Three-particle interactions from the lattice: a progress report

Steve Sharpe
University of Washington

Outline

- Motivations for studying 3 (or more) particles
- Status of formalism for (2 &) 3 particles
  - Examples of implementations
- Alternative derivation & new form of three-particle quantization condition (QC3)
- Equivalence of different QC3s
- Conclusions & outlook

3-particle papers

Max Hansen & SRS:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]

<u>Raúl Briceño, Max Hansen & SRS</u>:

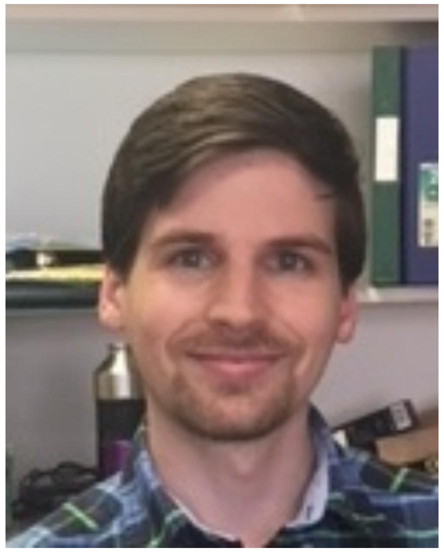
"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]

"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

<u>SRS</u>

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory," arXiv:1707.04279 (PRD) [SPT17]



<u>Tyler Blanton, Fernando Romero-López & SRS:</u>

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]

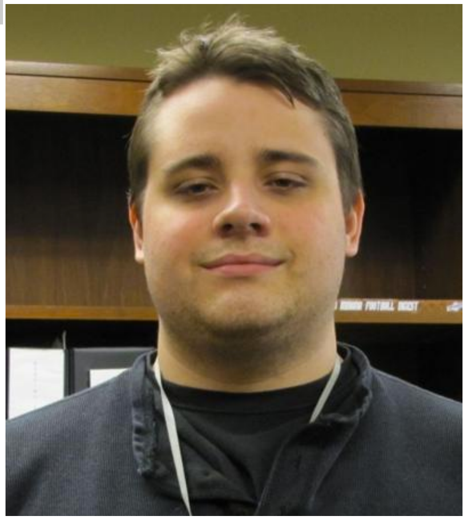
"I=3 three-pion scattering amplitude from lattice QCD," arXiv:1909.02973 (PRL) [BRS-PRL19]

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP) [BBHRS19]

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)



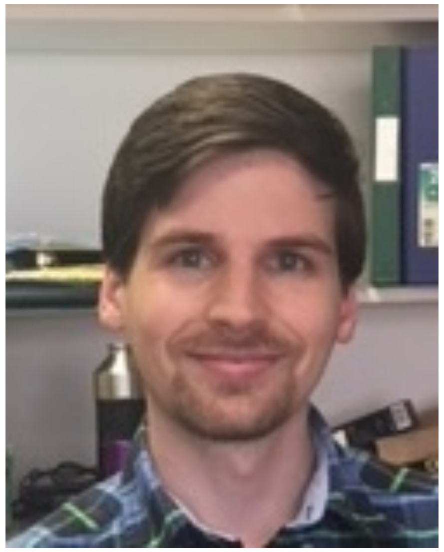
Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

"Generalizing the relativistic quantization condition to include all three-pion isospin channels", arXiv:2003.10974 (JHEP) [HRS20]

Focus for today



Tyler Blanton & SRS:

"Alternative derivation of the relativistic three-particle quantization condition,"

arXiv:2007.16188 (to appear in PRD) [BS20a]

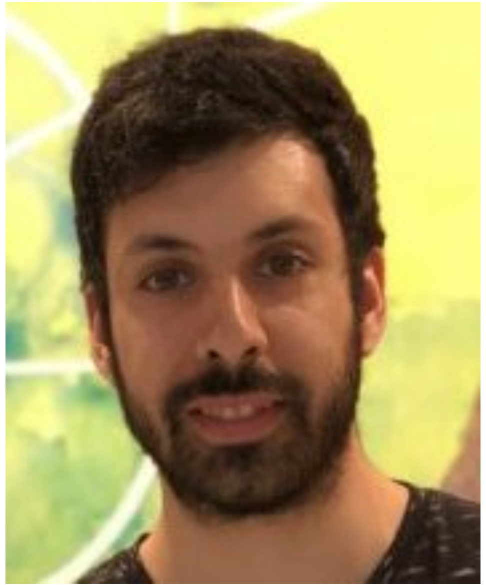
"Equivalence of relativistic three-particle quantization conditions,"

arXiv:2007.16190 (PRD under review) [BS20b]

