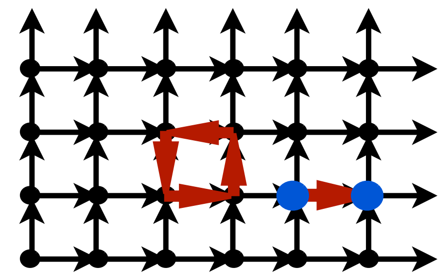


View from the front line:
simulations of quantum chromodynamics and the continuum limit

Stephen R. Sharpe
Physics Department
University of Washington

A fish out of water?

- Why someone might have mistaken me for an expert on quantization

PHYSICAL REVIEW D **103**, 054503 (2021)

Relativistic three-particle quantization condition for nondegenerate scalars

Tyler D. Blanton[*] and Stephen R. Sharpe[†]
Physics Department, University of Washington, Seattle, Washington 98195-1560, USA

PHYSICAL REVIEW D **102**, 054515 (2020)

Equivalence of relativistic three-particle quantization conditions

Tyler D. Blanton[*] and Stephen R. Sharpe[†]
Physics Department, University of Washington, Seattle, Washington 98195-1560, USA

PHYSICAL REVIEW D **102**, 054520 (2020)

Alternative derivation of the relativistic three-particle quantization condition

Tyler D. Blanton[*] and Stephen R. Sharpe[†]

PHYSICAL REVIEW D **98**, 014506 (2018)

Numerical study of the relativistic three-body quantization condition in the isotropic approximation

Raúl A. Briceño,[1,2,*] Maxwell T. Hansen,[3,†] and Stephen R. Sharpe[4,‡]

PHYSICAL REVIEW D **95**, 034501 (2017)

Applying the relativistic quantization condition to a three-particle bound state in a periodic box

Maxwell T. Hansen[1,*] and Stephen R. Sharpe[2,†]

PHYSICAL REVIEW D **90**, 116003 (2014)

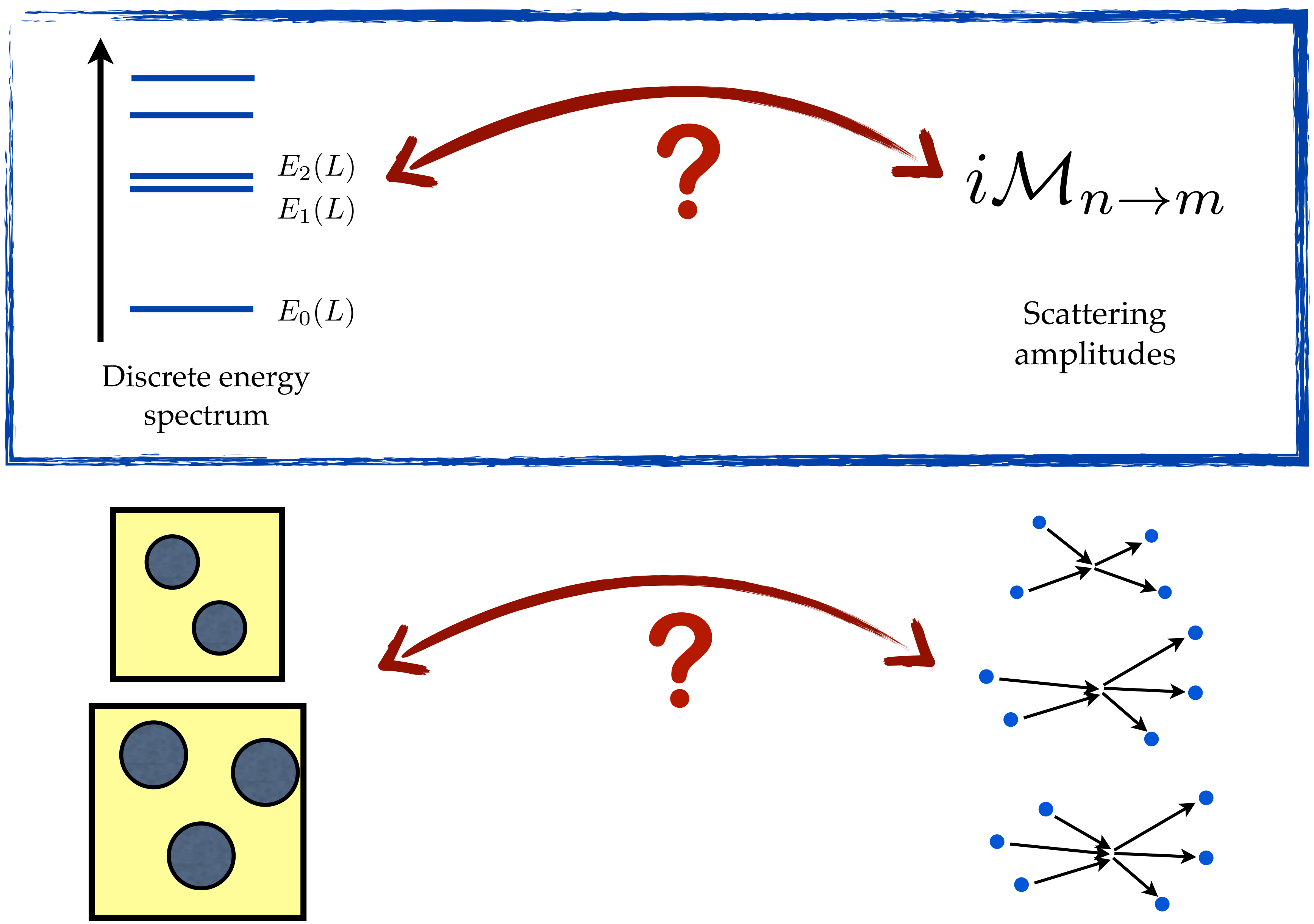
Relativistic, model-independent, three-particle quantization condition

Maxwell T. Hansen[*] and Stephen R. Sharpe[†]

- But this is the "trivial" quantization condition of particles in a box, albeit in the nontrivial context of a generic relativistic effective field theory

A fish out of water?

- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite-volume scattering amplitudes?

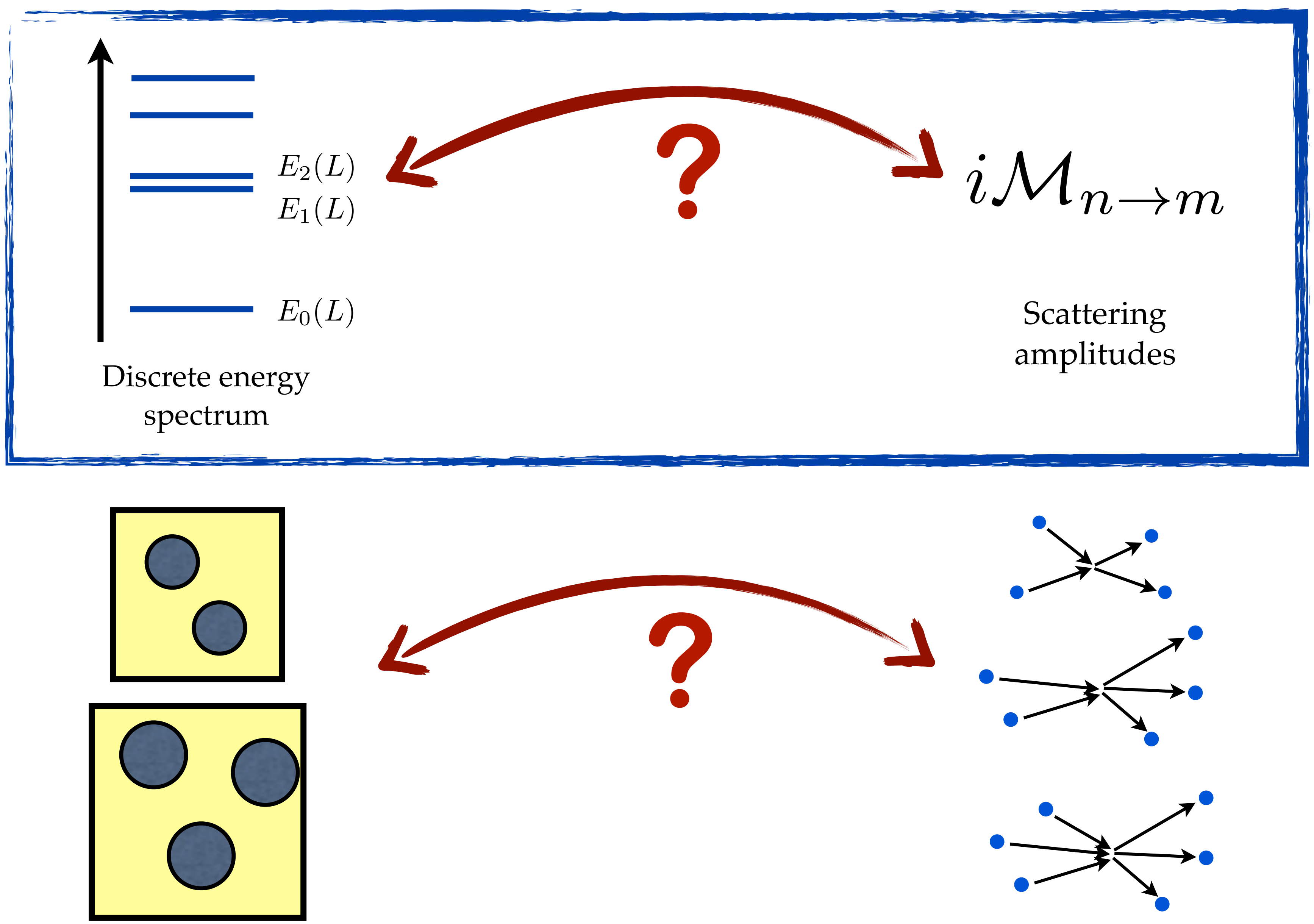


A fish out of water?

