theorem**)
3. Graphs with curvature bounded from below by $K > 0$ have a spectral gap $\lambda_2 \geq g(K) > 0$ (**Lichnerowicz Theorem**)

A matter of taste again. I **personally** also want

1. Many nice examples of graphs with positive curvature.
2. Everything happens on the graph.
3. Curvature lives in vertices, not edges (can compromise).

Things that you MAYBE want to be true

1. Graphs with curvature bounded below by $K > 0$ satisfy $\mathrm{diam}(G) \leq f(K)$ for some f. (**Bonnet-Myers Theorem**)
2. Equality cases in Bonnet-Myers ($\mathrm{diam}(G) = f(K)$) imply constant curvature (**Cheng Diameter Rigidity Theorem**)
3. Graphs with curvature bounded from below by $K > 0$ have a spectral gap $\lambda_2 \geq g(K) > 0$ (**Lichnerowicz Theorem**)

A matter of taste again. I **personally** also want

1. Many nice examples of graphs with positive curvature.
2. Everything happens on the graph.
3. Curvature lives in vertices, not edges (can compromise).
4. Cycle graph has positive curvature (can compromise).

There are many definitions of curvature on graphs.

There are many definitions of curvature on graphs.

1. Purely combinatorial definitions (Higuchi, Stone, Woess)

There are many definitions of curvature on graphs.

1. Purely combinatorial definitions (Higuchi, Stone, Woess)
2. Definitions based on the Laplacian (Bakry-Émery, Forman)

There are many definitions of curvature on graphs.

1. Purely combinatorial definitions (Higuchi, Stone, Woess)
2. Definitions based on the Laplacian (Bakry-Émery, Forman)
3. Definitions based on Optimal Transport, specifically the Ollivier curvature (2009) and the Lin-Lu-Yau curvature (2011)

There are many definitions of curvature on graphs.

1. Purely combinatorial definitions (Higuchi, Stone, Woess)
2. Definitions based on the Laplacian (Bakry-Émery, Forman)
3. Definitions based on Optimal Transport, specifically the Ollivier curvature (2009) and the Lin-Lu-Yau curvature (2011)

Wonderful reference: Norbert Peyerimhoff, Lecture Notes, Curvature Notions on Graphs, Summer School Leeds 2019

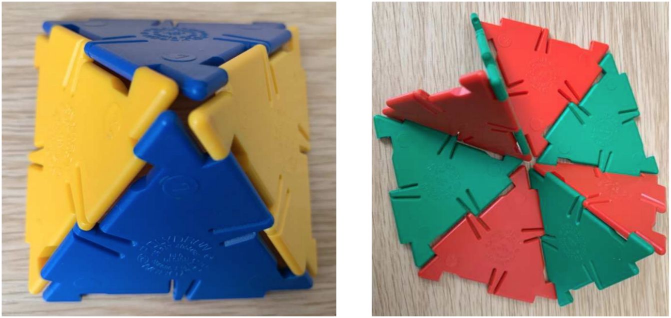


FIGURE 5. Triangle arrangement with positive vertex curvature $2\pi - \frac{4\pi}{3} = \frac{2\pi}{3}$ and with negative vertex curvature $2\pi - \frac{8\pi}{3} = -\frac{2\pi}{3}$

Combinatorial Curvature (Peyerimhoff Survey)

One definition could be

> *define curvature in a vertex locally so that things sum up to 360 degrees.*

Combinatorial Curvature (Peyerimhoff Survey)

One definition could be

> *define curvature in a vertex locally so that things sum up to 360 degrees.*

Do **not** try to parse the next definition.

Combinatorial Curvature (Peyerimhoff Survey)

One definition could be

> *define curvature in a vertex locally so that things sum up to 360 degrees.*

Do **not** try to parse the next definition.

Definition 2.1. *Let $\mathcal{T}$ be a tessellation of a surface S and $G = (V, E, F)$ be the combinatorial representation of $\mathcal{T}$, that is, we think of the faces $f \in F$ as regular Euclidean polygons of side length one with interior angles equals $\frac{(|f|-2)}{|f|}\pi$, where $|f|$ denotes the* degree of the face f*, that is, its number of sides. The* combinatorial curvature *of G is a function $K : V \to \mathbb{R}$ on the vertices and is defined by*

$$K(x) = 2\pi - \sum_{f:\, x \in f} \frac{|f| - 2}{|f|}\pi,$$

Combinatorial Curvature (Peyerimhoff Survey)

Good news

Combinatorial Curvature (Peyerimhoff Survey)

Good news

Theorem 2.2 (Discrete Global Gauss-Bonnet Theorem)**.** *Let* $G = (V, E, F)$ *be a combinatorial representation of a surface* S *and* $K : V \to \mathbb{R}$ *be its combinatorial curvature. Then we have*

$$\sum_{x \in V} K(x) = 2\pi\chi(S).$$

Combinatorial Curvature (Peyerimhoff Survey)

Good news

Theorem 2.2 (Discrete Global Gauss-Bonnet Theorem). *Let* $G = (V, E, F)$ *be a combinatorial representation of a surface* S *and* $K : V \to \mathbb{R}$ *be its combinatorial curvature. Then we have*

$$\sum_{x \in V} K(x) = 2\pi\chi(S).$$

More complicated news

Combinatorial Curvature (Peyerimhoff Survey)

Good news

Theorem 2.2 (Discrete Global Gauss-Bonnet Theorem). *Let $G = (V, E, F)$ be a combinatorial representation of a surface S and $K : V \to \mathbb{R}$ be its combinatorial curvature. Then we have*

$$\sum_{x \in V} K(x) = 2\pi\chi(S).$$

More complicated news

Informal Theorem (DeVos-Mohar, Ghidelli, Oldridge)

There aren't many graphs with positive (combinatorial) curvature.

Combinatorial Curvature (Peyerimhoff Survey)

Theorem (DeVos-Mohar, Ghidelli, Oldridge)

If a graph has positive (combinatorial) curvature, then

Combinatorial Curvature (Peyerimhoff Survey)

Theorem (DeVos-Mohar, Ghidelli, Oldridge)

If a graph has positive (combinatorial) curvature, then it is either a prism or an antiprism

Combinatorial Curvature (Peyerimhoff Survey)

Theorem (DeVos-Mohar, Ghidelli, Oldridge)

If a graph has positive (combinatorial) curvature, then it is either a prism or an antiprism

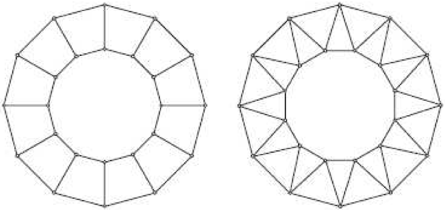


FIGURE 6. Examples of prisms and antiprisms

Combinatorial Curvature (Peyerimhoff Survey)

Theorem (DeVos-Mohar, Ghidelli, Oldridge)

If a graph has positive (combinatorial) curvature, then it is either a prism or an antiprism

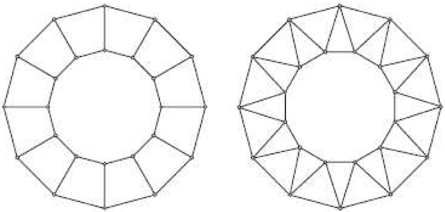


FIGURE 6. Examples of prisms and antiprisms

or it has at most 208 vertices.