Tyler Blanton, Drew Hanlon, Ben Hörz, Fernando Romero-López & SRS

"$3\pi^+$ & $3K^+$ interactions beyond leading order from lattice QCD,"

Work in progress



Motivations for studying three (or more) particles using LQCD

Determining resonance properties

- Most resonances have 3 (or more) particle decay channels
  - $\omega(782, I^G J^{PC} = 0^- 1^{--}) \to 3\pi$ (no subchannel resonances)
  - $a_2(1320, I^G J^{PC} = 1^- 2^{++}) \to \rho\pi \to 3\pi$
  - Roper: $N(1440) \to \Delta\pi \to N\pi\pi$ (branching ratio 25-50%)
  - $X(3872) \to J/\Psi\pi\pi$
  - $Z_c(3900) \to \pi J/\psi, \pi\pi\eta_c, \bar{D}D^*$ (studied by HALQCD)
- N.B. If a resonance has both 2- and 3-particle strong decays, then 2-particle methods fail—channels cannot be separated as they can in experiment

Predicting weak decay amplitudes

- Calculating weak decay amplitudes/form factors involving 3 particles, e.g. $K \to \pi\pi\pi$

- N.B. Can study weak $K \to \pi\pi$ decays independently of $K \to \pi\pi\pi$, since strong interactions do not mix these final states (in isospin-symmetric limit)

- Long-term goal is to develop methods to predict CP violation in $D \to \pi\pi, K\overline{K}, (\pi\pi\pi\pi), \ldots$ (as measured by LHCb in 2019)

Determining 3-body interactions

- Determining NN & NNN interactions
  - Input for effective field theory treatments of larger nuclei & nuclear matter
  - NNN interaction important for determining properties of neutron stars
- Similarly, $\pi\pi\pi$, $\pi K\bar{K}$, … interactions needed for study of pion/kaon condensation

The time is now!

Two- and three-pion finite-volume spectra at maximal isospin from lattice QCD

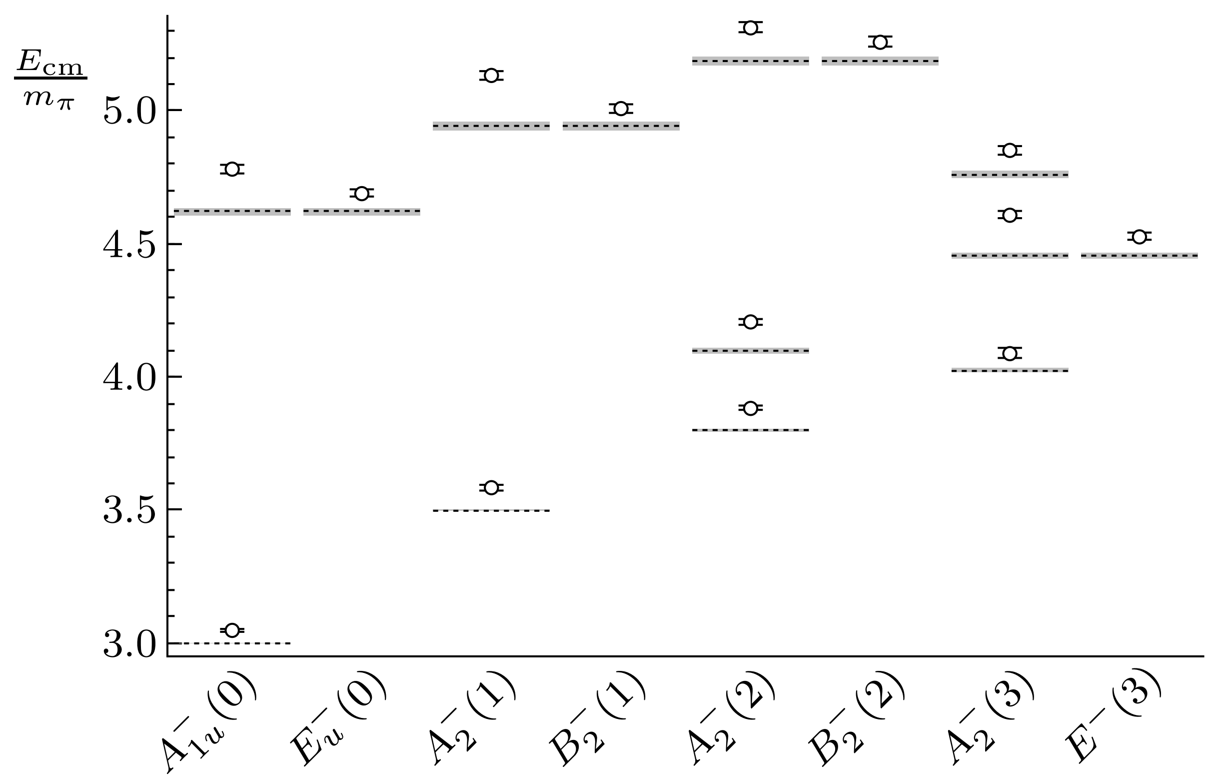
[arXiv:1905.04277]