- Lattice QCD can calculate energy levels of multiparticle systems in a box

!?

- How are these related to infinite-volume scattering amplitudes?



Today's topic: how can we claim that "lattice QCD can calculate…."?

- QCD=Quantum Chromodynamics=QFT describing the strong interactions (quarks & gluons)
- Lattice QCD $\Rightarrow$ discretize space-time $\Rightarrow$ computational method, implemented numerically
- (Also need to work in finite space-time volume, imaginary time, …)
- Key question for today: can we take the continuum limit (lattice spacing $a \to 0$)?
- Much numerical evidence, backed by some theoretical calculations, suggests that we can do so in a controlled way
- How rigorous can this be made?

State-of-the art LQCD results

Eur. Phys. J. C (2020) 80:113
https://doi.org/10.1140/epjc/s10052-019-7354-7

THE EUROPEAN PHYSICAL JOURNAL C

Check for updates

Review

FLAG Review 2019

Flavour Lattice Averaging Group (FLAG)

S. Aoki[1], Y. Aoki[2,3,34], D. Bečirević[4], T. Blum[3,5], G. Colangelo[6], S. Collins[7], M. Della Morte[8], P. Dimopoulos[9,10], S. Dürr[11,12], H. Fukaya[13], M. Golterman[14], Steven Gottlieb[15], R. Gupta[16], S. Hashimoto[2,17], U. M. Heller[18], G. Herdoiza[19], R. Horsley[20], A. Jüttner[21,a], T. Kaneko[2,17], C.-J. D. Lin[22,23], E. Lunghi[15], R. Mawhinney[24], A. Nicholson[25], T. Onogi[13], C. Pena[19], A. Portelli[20], A. Ramos[26], S. R. Sharpe[27], J. N. Simone[28], S. Simula[29], R. Sommer[30,31], R. Van de Water[28], A. Vladikas[32], U. Wenger[6], H. Wittig[33]

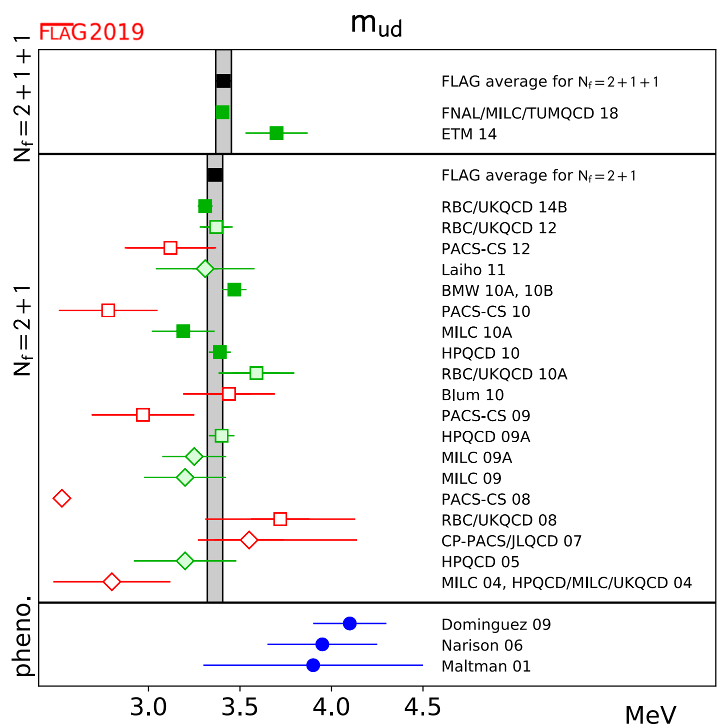
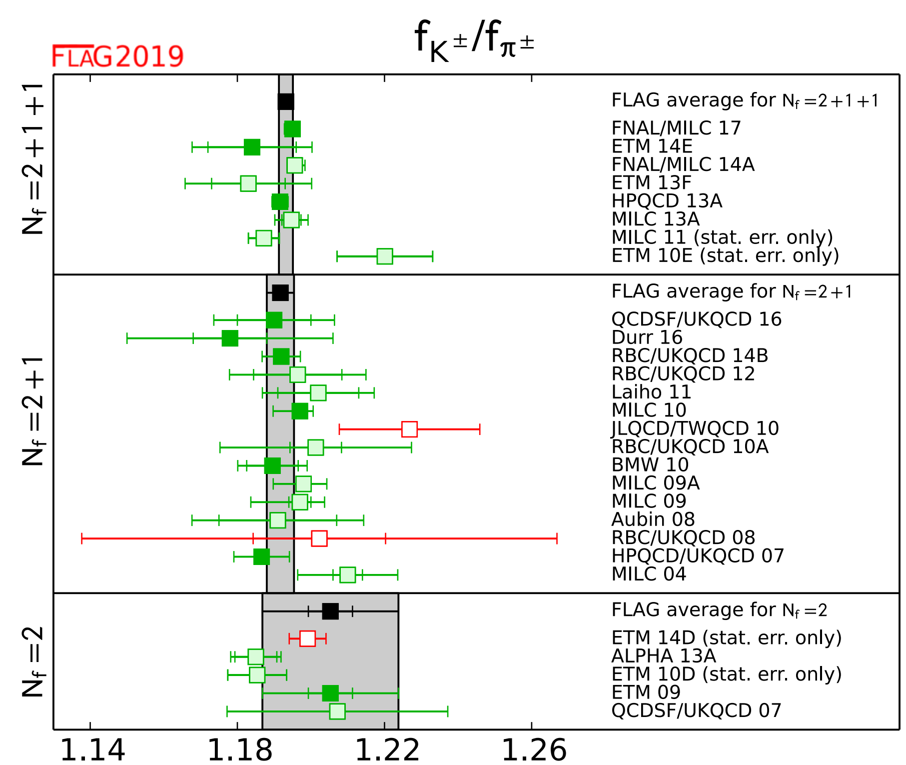


Fig. 2 Mean mass of the two lightest quarks, $m_{ud} = \frac{1}{2}(m_u + m_d)$. The bottom panel shows results based on sum rules [173,176,178] (for more details see Fig. 1)



What is QCD?

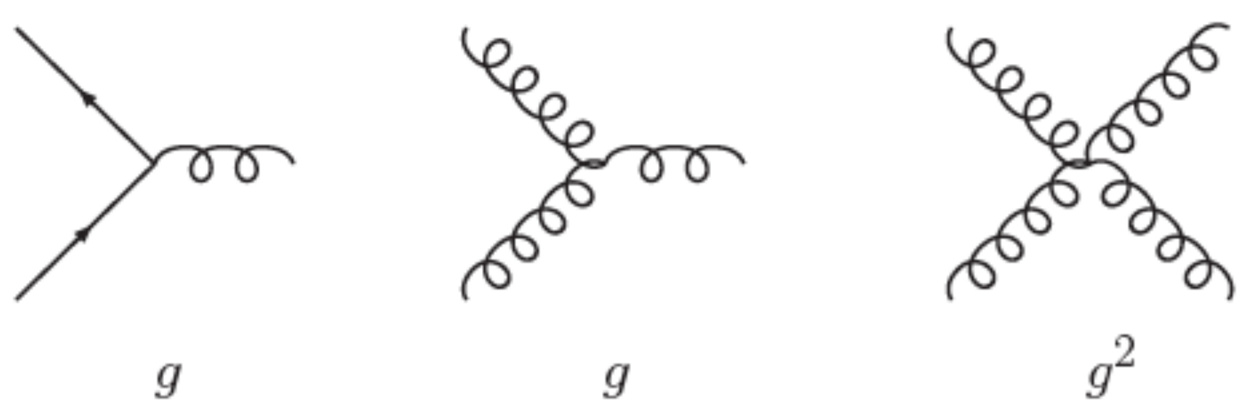
$$\mathcal{L}_{QCD} = \bar{q}_{a,i}\left[(i\gamma^\mu\partial_\mu - m_i)\,\delta_{ab}\delta_{ij}\right] q_{b,j} - g\; G^a_\mu\, \bar{q}_{i,b}\gamma^\mu T^a_{bc}\, q_{i,c} - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g\; f^{abc} G^b_\mu G^c_\nu$$

$q_{a,i}(x) =$ quark, 3 colors "a" and 6 flavors "i"

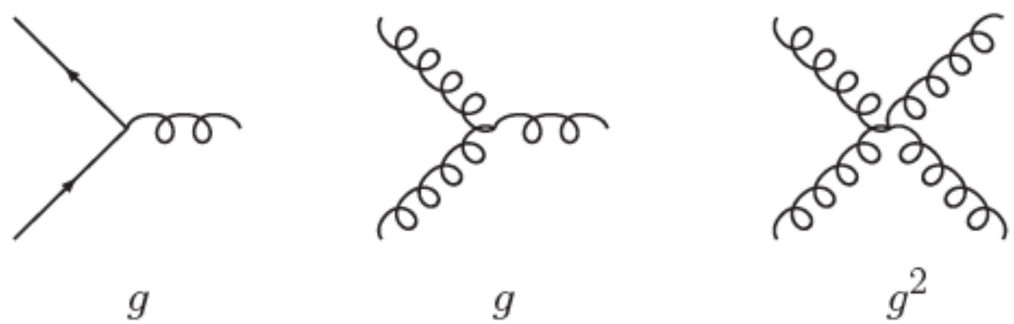
$G^a_\mu(x) =$ gluon, 8 colors "a"

g QCD coupling



Textbook quantization

- EITHER canonical quantization in term of quark and gluon field operators
  - Subtleties associated with gauge symmetry (redundant degrees of freedom)
- OR (more usual) use functional integral approach (Feynman path integral)
  - Non-abelian gauge symmetry leads to ghosts (spinless particles quantized as fermions)
- BOTH lead, formally, to standard Feynman rules (after gauge fixing)