Combinatorial Curvature (Peyerimhoff Survey)

belt of a fixed width around the equator. This example was discovered in 2011 by Ghidelli in private communications with J. Sneddon and later independently rediscovered by Oldridge [34].

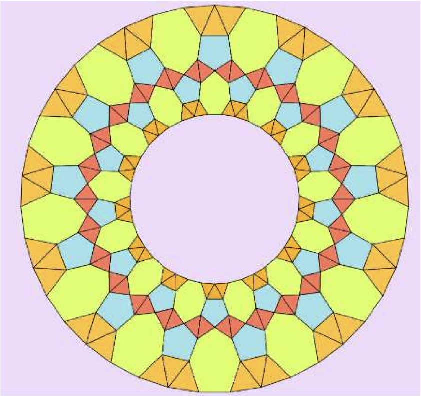


FIGURE 7. A planar graph with $|V| = 208$ and strictly positive combinatorial curvature in all vertices. Its faces have the degrees $3, 5, 7, 39$.

Next idea: Optimal Transport/Coupling of Random Walks

Idea behind Ollivier Curvature (Peyerimhoff Survey)

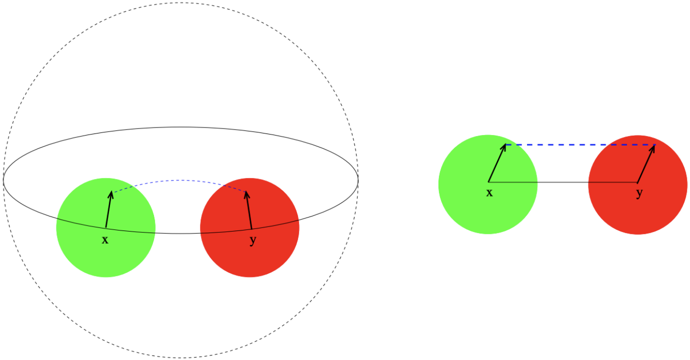


FIGURE 8. In the 2-sphere, corresponding points in small metric balls $B_\epsilon(x), B_\epsilon(y)$ in parallel directions have smaller distance than $d(x,y)$. In the Euclidean plane, they have the same distance $d(x,y)$.

Idea behind Ollivier Curvature

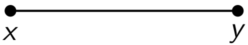
Ollivier curature is going to define the ‘curvature’ of an edge.

Idea behind Ollivier Curvature

Ollivier curature is going to define the 'curvature' of an edge. Let $G = (V, E)$ and let $u, v \in V$ be two adjacent vertices.

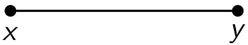
Idea behind Ollivier Curvature

Ollivier curature is going to define the 'curvature' of an edge. Let $G = (V, E)$ and let $u, v \in V$ be two adjacent vertices.



Idea behind Ollivier Curvature

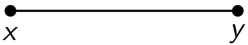
Ollivier curature is going to define the 'curvature' of an edge. Let $G = (V, E)$ and let $u, v \in V$ be two adjacent vertices.



The rules of the game are: transporting one unit mass across one edge costs 1.

Idea behind Ollivier Curvature

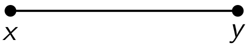
Ollivier curature is going to define the 'curvature' of an edge. Let $G = (V, E)$ and let $u, v \in V$ be two adjacent vertices.



The rules of the game are: transporting one unit mass across one edge costs 1. One unit of mass across 2 edges costs 2.

Idea behind Ollivier Curvature

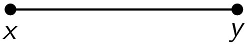
Ollivier curature is going to define the 'curvature' of an edge. Let $G = (V, E)$ and let $u, v \in V$ be two adjacent vertices.



The rules of the game are: transporting one unit mass across one edge costs 1. One unit of mass across 2 edges costs 2. Two units of mass across one edge costs 2.

Idea behind Ollivier Curvature

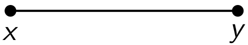
Ollivier curature is going to define the 'curvature' of an edge. Let $G = (V, E)$ and let $u, v \in V$ be two adjacent vertices.



The rules of the game are: transporting one unit mass across one edge costs 1. One unit of mass across 2 edges costs 2. Two units of mass across one edge costs 2. Cost of transporting δ_x to δ_y is

Idea behind Ollivier Curvature

Ollivier curature is going to define the 'curvature' of an edge. Let $G = (V, E)$ and let $u, v \in V$ be two adjacent vertices.

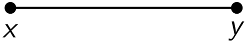


The rules of the game are: transporting one unit mass across one edge costs 1. One unit of mass across 2 edges costs 2. Two units of mass across one edge costs 2. Cost of transporting δ_x to δ_y is

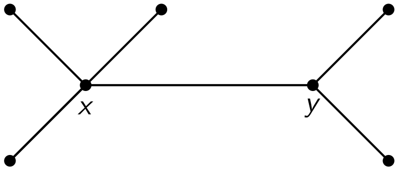
$$W_1(\delta_x, \delta_y) = 1.$$

Idea behind Ollivier Curvature

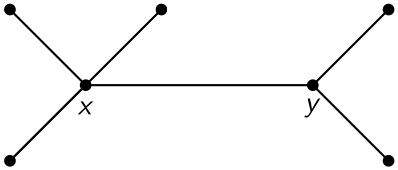
The next step is to consider the neighbors of x and y as well.



Idea behind Ollivier Curvature

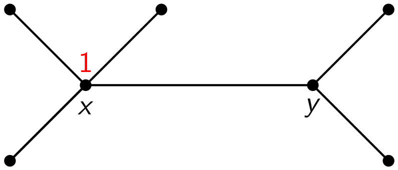


Idea behind Ollivier Curvature



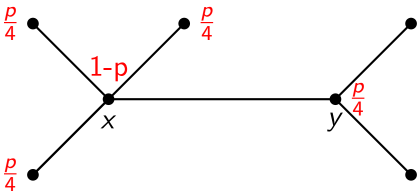
Instead of having a single Dirac measure in x

Idea behind Ollivier Curvature



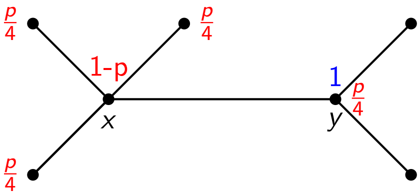
Instead of having a single Dirac measure in x

Idea behind Ollivier Curvature



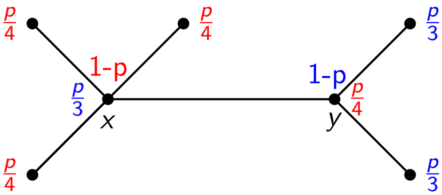
Instead of having a single Dirac measure in x, we share.