Ben Hörz*
Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

Andrew Hanlon†
Helmholtz-Institut Mainz, Johannes Gutenberg-Universität, 55099 Mainz, Germany

(Dated: May 13, 2019)

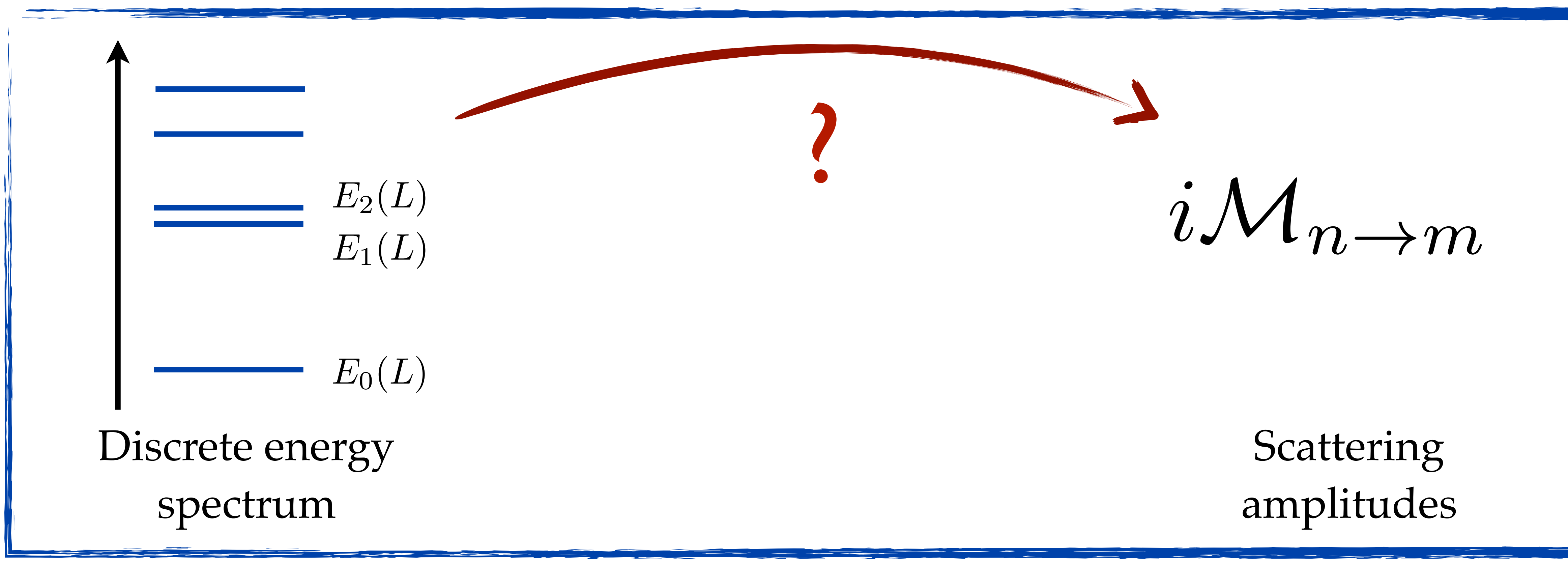
We present the three-pion spectrum with maximum isospin in a finite volume determined from lattice QCD, including, for the first time, excited states across various irreducible representations at zero and nonzero total momentum, in addition to the ground states in these channels. The required correlation functions, from which the spectrum is extracted, are computed using a newly implemented algorithm which reduces the number of operations, and hence speeds up the computation by more than an order of magnitude. The results for the $I = 3$ three-pion and the $I = 2$ two-pion spectrum are publicly available, including all correlations, and can be used to test the available three-particle finite-volume approaches to extracting three-pion interactions.