+ ghost vertices

- Calculation of loop contributions to vertex functions & scattering amplitudes leads to UV (and IR) divergences
  - Regularize (e.g. dim. reg., introduces a scale), renormalize with counterterms (e.g. $\overline{\mathrm{MS}}$ scheme?)
  - Leads to method for calculating order by order in perturbation theory (extremely successful for QED, although series known not to converge)

Wilsonian view—EFTs

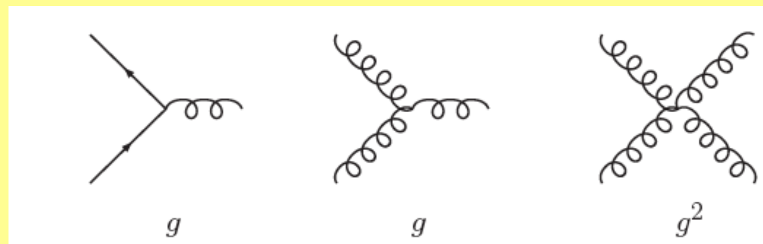
- Theory is defined with an UV cutoff on momenta ($\sim 1/a$ in a lattice theory)
- As the cutoff is changed, couplings and masses change ("run") to account for the modes that have been "integrated out/in"
    - Leads to differential equations (RG equations) for couplings (and masses)
    - Physically more appealing (and mathematically more solid) as only claiming to control a range of momenta
- Running, in general, takes place in a space with an infinite number of couplings (EFT)
    - Couplings are organized by their dimensions, n, with contributions, in general, $\sim (q/\Lambda)^{n-d}$ with $q \sim$ momentum scale of process, and $d =$ space-time dimension
    - Renormalizable couplings have $n \leq d$, and are those that remain as $\Lambda \to \infty$ (e.g. in QCD: gluon coupling g and quark masses)
- Caveats: in general, can only work out explicitly in perturbation theory; there may be large anomalous dimensions (due to interactions); "triviality" (couplings diverge at finite Λ)

Wilsonian view of QCD

$\Lambda \sim 1/a$

Asymptotic Freedom

$$\alpha_s(\Lambda) = \frac{g(\Lambda)^2}{4\pi} \propto \frac{1}{\log(\Lambda/\Lambda_{\rm QCD})}$$



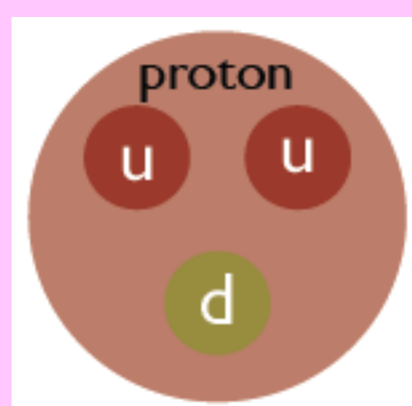
Perturbative region

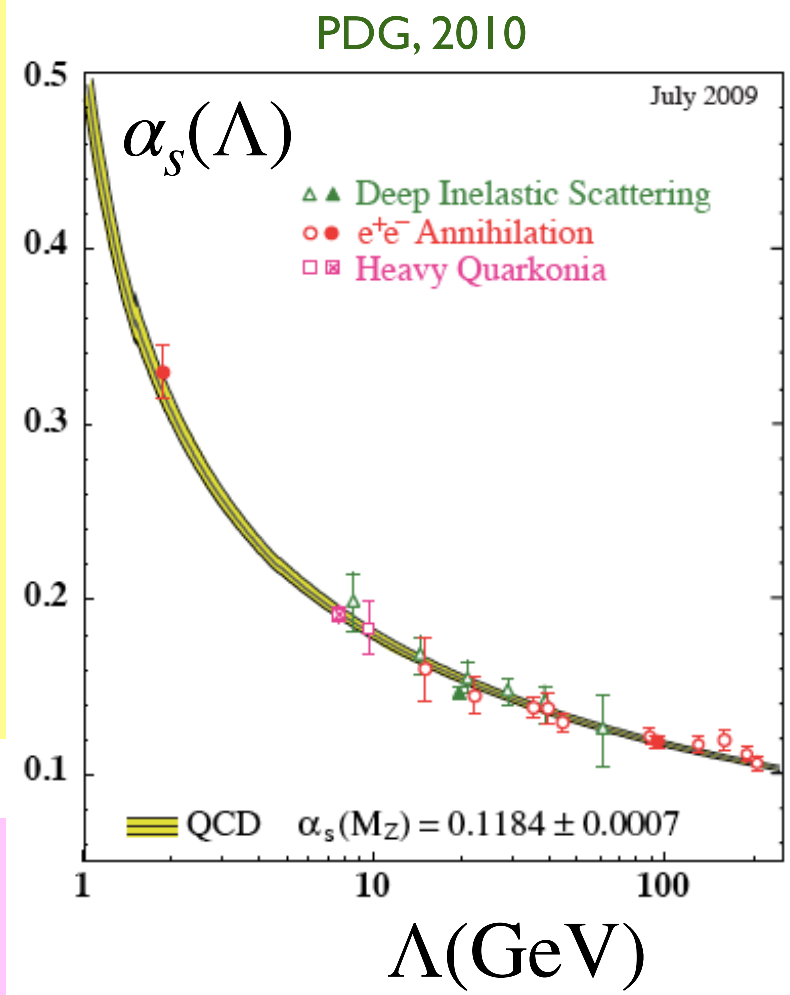
~ 1 GeV

Nonperturbative region

$\Lambda_{\rm QCD} \approx 300$ MeV
$\approx 1/\text{fm}$

Confinement





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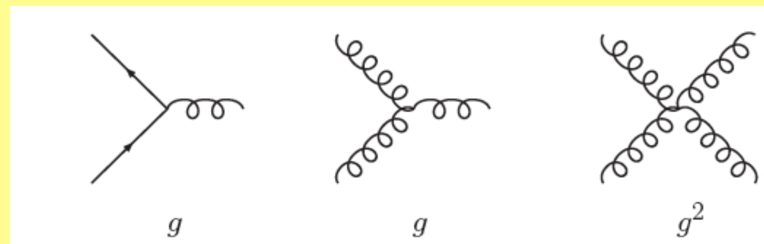

Perturbative region

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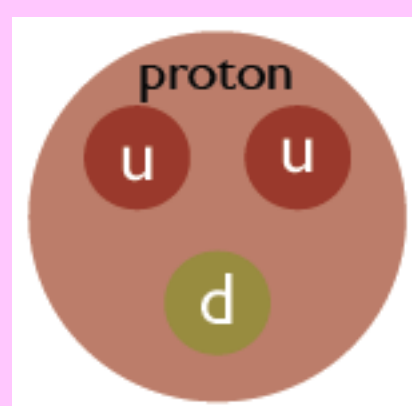
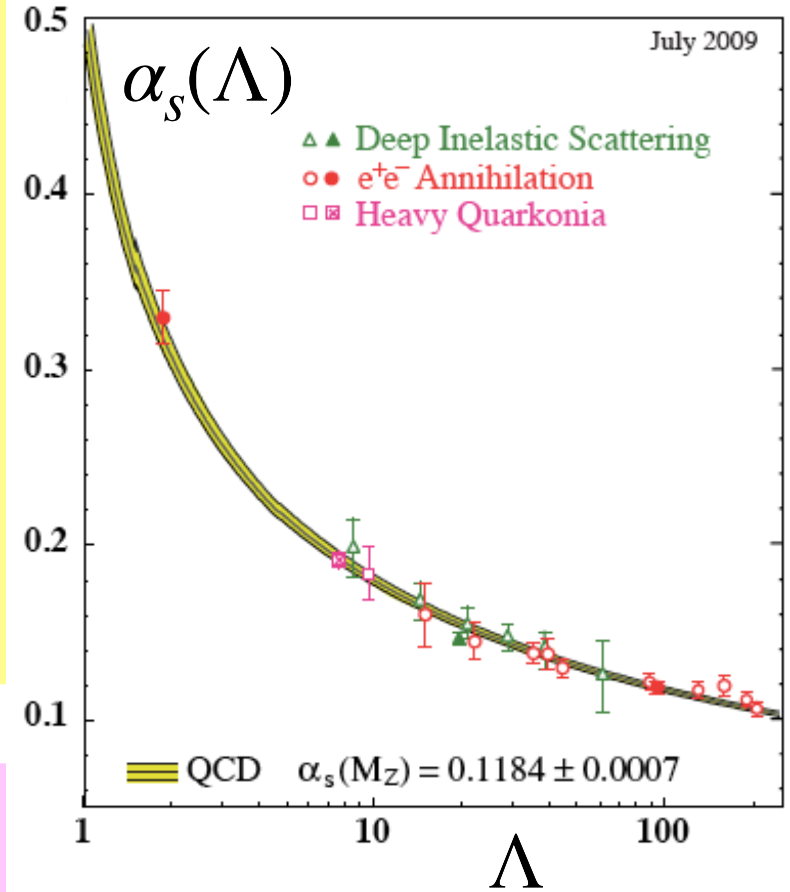