Idea behind Ollivier Curvature



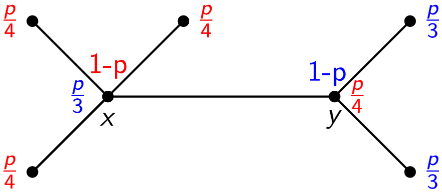
Instead of transporting directly to y.

Idea behind Ollivier Curvature



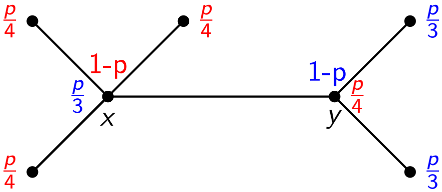
Instead of transporting directly to y, we share.

Idea behind Ollivier Curvature



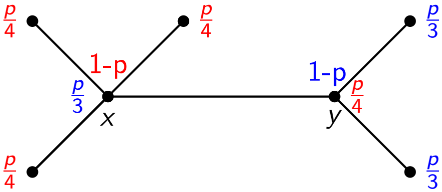
What is the transport cost of μ_p to ν_p?

Idea behind Ollivier Curvature



What is the transport cost of μ_p to ν_p? In this example, one would expect it to be slightly larger than 1.

Idea behind Ollivier Curvature



What is the transport cost of μ_p to ν_p? In this example, one would expect it to be slightly larger than 1.

Definition (Ollivier 2009)

The $p-$curvature of the edge (x, y) is given by

$$K_p(x, y) = 1 - W^1(\mu_p, \nu_p).$$

Ollivier curvature

Yann Ollivier

1. Parameter p.

Ollivier curvature

Yann Ollivier

1. Parameter p.
2. Computation requires solving optimal transport problem (linear programming).

Ollivier curvature

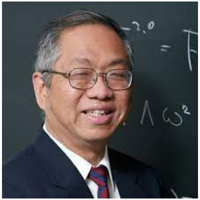
Yann Ollivier

1. Parameter p.
2. Computation requires solving optimal transport problem (linear programming).
3. Has many nice properties!

Lin-Lu-Yau curvature

Yong Lin

Linyuan Lu

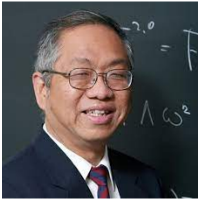


Shing-Tung Yau

Lin-Lu-Yau curvature

Yong Lin

Linyuan Lu



Shing-Tung Yau

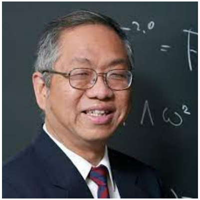
Definition (Lin-Lu-Yau 2011)

The LLY-curvature of the edge (x, y) is given by

Lin-Lu-Yau curvature

Yong Lin

Linyuan Lu



Shing-Tung Yau

Definition (Lin-Lu-Yau 2011)

The LLY-curvature of the edge (x, y) is given by

$$K_{LLY}(x, y) = \frac{\max(\deg(x), \deg(y)) + 1}{\max(\deg(x), \deg(y))} \cdot K_{\frac{1}{\max(\deg(x),\deg(y))+1}}(x, y).$$

No need to remember these formulas.

A very simple notion of curvature

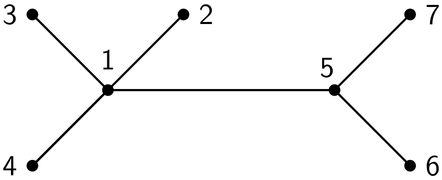
Given a connected graph G on n vertices, define the *graph distance matrix*

$$D_{ij} = d(v_i, v_j).$$

A very simple notion of curvature

Given a connected graph G on n vertices, define the *graph distance matrix*

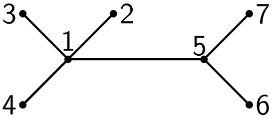
$$D_{ij} = d(v_i, v_j).$$



A very simple notion of curvature

Given a connected graph G on n vertices, define the *graph distance matrix*

$$D_{ij} = d(v_i, v_j).$$



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 & 3 & 3 \\ 1 & 2 & 0 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 0 & 2 & 3 & 3 \\ 1 & 2 & 2 & 2 & 0 & 1 & 1 \\ 2 & 3 & 3 & 3 & 1 & 0 & 2 \\ 2 & 3 & 3 & 3 & 1 & 2 & 0 \end{pmatrix}$$

A very simple notion of curvature

Now solve the quadratic linear system of equations

$$Dx = \mathbf{n} = (n, n, n, \ldots, n).$$

A very simple notion of curvature

Now solve the quadratic linear system of equations

$$Dx = \mathbf{n} = (n, n, n, \ldots, n).$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 & 3 & 3 \\ 1 & 2 & 0 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 0 & 2 & 3 & 3 \\ 1 & 2 & 2 & 2 & 0 & 1 & 1 \\ 2 & 3 & 3 & 3 & 1 & 0 & 2 \\ 2 & 3 & 3 & 3 & 1 & 2 & 0 \end{pmatrix} \cdot x = \begin{pmatrix} 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \end{pmatrix}$$

A very simple notion of curvature

Now solve the quadratic linear system of equations

$$Dx = \mathbf{n} = (n, n, n, \ldots, n).$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 & 3 & 3 \\ 1 & 2 & 0 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 0 & 2 & 3 & 3 \\ 1 & 2 & 2 & 2 & 0 & 1 & 1 \\ 2 & 3 & 3 & 3 & 1 & 0 & 2 \\ 2 & 3 & 3 & 3 & 1 & 2 & 0 \end{pmatrix} \cdot x = \begin{pmatrix} 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \end{pmatrix}$$

$$x = \left(-\frac{7}{3}, \frac{7}{6}, \frac{7}{6}, \frac{7}{6}, -\frac{7}{6}, \frac{7}{6}, \frac{7}{6}\right)$$