Status of formalism for (2 &) 3 particles

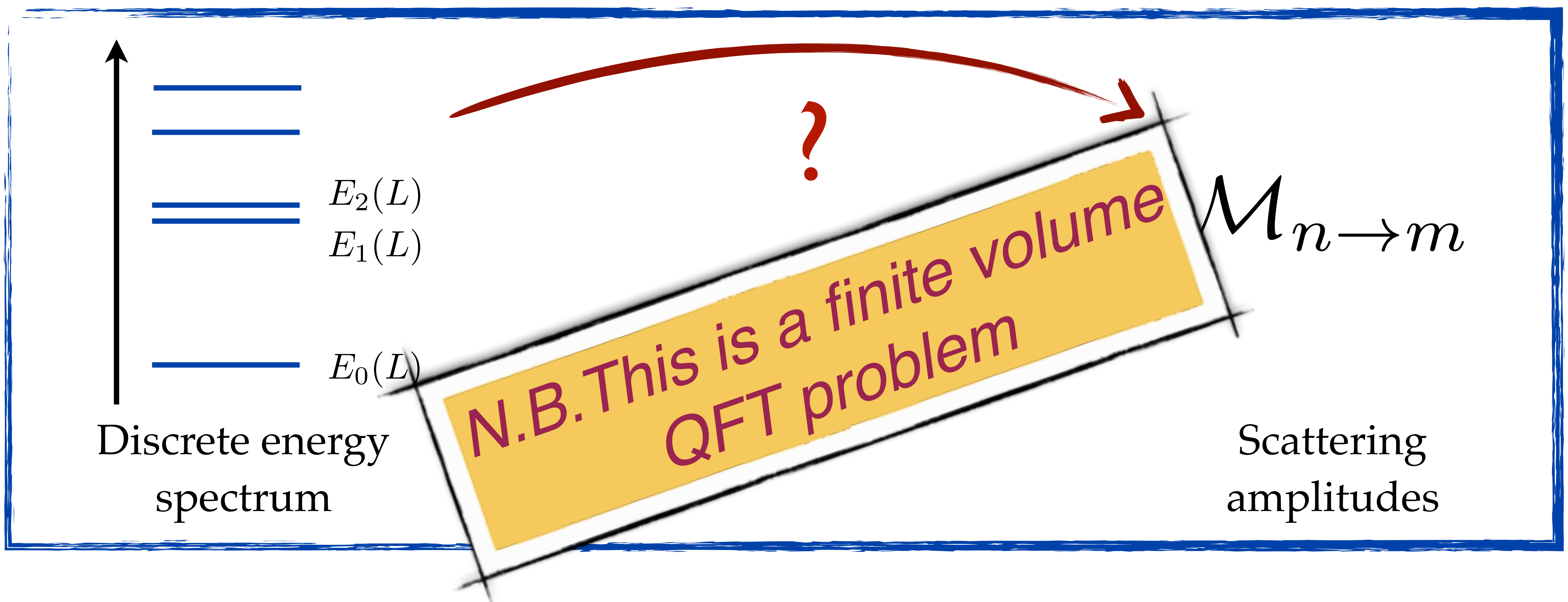
The fundamental issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?