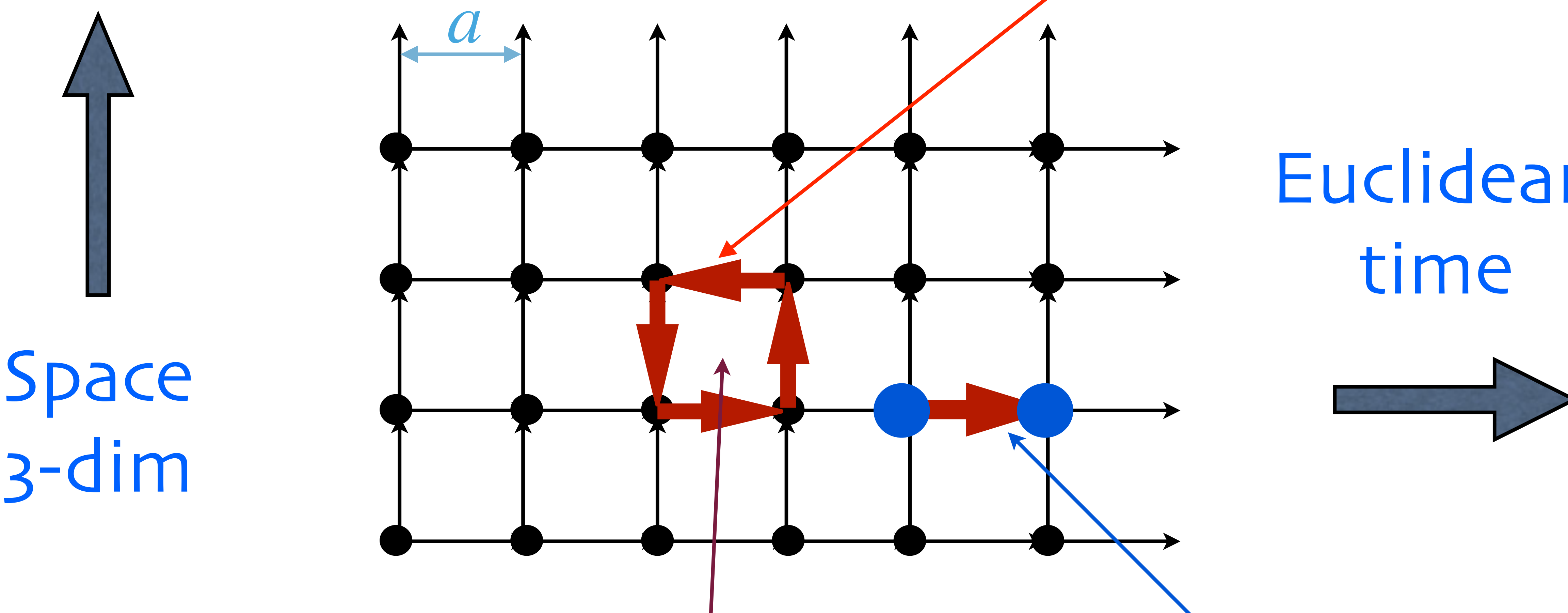
PDG, 2010



How do we test this ?
Can we control the $a \to 0$ limit?

Lattice QCD

$U_{n,\nu} \simeq e^{iagG^b_{n,\nu}T^b} \in SU(3)$



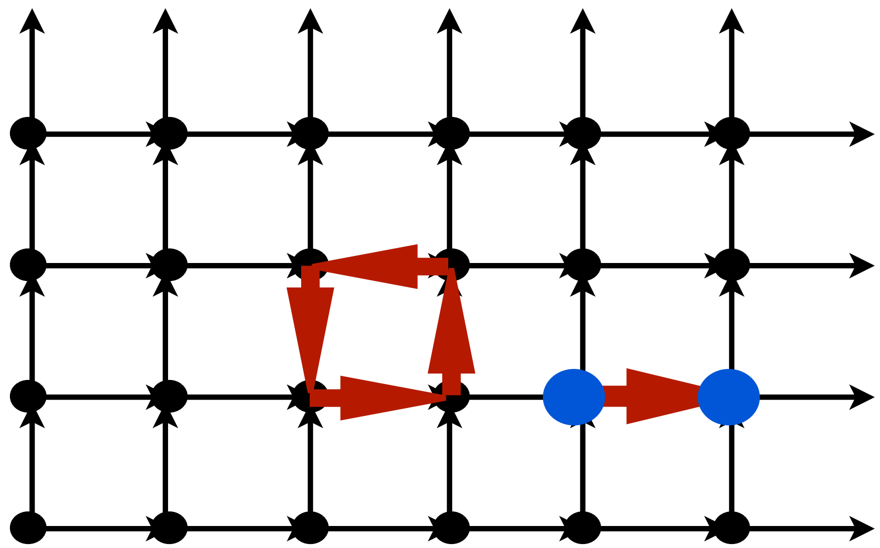
$$S_E^{\mathrm{latt}} = -\sum_{\Box} \frac{6}{g^2} \mathrm{Re}\,\mathrm{tr}_N(U_{\Box,\mu\nu}) - \sum_q \bar{q}(D_\mu^{\mathrm{lat}}\gamma_\mu + am_q)q$$

Wilson gauge action

Lattice fermion action

Gauge symmetry preserved; Nonperturbative formulation

Quantizing Lattice QCD



Use Feynman path integral definition of QM

Grassmann variables

$$Z_{\mathrm{QCD}} = \int \prod dU\, d\bar{q}\, dq\, e^{-S_E^{\mathrm{lat}}}$$

No need for gauge fixing

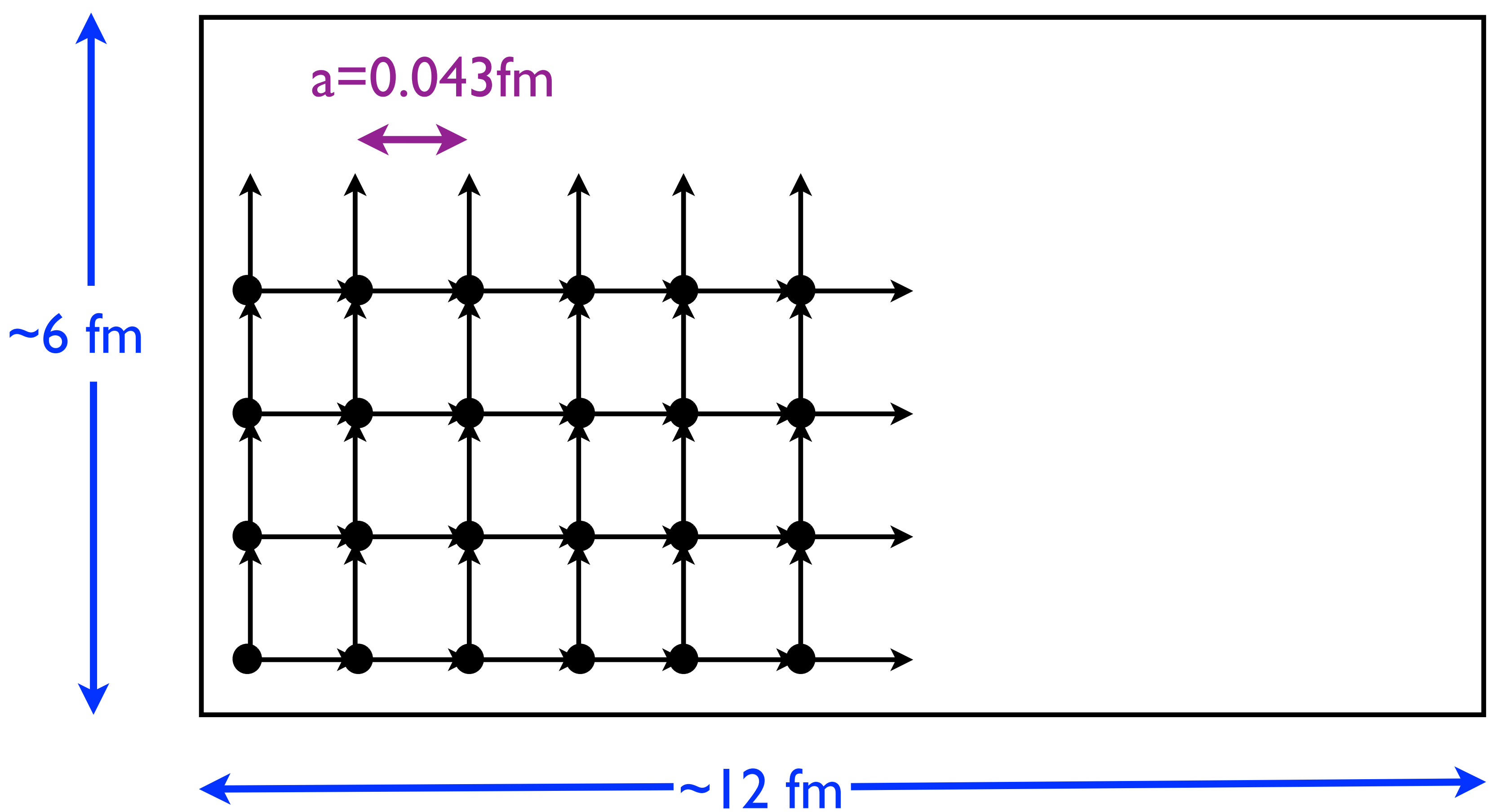
$$= \int dU\, e^{-S_{\mathrm{glue}}^{\mathrm{lat}}} \prod_q \det\left(D_\mu^{\mathrm{lat}} \gamma_\mu + m\right)$$

- Countably-infinite set of coupled QM systems
- In classical continuum limit action goes over to desired continuum form
    - **Define** QFT as $a \to 0$ limit of this QM system

Lattice QCD in practice

- Restrict to finite volume ⇒ finite QM system
- Using Monte-Carlo methods to simulate
  - State of the art: $144^3 \times 288$ lattice [MILC collaboration]