A very simple notion of curvature

Now solve the quadratic linear system of equations

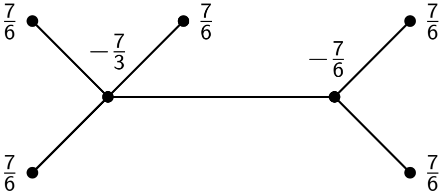
$$Dx = \mathbf{n} = (n, n, n, \ldots, n).$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 & 3 & 3 \\ 1 & 2 & 0 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 0 & 2 & 3 & 3 \\ 1 & 2 & 2 & 2 & 0 & 1 & 1 \\ 2 & 3 & 3 & 3 & 1 & 0 & 2 \\ 2 & 3 & 3 & 3 & 1 & 2 & 0 \end{pmatrix} \cdot x = \begin{pmatrix} 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \end{pmatrix}$$

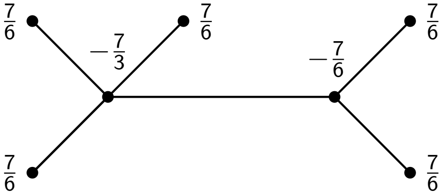
$$x = \left(-\frac{7}{3}, \frac{7}{6}, \frac{7}{6}, \frac{7}{6}, -\frac{7}{6}, \frac{7}{6}, \frac{7}{6} \right)$$

and these are defined to be the curvatures in the vertices.

A very simple notion of curvature

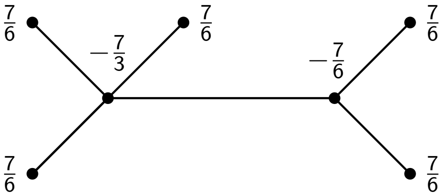


A very simple notion of curvature



Solving a linear system: existence? uniqueness? (*Later.*)

A very simple notion of curvature



Solving a linear system: existence? uniqueness? (*Later.*)

Motivation. Mass equilibrium. Signed measure $x : V \to \mathbb{R}$

$$\sum_{j \in V} d(i,j) \cdot x_j \qquad \text{independent of } i.$$

Examples

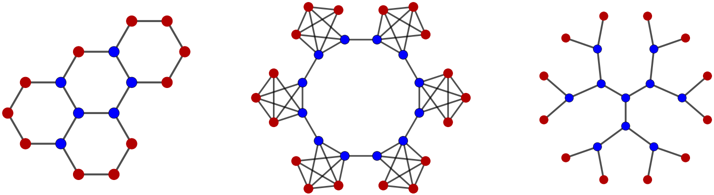
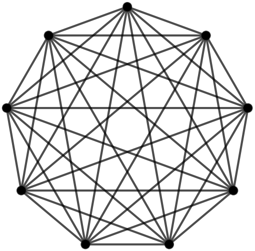


Figure: Vertices colored by curvature (red if positive, blue if negative).

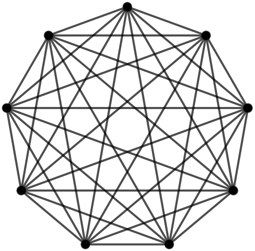
Examples: the complete graph K_n



$K(K_n) = \frac{n}{n-1}$

- constant curvature

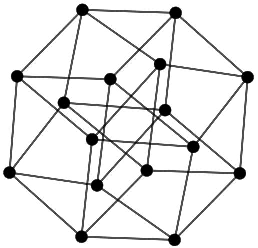
Examples: the complete graph K_n



$$K(K_n) = \frac{n}{n-1}$$

- constant curvature
- same as the Lin-Lu-Yau curvature

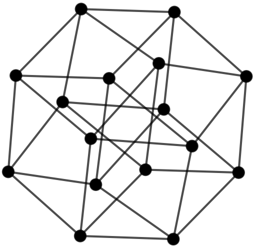
Examples: the hypercube graph Q_n



$K(Q_n) = \frac{2}{n}$

- constant curvature

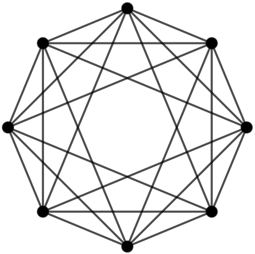
Examples: the hypercube graph Q_n



$$K(Q_n) = \frac{2}{n}$$

- constant curvature
- same as Lin-Lu-Yau curvature

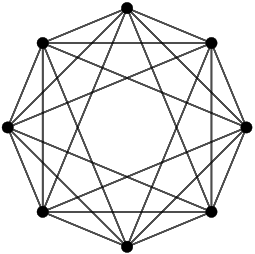
Examples: the cocktail party graph CP_n



$$K(CP_n) = 1$$

- constant curvature

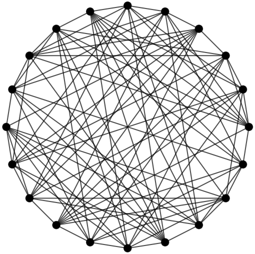
Examples: the cocktail party graph CP_n



$K(CP_n) = 1$

- constant curvature
- same as Ollivier curvature

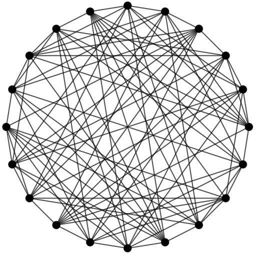
Examples: the Johnson graph $J_{n,k}$



$K(J_{n,k}) = \frac{n}{(n-k)k}$

- vertices are $k-$element subsets of n element set and connected if intersection is size $k-1$

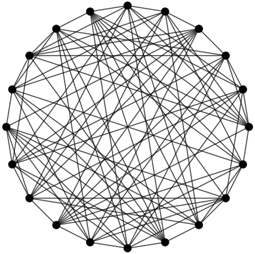
Examples: the Johnson graph $J_{n,k}$



$$K(J_{n,k}) = \frac{n}{(n-k)k}$$

- vertices are $k-$element subsets of n element set and connected if intersection is size $k-1$
- constant curvature

Examples: the Johnson graph $J_{n,k}$

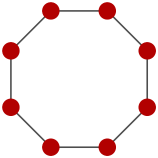


$$K(J_{n,k}) = \frac{n}{(n-k)k}$$

- vertices are $k-$element subsets of n element set and connected if intersection is size $k-1$
- constant curvature
- same as Ollivier curvature

‘archimedes drawing a circle in the sand by johannes vermeer’