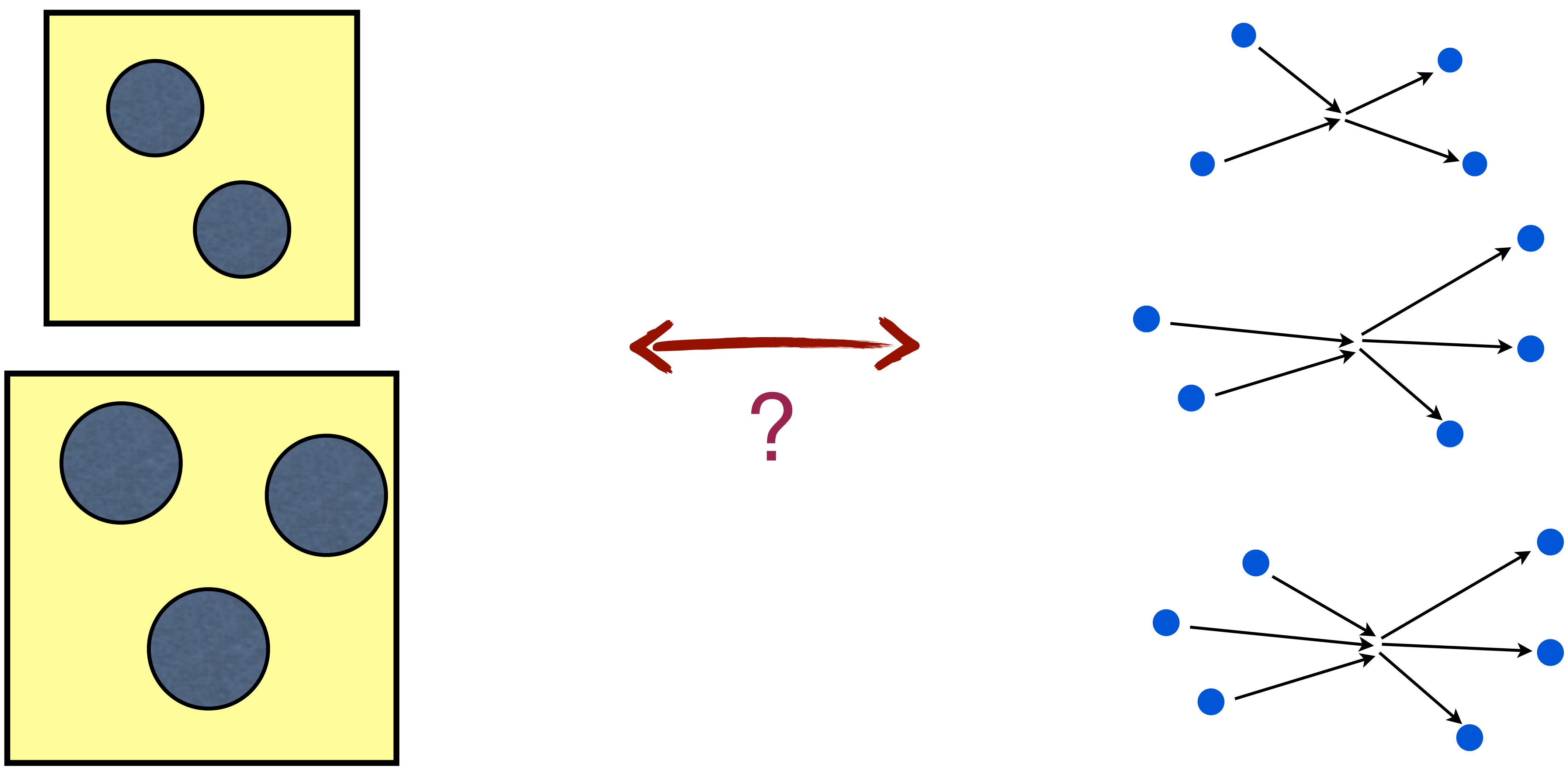
The fundamental issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box
- How are these related to infinite-volume scattering amplitudes (which determine resonance properties)?