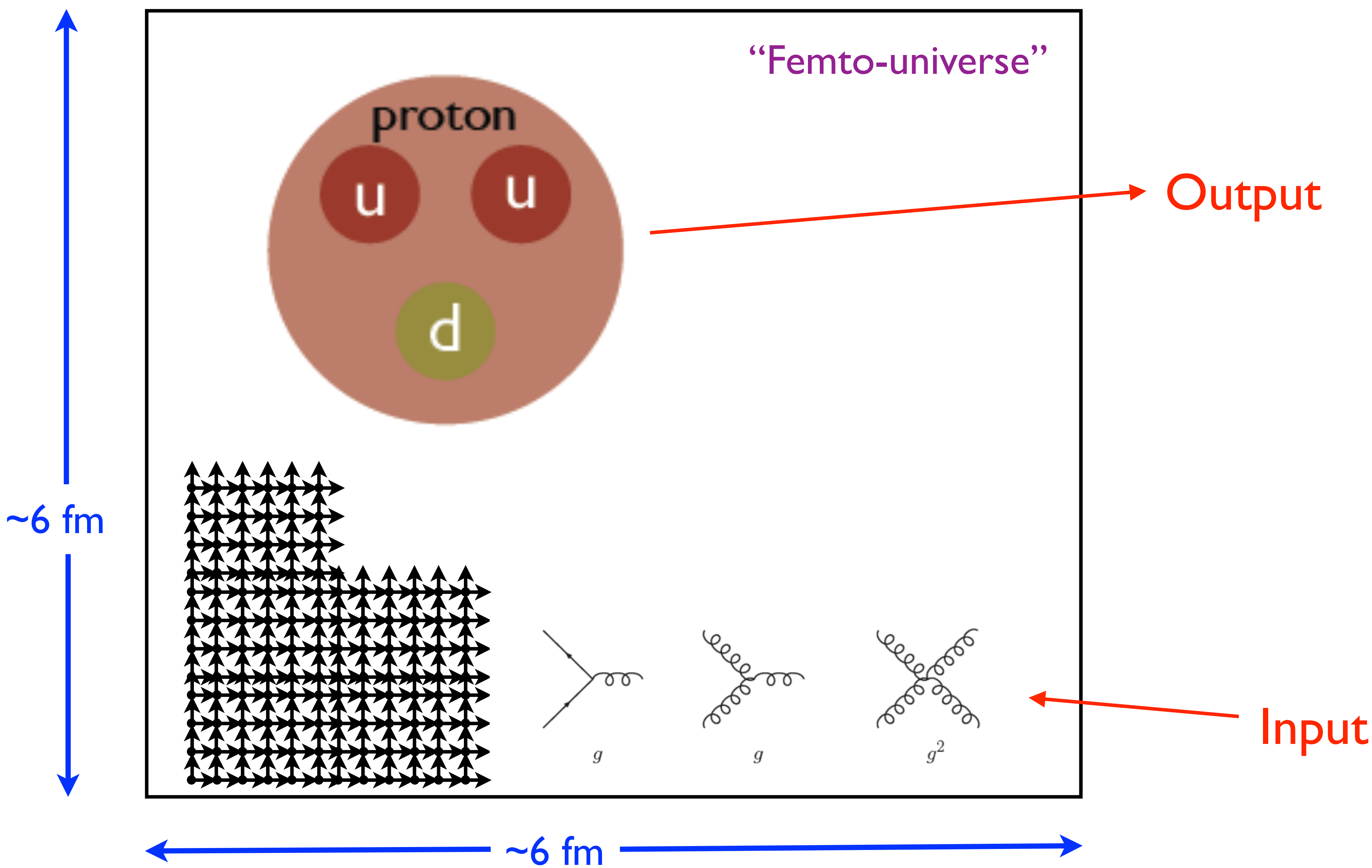
Highly Improved Staggered (HISQ) fermions
Physical quark masses (in isospin limit: $m_u=m_d$)



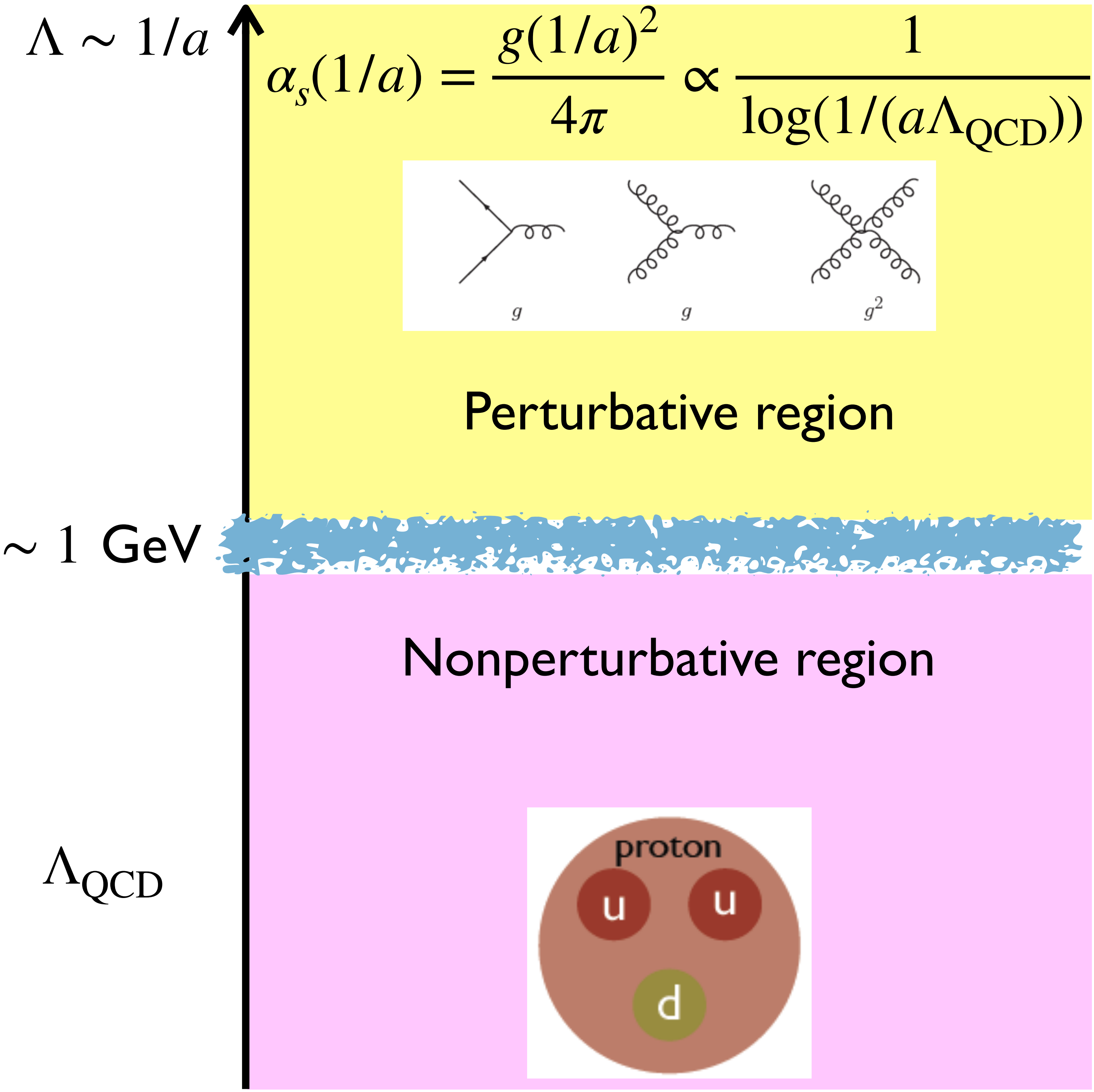
Need to invert matrices of size ~ $(3\mathrm{x}10^9) \times (3\mathrm{x}10^9)$

Wilsonian view again

- Finite-volume physics must accommodate and reproduce both short-distance weakly interacting quarks & gluons AND long-distance confinement