The Cycle Graph C_n



has Ollivier and LLY curvature 0 when $n \geq 6$ but

$$K = \frac{n}{\left\lfloor \frac{n^2}{4} \right\rfloor} \sim \frac{4}{n}.$$

The Bonnet-Myers Theorem

Let (M, g) be a complete connected $n-$dimensional Riemannian manifold with Ricci curvature bounded below by $K > 0$, then

$$\text{diam}(M) \leq \frac{\pi}{\sqrt{K}}.$$

The Bonnet-Myers Theorem

Let (M, g) be a complete connected $n-$dimensional Riemannian manifold with Ricci curvature bounded below by $K > 0$, then

$$\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{K}}.$$

positive curvature $\rightarrow$ small diameter

The Bonnet-Myers Theorem

Let (M, g) be a complete connected $n-$dimensional Riemannian manifold with Ricci curvature bounded below by $K > 0$, then

$$\text{diam}(M) \leq \frac{\pi}{\sqrt{K}}.$$

positive curvature $\rightarrow$ small diameter

large diameter $\rightarrow$ curvature somewhere small

The Bonnet-Myers Theorem, 1941

Let (M, g) be a complete connected $n-$dimensional Riemannian manifold with Ricci curvature bounded below by $K > 0$, then

$$\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{K}}.$$

The Bonnet-Myers Theorem, 1941

Let (M, g) be a complete connected $n-$dimensional Riemannian manifold with Ricci curvature bounded below by $K > 0$, then

$$\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{K}}.$$

Bonnet-Myers on Graphs (Ollivier 2009, Lin-Lu-Yau 2011)

If G has Ollivier or Lin-Lu-Yau curvature bounded from below by $K > 0$, then

$$\mathrm{diam}(G) \leq \frac{2}{K}.$$

This is known to be sharp in some cases.

A linear system of equations, like

$$Dx = \mathbf{n} = (n, n, n, \ldots, n)$$

need not have a unique solution.

A linear system of equations, like

$$Dx = \mathbf{n} = (n, n, n, \dots, n)$$

need not have a unique solution. So what is the curvature?

A linear system of equations, like

$$Dx = \mathbf{n} = (n, n, n, \ldots, n)$$

need not have a unique solution. So what is the curvature?

Proposition (Invariance of Total Curvature)

Let G be a connected graph and suppose $Dw_1 = \mathbf{n} = Dw_2$ for two vectors $w_1, w_2 \in \mathbb{R}^n_{\geq 0}$.

A linear system of equations, like

$$Dx = \mathbf{n} = (n, n, n, \ldots, n)$$

need not have a unique solution. So what is the curvature?

Proposition (Invariance of Total Curvature)

Let G be a connected graph and suppose $Dw_1 = \mathbf{n} = Dw_2$ for two vectors $w_1, w_2 \in \mathbb{R}^n_{\geq 0}$. Then $\|w_1\|_{\ell^1} = \|w_2\|_{\ell^1}$.

Proposition (Invariance of Total Curvature)

Let G be a connected graph and suppose $Dw_1 = \mathbf{n} = Dw_2$ for two vectors $w_1, w_2 \in \mathbb{R}^n_{\geq 0}$. Then $\|w_1\|_{\ell^1} = \|w_2\|_{\ell^1}$.

Proposition (Invariance of Total Curvature)

Let G be a connected graph and suppose $Dw_1 = \mathbf{n} = Dw_2$ for two vectors $w_1, w_2 \in \mathbb{R}^n_{\geq 0}$. Then $\|w_1\|_{\ell^1} = \|w_2\|_{\ell^1}$.

Theorem (Bonnet-Myers Theorem)

Let G be connected and suppose $Dw = \mathbf{n}$. If $w_i \geq K$, then

Proposition (Invariance of Total Curvature)

Let G be a connected graph and suppose $Dw_1 = \mathbf{n} = Dw_2$ for two vectors $w_1, w_2 \in \mathbb{R}^n_{\geq 0}$. Then $\|w_1\|_{\ell^1} = \|w_2\|_{\ell^1}$.

Theorem (Bonnet-Myers Theorem)

Let G be connected and suppose $Dw = \mathbf{n}$. If $w_i \geq K$, then

$$\operatorname{diam}(G) \leq \frac{2n}{\|w\|_{\ell^1}} \leq \frac{2}{K}.$$

Proposition (Invariance of Total Curvature)

Let G be a connected graph and suppose $Dw_1 = \mathbf{n} = Dw_2$ for two vectors $w_1, w_2 \in \mathbb{R}^n_{\geq 0}$. Then $\|w_1\|_{\ell^1} = \|w_2\|_{\ell^1}$.

Theorem (Bonnet-Myers Theorem)

Let G be connected and suppose $Dw = \mathbf{n}$. If $w_i \geq K$, then

$$\operatorname{diam}(G) \leq \frac{2n}{\|w\|_{\ell^1}} \leq \frac{2}{K}.$$

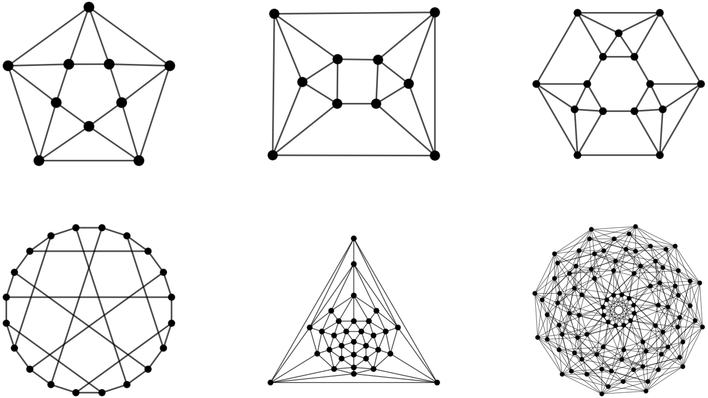
Corollary (Cheng Diameter Rigidity Theorem)

Let G be connected and suppose $Dw = \mathbf{n}$. If $w_i \geq K$ and

$$\operatorname{diam}(G) = \frac{2}{K}, \text{ then } \quad w_i = K.$$

Examples of graphs for which the Theorem is sharp

$$\mathrm{diam}(G) = \frac{2}{K}.$$



Theorem (A Bonnet-Myers Inequality)

Let G be connected and suppose $Dw = \mathbf{n}$. If $w_i \geq K$, then

$$\operatorname{diam}(G) \leq \frac{2n}{\|w\|_{\ell^1}} \leq \frac{2}{K}.$$

positive curvature $\rightarrow$ small diameter

Theorem (A Bonnet-Myers Inequality)

Let G be connected and suppose $Dw = \mathbf{n}$. If $w_i \geq K$, then

$$\operatorname{diam}(G) \leq \frac{2n}{\|w\|_{\ell^1}} \leq \frac{2}{K}.$$

positive curvature $\rightarrow$ small diameter

small diameter $\rightarrow$ ‘graph is very curved’

Theorem (A Bonnet-Myers Inequality)