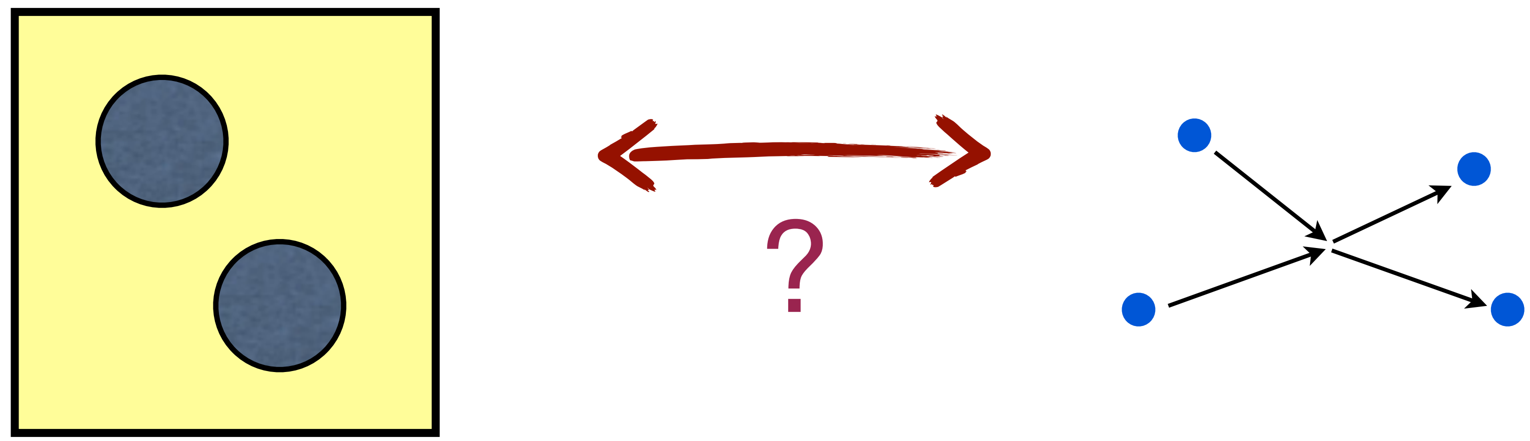
The fundamental issue

- Lattice simulations are done in finite volumes; experiments are not



How do we connect these?

2-particle quantization condition



- Two particles in cubic box of size L with PBC and total momentum **P**
- Below inelastic threshold (4 pions if have Z_2 symmetry), the finite-volume spectrum E_1, E_2, ... is given by solutions to an equation in partial-wave (*l*,*m*) space

QC2: $$\det\left[F_{PV}(E,\mathbf{P},L)^{-1} + \mathcal{K}_2(E^*)\right] = 0$$

[Lüscher 86 & 91; Rummukainen & Gottlieb 85; Kim, Sachrajda & SRS 05; …]

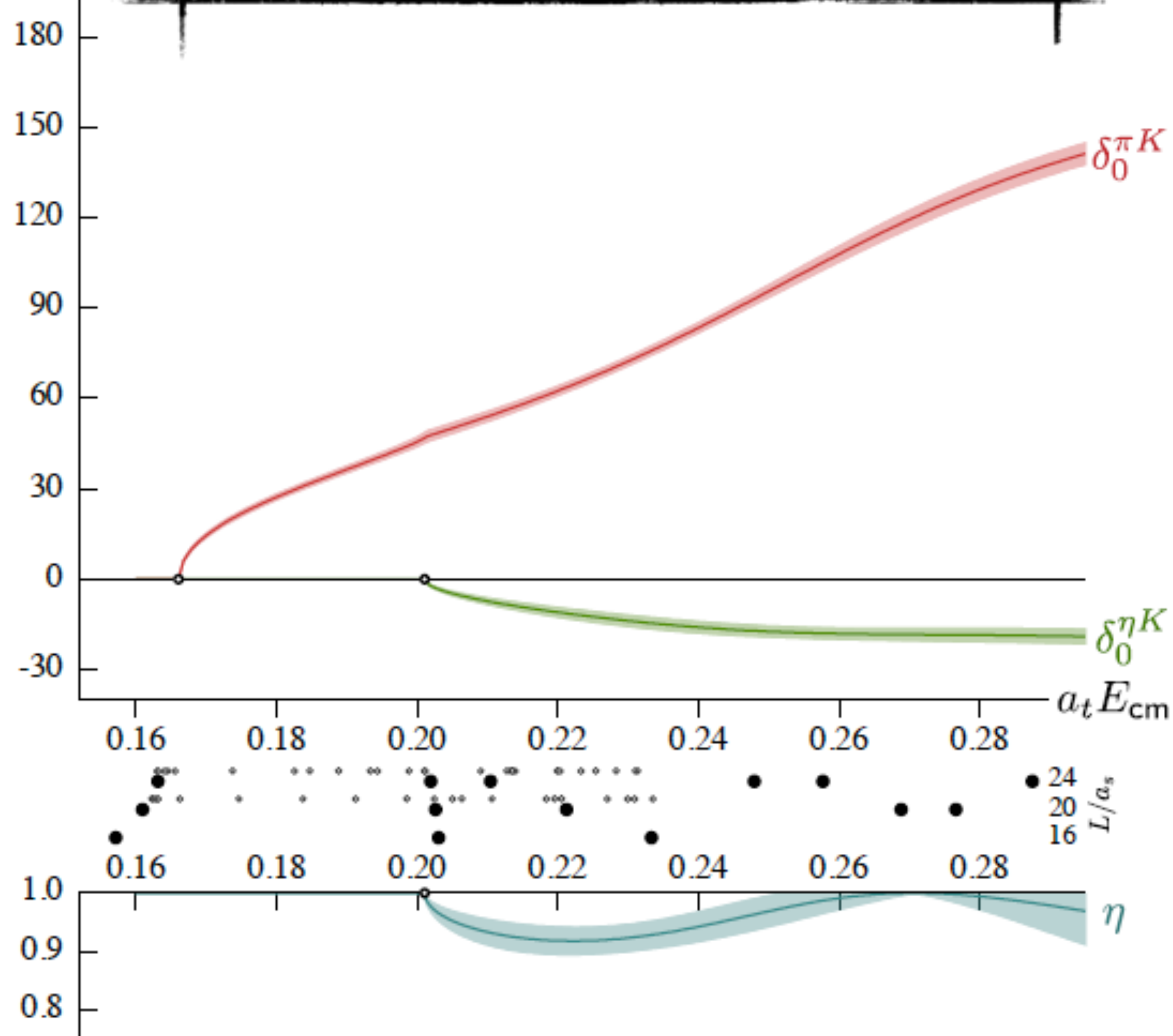
- $\mathcal{K}_2 \sim \tan\delta/q$ is the two-particle K-matrix, which is diagonal in *l*,*m*
- F_{PV} is a known kinematical "zeta-function", depending on the box shape & E; It is off-diagonal in l,m, since the box violates rotation symmetry
- Valid up to corrections $\sim e^{-ML}$
- Generalized to arbitrary masses, spins and multiple channels