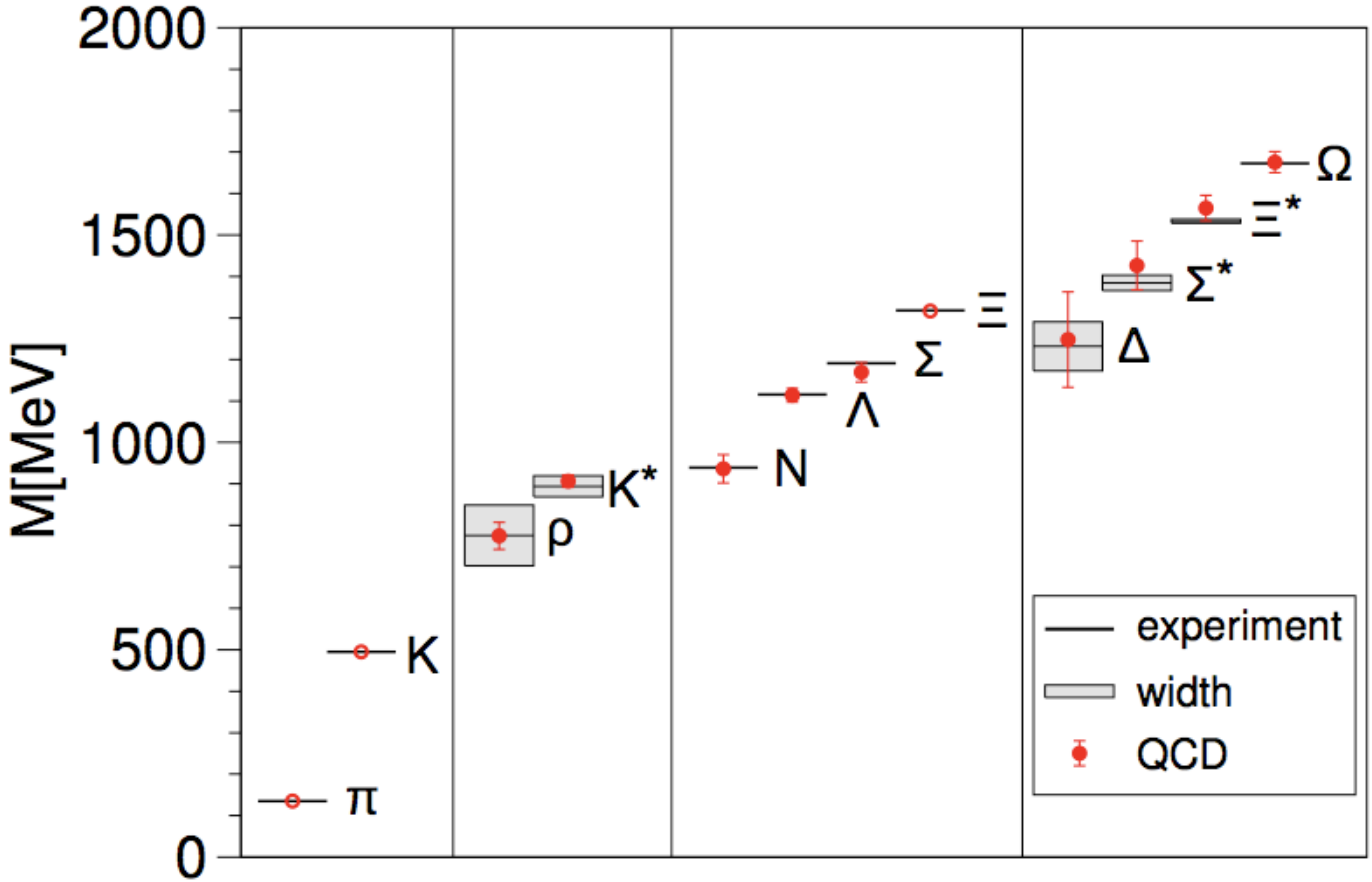
Key questions



- SCALING. Can we reduce $a \to 0$ (by varying g appropriately) while the long-distance non-perturbative physics remains fixed?
- ASYMPTOTIC SCALING. If scaling holds does the required $g(1/a)$ depend on a in the manner predicted by perturbation theory?
- CORRECTIONS TO SCALING. Can we predict the form of the a^n corrections to scaling, and do the numerical results agree with the predictions?
- Do the results agree with nature?

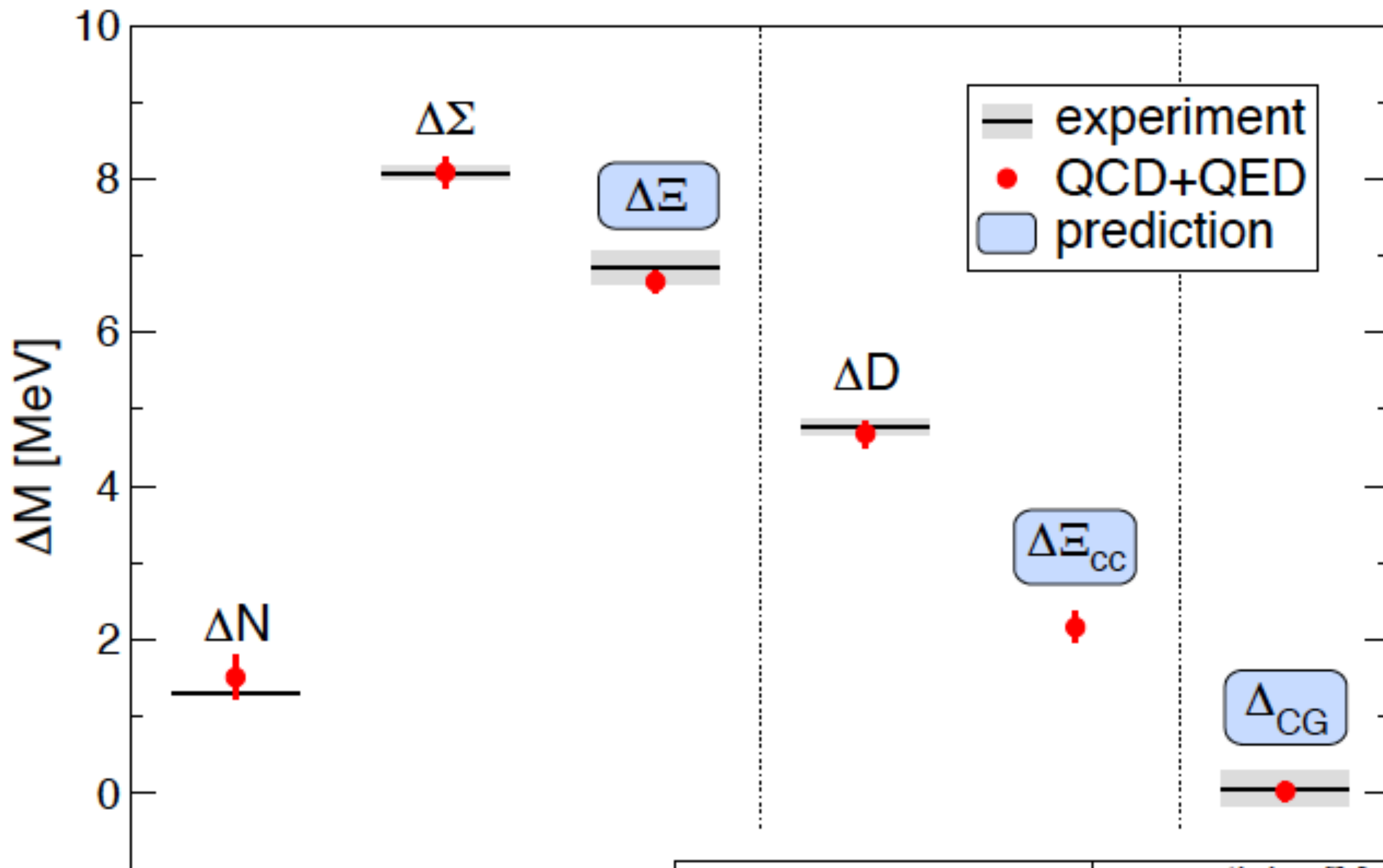
Agreement with nature

- Spectrum of light hadrons in QCD with $m_u = m_d$ (after $a \to 0$ extrapolation)



BMW collab. Science 322 (2008) 1224

Isospin splittings



BMW Collaboration
2014

u, d, s & c in loops
$m_u \neq m_d$
QED included

quark masses & scale
determined using
π^+, K^+, K^0, D^0, Ω

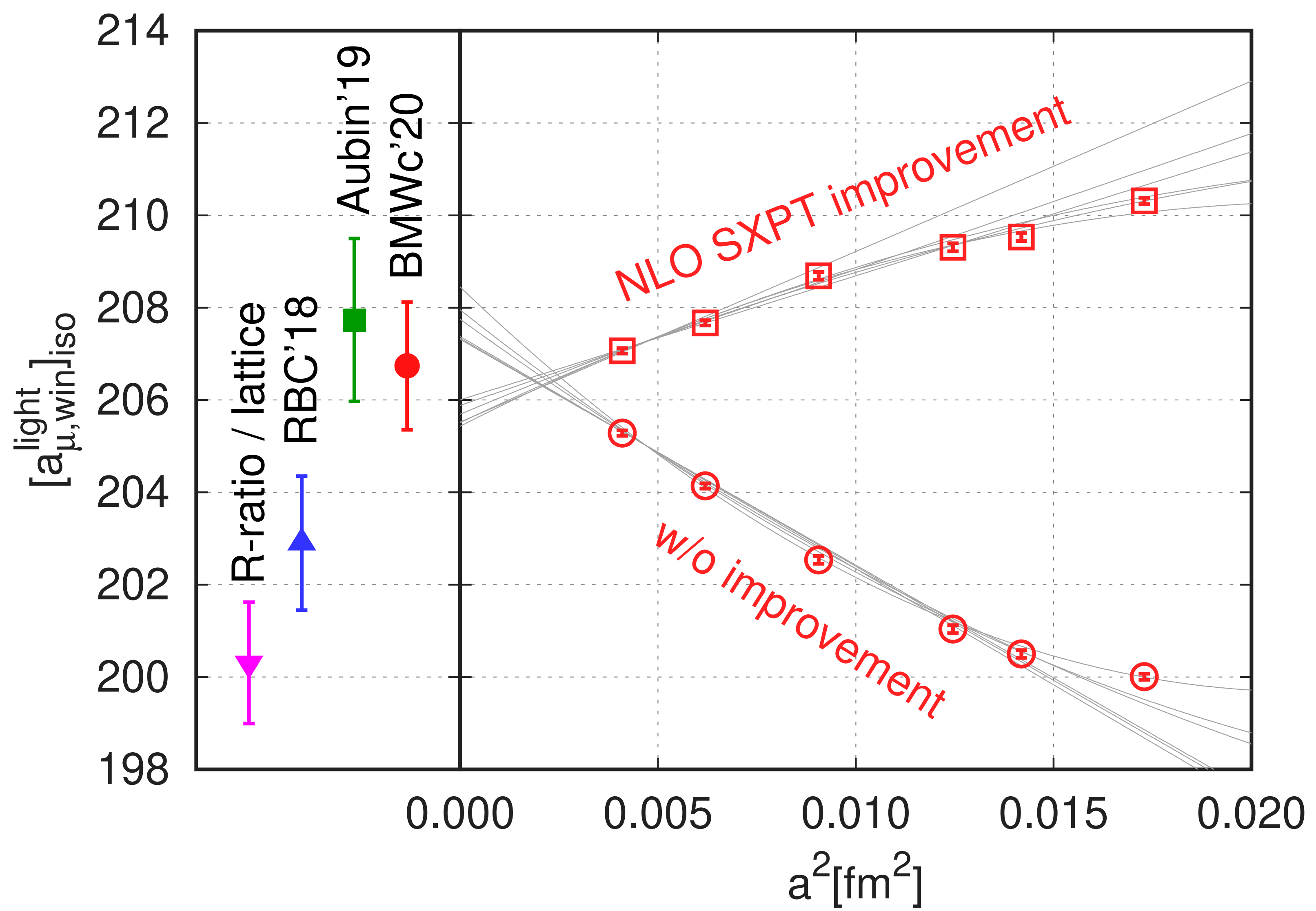
Errors ~ 0.2 MeV !