Let G be connected and suppose $Dw = \mathbf{n}$. If $w_i \geq K$, then

$$\operatorname{diam}(G) \leq \frac{2n}{\|w\|_{\ell^1}} \leq \frac{2}{K}.$$

positive curvature $\rightarrow$ small diameter

small diameter $\rightarrow$ 'graph is very curved'

Theorem (*Reverse* Bonnet-Myers)

Let G be connected and suppose $Dw = \mathbf{n}$ with $w_i \geq 0$. Then

Theorem (A Bonnet-Myers Inequality)

Let G be connected and suppose $Dw = \mathbf{n}$. If $w_i \geq K$, then

$$\operatorname{diam}(G) \leq \frac{2n}{\|w\|_{\ell^1}} \leq \frac{2}{K}.$$

positive curvature $\rightarrow$ small diameter

small diameter $\rightarrow$ 'graph is very curved'

Theorem (*Reverse* Bonnet-Myers)

Let G be connected and suppose $Dw = \mathbf{n}$ with $w_i \geq 0$. Then

$$\|w\|_{\ell^1} \geq \frac{n^2}{n-1} \frac{1}{\operatorname{diam}(G)}$$

with equality if and only if $G = K_n$.

Theorem (Lichnerowicz, 1958)

Let (M, g) be an $n-$dimensional manifold with Ricci curvature bounded below by K, then $\lambda_1 \geq n/(n-1) \cdot K$.

Theorem (Lichnerowicz, 1958)

Let (M, g) be an $n-$dimensional manifold with Ricci curvature bounded below by K, then $\lambda_1 \geq n/(n-1) \cdot K$.

Theorem (Ollivier, Lin-Lu-Yau)

If G has (O/LLY)-curvature bounded below by K, then the first eigenvalue of the Laplacian satisfies

$$\inf_{\int f=0} \frac{\sum_{(u,v)\in E}(f(u)-f(v))^2}{\sum f(v)^2} = \lambda_1 \geq K.$$

Theorem (Lichnerowicz, 1958)

Let (M, g) be an $n-$dimensional manifold with Ricci curvature bounded below by K, then $\lambda_1 \geq n/(n-1) \cdot K$.

Theorem (Ollivier, Lin-Lu-Yau)

If G has (O/LLY)-curvature bounded below by K, then the first eigenvalue of the Laplacian satisfies

$$\inf_{\int f=0} \frac{\sum_{(u,v)\in E}(f(u)-f(v))^2}{\sum f(v)^2} = \lambda_1 \geq K.$$

Proposition (S)

If G has curvature bounded below by K, then the first eigenvalue of the Laplacian satisfies

$$\lambda_1 \geq \frac{K}{2n}.$$

Theorem (Lichnerowicz, 1958)

Let (M, g) be an $n-$dimensional manifold with Ricci curvature bounded below by K, then $\lambda_1 \geq n/(n-1) \cdot K$.

Theorem (Ollivier, Lin-Lu-Yau)

If G has (O/LLY)-curvature bounded below by K, then the first eigenvalue of the Laplacian satisfies

$$\inf_{\int f=0} \frac{\sum_{(u,v)\in E}(f(u)-f(v))^2}{\sum f(v)^2} = \lambda_1 \geq K.$$

Proposition (S)

If G has curvature bounded below by K, then the first eigenvalue of the Laplacian satisfies

$$\lambda_1 \geq \frac{K}{2n}.$$

Sharp up to constants (cycle graph).

Special Case (Oliver Alfred Gross (RAND?), 1964)

Let $0 \leq x_1, \ldots, x_n \leq 1$. There exists $0 \leq x \leq 1$ such that

$$\frac{1}{n} \sum_{i=1}^{n} |x - x_i| = \frac{1}{2}.$$

Special Case (Oliver Alfred Gross (RAND?), 1964)

Let $0 \leq x_1, \ldots, x_n \leq 1$. There exists $0 \leq x \leq 1$ such that

$$\frac{1}{n} \sum_{i=1}^{n} |x - x_i| = \frac{1}{2}.$$

Proof.

Set $f(x) = \frac{1}{n} \sum_{i=1}^{n} |x - x_i|$.

Special Case (Oliver Alfred Gross (RAND?), 1964)

Let $0 \leq x_1, \ldots, x_n \leq 1$. There exists $0 \leq x \leq 1$ such that

$$\frac{1}{n} \sum_{i=1}^{n} |x - x_i| = \frac{1}{2}.$$

Proof.

Set $f(x) = \frac{1}{n} \sum_{i=1}^{n} |x - x_i|$. Then

$$f(0) + f(1) = \frac{1}{n} \sum_{i=1}^{n} |x_i| + |1 - x_i| = 1.$$

Special Case (Oliver Alfred Gross (RAND?), 1964)

Let $0 \le x_1, \ldots, x_n \le 1$. There exists $0 \le x \le 1$ such that

$$\frac{1}{n}\sum_{i=1}^{n} |x - x_i| = \frac{1}{2}.$$

Proof.

Set $f(x) = \frac{1}{n}\sum_{i=1}^{n} |x - x_i|$. Then

$$f(0) + f(1) = \frac{1}{n}\sum_{i=1}^{n} |x_i| + |1 - x_i| = 1.$$

Either $f(0) = f(1) = 1/2$ or one is smaller and one is bigger and the intermediate value theorem. □

Theorem (Gross, 1964)

Let (X, d) be a compact, connected metric space.

Theorem (Gross, 1964)

Let (X, d) be a compact, connected metric space. There exists a unique $r > 0$

Theorem (Gross, 1964)

Let (X, d) be a compact, connected metric space. There exists a unique $r > 0$ such that for all $x_1, \dots, x_n \in X$ there exists $x \in X$ such that

$$\frac{1}{n}\sum_{i=1}^{n} d(x, x_i) = r.$$

Theorem (Gross, 1964)

Let (X, d) be a compact, connected metric space. There exists a unique $r > 0$ such that for all $x_1, \ldots, x_n \in X$ there exists $x \in X$ such that

$$\frac{1}{n}\sum_{i=1}^{n} d(x, x_i) = r.$$

These numbers $r > 0$ are only known in special cases (easy to approximate though).

Theorem (Gross, 1964)

Let (X, d) be a compact, connected metric space. There exists a unique $r > 0$ such that for all $x_1, \ldots, x_n \in X$ there exists $x \in X$ such that

$$\frac{1}{n}\sum_{i=1}^{n} d(x, x_i) = r.$$

These numbers $r > 0$ are only known in special cases (easy to approximate though). Proof uses Glicksberg Fixed Point Theorem (Glicksberg $\rightarrow$ Garnett $\rightarrow$ Jones).

Theorem (Gross, 1964)

Let (X, d) be a compact, connected metric space. There exists a unique $r > 0$ such that for all $x_1, \ldots, x_n \in X$ there exists $x \in X$ such that

$$\frac{1}{n}\sum_{i=1}^{n} d(x, x_i) = r.$$

These numbers $r > 0$ are only known in special cases (easy to approximate though). Proof uses Glicksberg Fixed Point Theorem (Glicksberg $\rightarrow$ Garnett $\rightarrow$ Jones). We will now do this on graphs (compact metric space but not connected metric space).

Theorem (Total Curvature Minimax)

Let G be nonnegatively curved with total curvature $\|w\|_{\ell^1}$.

Theorem (Total Curvature Minimax)

Let G be nonnegatively curved with total curvature $\|w\|_{\ell^1}$. Then, for any list of vertices $v_1, \ldots, v_m$,

Theorem (Total Curvature Minimax)

Let G be nonnegatively curved with total curvature $\|w\|_{\ell^1}$. Then, for any list of vertices $v_1, \ldots, v_m$, there exist $a, b \in V$

Theorem (Total Curvature Minimax)

Let G be nonnegatively curved with total curvature $\|w\|_{\ell^1}$. Then, for any list of vertices $v_1, \ldots, v_m$, there exist $a, b \in V$

$$\frac{1}{m}\sum_{i=1}^{m} d(a, v_m) \leq \frac{n}{\|w\|_{\ell^1}} \leq \frac{1}{m}\sum_{i=1}^{m} d(b, v_m)$$

Theorem (Total Curvature Minimax)

Let G be nonnegatively curved with total curvature $\|w\|_{\ell^1}$. Then, for any list of vertices $v_1, \ldots, v_m$, there exist $a, b \in V$

$$\frac{1}{m}\sum_{i=1}^{m} d(a, v_m) \leq \frac{n}{\|w\|_{\ell^1}} \leq \frac{1}{m}\sum_{i=1}^{m} d(b, v_m)$$

Moreover, $n/\|w\|_{\ell^1}$ is the *unique* number with this property.

Theorem (Total Curvature Minimax)

Let G be nonnegatively curved with total curvature $\|w\|_{\ell^1}$. Then, for any list of vertices $v_1, \ldots, v_m$, there exist $a, b \in V$

$$\frac{1}{m}\sum_{i=1}^{m} d(a, v_m) \leq \frac{n}{\|w\|_{\ell^1}} \leq \frac{1}{m}\sum_{i=1}^{m} d(b, v_m)$$

Moreover, $n/\|w\|_{\ell^1}$ is the *unique* number with this property.

Proof of Bonnet-Myers.

Theorem (Total Curvature Minimax)

Let G be nonnegatively curved with total curvature $\|w\|_{\ell^1}$. Then, for any list of vertices $v_1, \ldots, v_m$, there exist $a, b \in V$

$$\frac{1}{m}\sum_{i=1}^{m} d(a, v_m) \leq \frac{n}{\|w\|_{\ell^1}} \leq \frac{1}{m}\sum_{i=1}^{m} d(b, v_m)$$

Moreover, $n/\|w\|_{\ell^1}$ is the *unique* number with this property.

Proof of Bonnet-Myers. Take $d(v_1, v_2) = \mathrm{diam}(G)$.

Theorem (Total Curvature Minimax)

Let G be nonnegatively curved with total curvature $\|w\|_{\ell^1}$. Then, for any list of vertices $v_1, \ldots, v_m$, there exist $a, b \in V$

$$\frac{1}{m}\sum_{i=1}^{m} d(a, v_m) \leq \frac{n}{\|w\|_{\ell^1}} \leq \frac{1}{m}\sum_{i=1}^{m} d(b, v_m)$$

Moreover, $n/\|w\|_{\ell^1}$ is the *unique* number with this property.

Proof of Bonnet-Myers. Take $d(v_1, v_2) = \mathrm{diam}(G)$. Then

$$\frac{1}{2}(d(a, v_1) + d(a, v_2)) \leq \frac{n}{\|w\|_{\ell^1}}.$$

Theorem (Total Curvature Minimax)

Let G be nonnegatively curved with total curvature $\|w\|_{\ell^1}$. Then, for any list of vertices $v_1, \ldots, v_m$, there exist $a, b \in V$

$$\frac{1}{m}\sum_{i=1}^{m} d(a, v_m) \leq \frac{n}{\|w\|_{\ell^1}} \leq \frac{1}{m}\sum_{i=1}^{m} d(b, v_m)$$

Moreover, $n/\|w\|_{\ell^1}$ is the *unique* number with this property.

Proof of Bonnet-Myers. Take $d(v_1, v_2) = \mathrm{diam}(G)$. Then

$$\frac{1}{2}(d(a, v_1) + d(a, v_2)) \leq \frac{n}{\|w\|_{\ell^1}}.$$

Triangle inequality $\mathrm{diam}(G) = d(v_1, v_2) \leq d(a, v_1) + d(a, v_2)$.

von Neumann Minimax (1928)

von Neumann Minimax (1928)

Let $A \in \mathbb{R}^{n \times n}$ by a symmetric matrix.

von Neumann Minimax (1928)

Let $A \in \mathbb{R}^{n \times n}$ by a symmetric matrix. There exists a unique $\alpha \in \mathbb{R}$

von Neumann Minimax (1928)

Let $A \in \mathbb{R}^{n \times n}$ by a symmetric matrix. There exists a unique $\alpha \in \mathbb{R}$ such that for all $(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$ satisfying $x_1 + \cdots + x_n = 1$

von Neumann Minimax (1928)

Let $A \in \mathbb{R}^{n \times n}$ by a symmetric matrix. There exists a unique $\alpha \in \mathbb{R}$ such that for all $(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$ satisfying $x_1 + \cdots + x_n = 1$

$$\min_{1 \leq i \leq n} (Ax)_i \leq \alpha \leq \max_{1 \leq i \leq n} (Ax)_i .$$

von Neumann Minimax (1928)

Let $A \in \mathbb{R}^{n \times n}$ by a symmetric matrix. There exists a unique $\alpha \in \mathbb{R}$ such that for all $(x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0}$ satisfying $x_1 + \cdots + x_n = 1$

$$\min_{1 \leq i \leq n} (Ax)_i \leq \alpha \leq \max_{1 \leq i \leq n} (Ax)_i.$$

In our case

$$\alpha = \frac{n}{\|w\|_{\ell^1}}$$

Existence?

When does the equation

$$Dw = \mathbf{n}$$

have a solution?

Existence?

When does the equation

$$Dw = \mathbf{n}$$

have a solution? **Not always.**

Existence?

When does the equation

$$Dw = \mathbf{n}$$

have a solution? **Not always. But frustratingly often.**

Existence?