	mass splitting [MeV]	QCD [MeV]	QED [MeV]
$\Delta N = n - p$	1.51(16)(23)	2.52(17)(24)	-1.00(07)(14)
$\Delta\Sigma = \Sigma^- - \Sigma^+$	8.09(16)(11)	8.09(16)(11)	0
$\Delta\Xi = \Xi^- - \Xi^0$	6.66(11)(09)	5.53(17)(17)	1.14(16)(09)
$\Delta D = D^\pm - D^0$	4.68(10)(13)	2.54(08)(10)	2.14(11)(07)
$\Delta\Xi_{cc} = \Xi_{cc}^{++} - \Xi_{cc}^{+}$	2.16(11)(17)	-2.53(11)(06)	4.69(10)(17)
$\Delta_{CG} = \Delta N - \Delta\Sigma + \Delta\Xi$	0.00(11)(06)	-0.00(13)(05)	0.00(06)(02)

Corrections to scaling

- Generic expectation with "improved actions": dimensionless quantities approach continuum limit as $c_1 a^2 + c_2 a^4 + \ldots$ (up to $\log(a)$ corrections), with c_i unknown

- Example from recent LQCD calculation of anomalous magnetic moment of muon



BMW collab. Nature (2021) arXiv:2002.12347

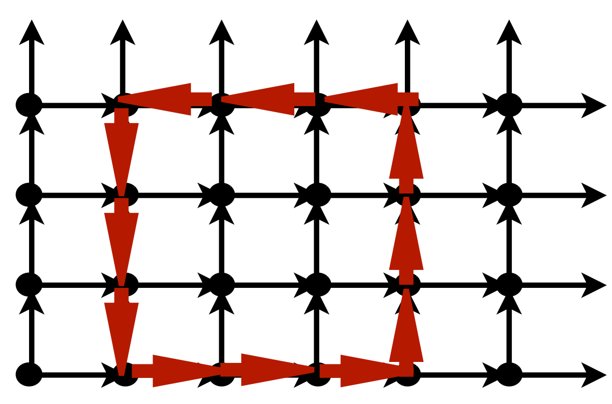
Evidence for asymptotic scaling

- Does bare coupling $g(1/a)$ run with a as predicted by perturbation theory (PT) ?
    - Accuracy of PT improves as a decreases (asymptotic freedom)

$$\frac{dg(1/a)}{d\log(a)} = \beta_0 g(1/a)^3 + \beta_1 g(1/a)^5 + \mathcal{O}(g^7) + \mathcal{O}(a^2)$$

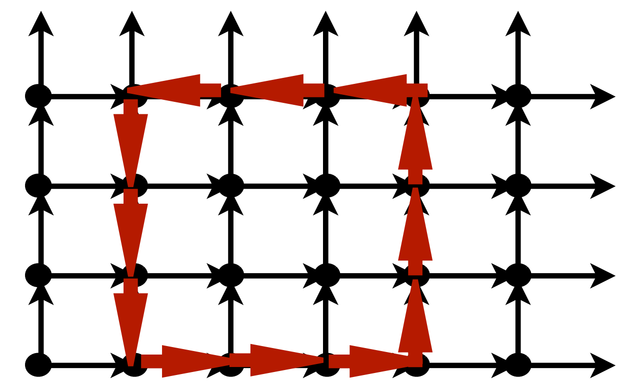
β_0, β_1 known and both positive

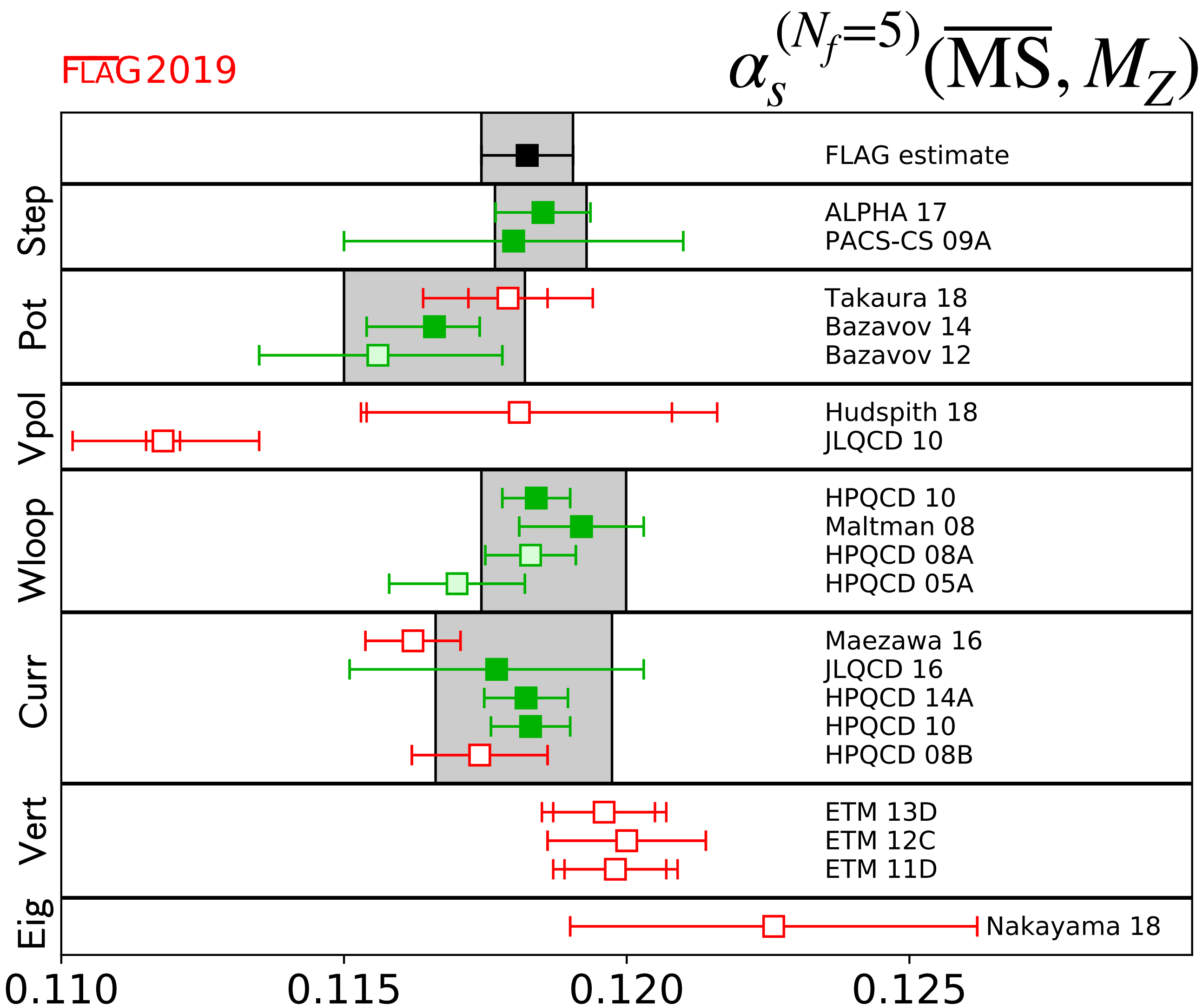
    - Tricky issue as there are a^n corrections that must be disentangled
- Can test indirectly by checking that PT correctly reproduces short-distance observables
    - Wilson loops, short-distance correlation functions, …



Evidence for asymptotic scaling

- Can test indirectly by checking that PT correctly reproduces short-distance observables determined using lattice QCD
    - Wilson loops, short-distance correlation functions, …