When does the equation

$$Dw = \mathbf{n}$$

have a solution? **Not always. But frustratingly often.**

Mathematica's list of 10.000 graphs contains

Existence?

When does the equation

$$Dw = \mathbf{n}$$

have a solution? **Not always. But frustratingly often.**

Mathematica's list of 10.000 graphs contains roughly 5000 examples where $\text{rank}(D) < n$

Existence?

When does the equation

$$Dw = \mathbf{n}$$

have a solution? **Not always. But frustratingly often.**

Mathematica's list of 10.000 graphs contains roughly 5000 examples where $\text{rank}(D) < n$ but only 5 examples where $Dw = \mathbf{n}$ does not have a solution.

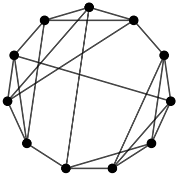
Existence?

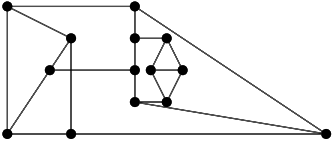
When does the equation

$$Dw = \mathbf{n}$$

have a solution? **Not always. But frustratingly often.**

Mathematica's list of 10.000 graphs contains roughly 5000 examples where $\text{rank}(D) < n$ but only 5 examples where $Dw = \mathbf{n}$ does not have a solution.





Very strange phenomenon....

$$Dx = \mathbf{n} \qquad \textit{where} \qquad D_{ij} = d(v_i, v_j).$$

$$Dx = \mathbf{n} \qquad \textit{where} \qquad D_{ij} = d(v_i, v_j).$$

But as we know: the geodesic distance on a graph might be not be a good way of measuring distances.

$$Dx = \mathbf{n} \qquad \textit{where} \qquad D_{ij} = d(v_i, v_j).$$

But as we know: the geodesic distance on a graph might be not be a good way of measuring distances. **Much of the theory is robust:** pick your favorite metric!

Resistance Curvature!

Karel Devriendt
(MPI Leipzig)

Andrea Ottolini
(UW Math)

Idea: replace distance by a graph-adapted notion of distance.
Random walks but symmetrized.

Idea: replace distance by a graph-adapted notion of distance.
Random walks but symmetrized.

Resistance Distance

Let $\Omega \in \mathbb{R}^{n \times n}$ be the matrix of effective resistances

$$\Omega_{ij} = \frac{\text{commute time between } v_i \text{ and } v_j}{2 \cdot |E|}.$$

Resistance Curvature

Let $\Omega \in \mathbb{R}^{n \times n}$ be the matrix of effective resistances. Define resistance curvature κ as the solution of

$$\Omega\kappa = \mathbf{1}.$$

Resistance Curvature

Let $\Omega \in \mathbb{R}^{n \times n}$ be the matrix of effective resistances. Define resistance curvature κ as the solution of

$$\Omega\kappa = \mathbf{1}.$$

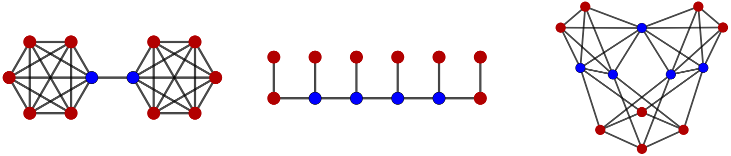
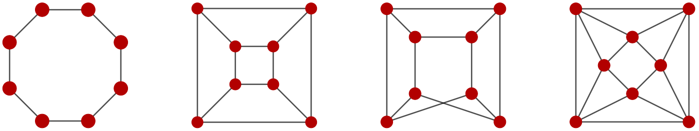


Figure: Vertices of graphs colored by the sign of the resistance curvature (red if positive, blue if negative).



$K \sim 0.1$ $K \sim 0.206$ $K \sim 0.208$ $K \sim 0.289$

Figure: Graphs with $\#V = 8$ and constant resistance curvature: the cycle C_8, the cube Q_3, the Wagner Graph and Antiprism$_4$. As curvature increases the average commute time between vertices decreases.

Theorem (KOS, 2023)

Let $G = (V, E)$ be a connected graph with maximal degree Δ and resistance curvature bounded from below by $K > 0$. Then

$$\operatorname{diam}(G) \leq \sqrt{\frac{\Delta}{K}} \cdot \log |V|$$

Theorem (KOS, 2023)

Let $G = (V, E)$ be a connected graph with maximal degree Δ and resistance curvature bounded from below by $K > 0$. Then

$$\mathrm{diam}(G) \leq \sqrt{\frac{\Delta}{K}} \cdot \log |V|$$

Conjecture (Bonnet-Myers)

Let $G = (V, E)$ be a connected graph with resistance curvature bounded from below by $K > 0$. Then

$$\mathrm{diam}(G) \leq \frac{100}{\sqrt{K}}.$$

Lichnerowicz Inequality (KOS, 2023)

Suppose $G = (V, E)$ has resistance curvature bounded from below by $K > 0$, then the smallest positive eigenvalue of $D - A$ satisfies

$$\lambda_2 \geq 2K.$$

Lichnerowicz Inequality (KOS, 2023)

Suppose $G = (V, E)$ has resistance curvature bounded from below by $K > 0$, then the smallest positive eigenvalue of $D - A$ satisfies

$$\lambda_2 \geq 2K.$$

Commute Time Pinching (KOS, 2023)

Suppose $G = (V, E)$ has curvature bounded from below by $K > 0$ and bounded from above by K_2.

Lichnerowicz Inequality (KOS, 2023)

Suppose $G = (V, E)$ has resistance curvature bounded from below by $K > 0$, then the smallest positive eigenvalue of $D - A$ satisfies

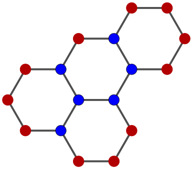
$$\lambda_2 \geq 2K.$$

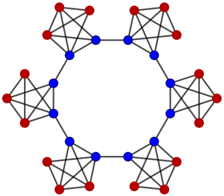
Commute Time Pinching (KOS, 2023)

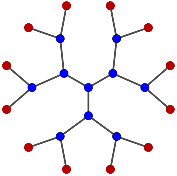
Suppose $G = (V, E)$ has curvature bounded from below by $K > 0$ and bounded from above by K_2. Then, for all vertices $x \in V$,

$$\frac{2}{K_2}\frac{|E|}{|V|} \leq \max_{y \in V} \text{ commute}(x, y) \leq \max_{y,z \in V} \text{ commute}(y, z) \leq \frac{4}{K}\frac{|E|}{|V|}.$$

1. Curvature on Graphs via Equilibrium Measures, *J. Graph Theory*
2. K. Devriendt, A. Ottolini and S, Graph curvature via resistance distance, arXiv:2302.06021







THANK YOU!