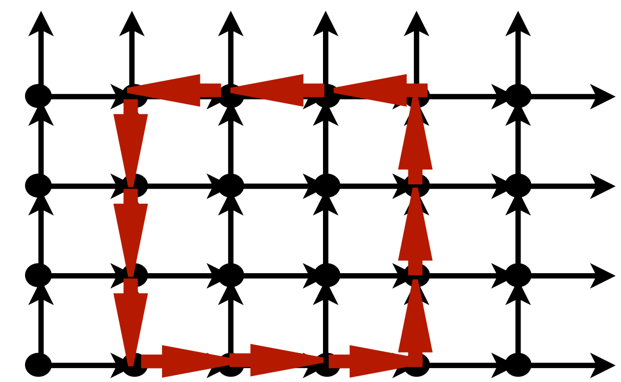




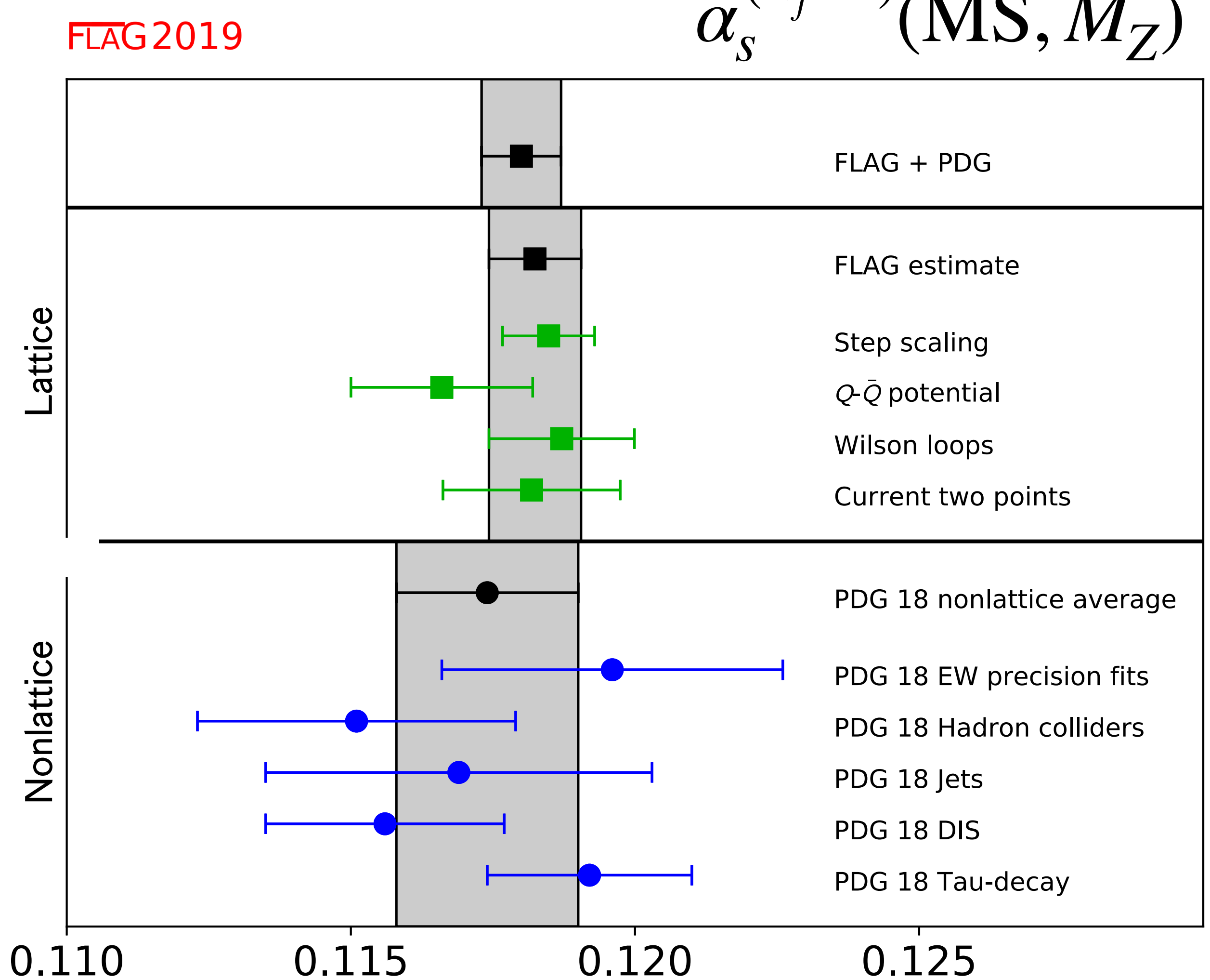
Methods based on different quantities, and a range of scales above $\sim 2\ \mathrm{GeV}$, give consistent results

Evidence for asymptotic scaling

- Can test indirectly by checking that PT correctly reproduces short-distance observables determined using lattice QCD
    - Wilson loops, short-distance correlation functions, …



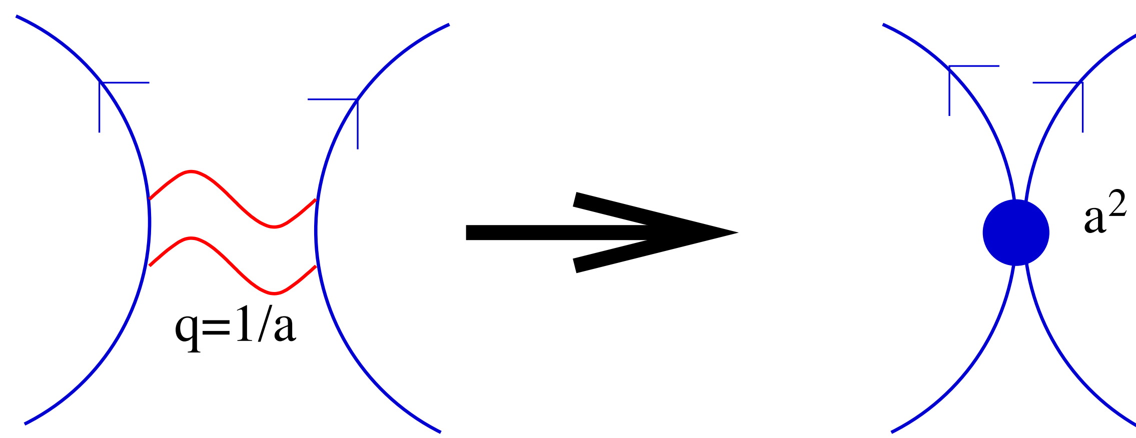
$\alpha_s^{(N_f=5)}(\overline{\mathrm{MS}}, M_Z)$



Lattice-based result agrees with those from high-energy experiments fit to perturbative QCD expressions

Theory of scaling violations

- Based on Symanzik effective field theory (SEFT), known to be valid to all orders in PT
  - Continuum EFT that reproduces the discretization effects of the lattice theory



$g = g(1/a)$

$$\mathscr{L}_{\mathrm{SEFT}} = \mathscr{L}_{\mathrm{QCD}} + a^2 \sum_i c_i(g) O_i + \mathcal{O}(a^4)$$

All operators of dimension 6 consistent with symmetries of lattice QCD

$$\mathscr{L}_{\mathrm{QCD}} = -\frac{1}{2}\mathrm{tr} \sum_{\mu\nu} \left(G_{\mu\nu} G_{\mu\nu} \right)$$

$$O_1 = \sum_{\mu\nu\rho} \mathrm{Tr}\left([D_\mu, G_{\nu\rho}][D_\mu, G_{\nu\rho}] \right), \quad O_2 = \sum_{\mu\nu} \mathrm{Tr}\left([D_\mu, G_{\mu\nu}][D_\mu, G_{\mu\nu}] \right), \quad \ldots$$

State of the art

- Result for physical (dimensionless) quantities in renormalization group improved PT

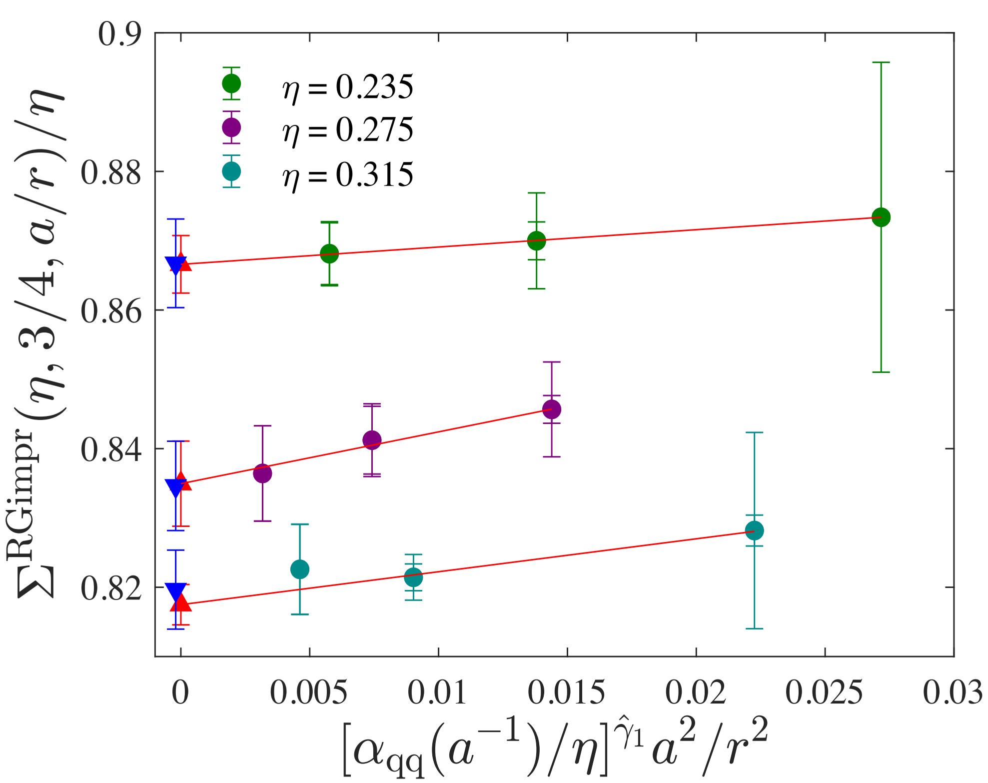
$$Q(a) = Q(0) + \sum_i d_i^{(0)} a^2 g(1/a)^n g(1/a)^{2\gamma_i}\left[1 + \mathcal{O}(g^2)\right] + \ldots$$

unknown coefficients

n>0 if calculation "improved"

Operator-dependent logarithmic correction to scaling, determined by anomalous dimension γ_i, which is known in PT

$$\left[\frac{1}{\log(1/(a\Lambda_{\mathrm{QCD}}))}\right]^{\gamma_i}$$



Husung, Nada, Sommer, PoS(Lattice2019)263

Example of fitting including logarithmic correction to power-law scaling in QCD without fermions

Closing comments

- Asymptotic freedom (AF) allows theoretical control of continuum limit, assuming no short-distance nonperturbative effects
  - Supported by numerical evidence
  - Extends to other AF theories (varying numbers of colors and fermions)
- Corrections to scaling in lower-dimensional theories (e.g. O(3) sigma model) studied in great detail
  - Logarithmic corrections turn a^2 dependence almost into a linear dependence
- For SUSY theories, there are many exact results assuming the theory exists
  - Hard to confirm with lattice methods, although some progress
- QIS approaches that aim to solve sign problems in lattice theories (e.g. studying real time processes) are based on the Hamiltonian approach with spatial discretization
  - Face the same issue of understanding the spatial continuum limit

Thank you.
Questions?