State of the art: coupled 2-body channels

$$\boxed{\det\left[(F_{\rm PV})^{-1}+\mathcal{K}_2\right]=0}$$

Same form of quantization condition holds, but matrices include extra channel index

[He, Feng, Liu 05;
Meißner et al. 09-11;
Briceño & Davoudi 12;
Hansen & SRS 12]



[Dudek, Edwards, Thomas & Wilson 14]

Practical implementation requires truncation of ℓ, m indices

Status for three particles

- Applied so far only to systems of three spin-0 particles
- Three approaches
  - All orders diagrammatic derivation in generic relativistic EFT (RFT) [Hansen & SRS 14; Briceño, Hansen & SRS 17; …]
    - Control all sources of $1/L^n$ volume dependence, while neglecting terms $\propto e^{-ML}$
    - Complicated derivation, but general result: holds for all 2-particle ("dimer") partial waves
    - Originally derived for 3 identical particles with $\mathbb{Z}_2$ symmetry; generalized to allow $2 \leftrightarrow 3$ transitions, and nonidentical but degenerate particles (e.g. 3 pions with any allowed isospin)
  - Nonrelativistic EFT [Hammer & Rusetsky 17; …]
    - Greatly simplified derivation; applied so far only for s-wave dimers; nonrelativistic kinematics
  - ``Finite-volume unitarity" (FVU) [Mai & Döring 17; …]
    - Based on infinite-volume unitary representation of three-particle amplitude $\mathscr{M}_3$ in terms of R matrix (generalization of $\mathscr{K}_2$)
    - Relativistic, but obtained so far only for s-wave dimers

Refs for alternate approaches

★ **NREFT approach**

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, 1706.07700, JHEP & 1707.02176, JHEP [Formalism & examples]
- M. Döring et al., 1802.03362, PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111, PRD [large volume expansion for excited levels]

★ **Finite-volume unitarity (FVU) approach**

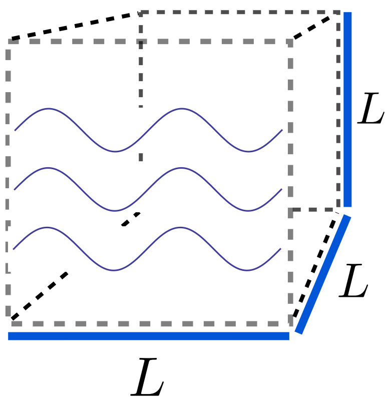
- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of M_3 involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749,PRD [applying FVU approach to $3\pi^+$spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]

★ **HALQCD approach**

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]

Two-step method

2 & 3 particle
Spectra from LQCD



Quantization conditions

QC2: $\det\left[F_2^{-1} + \mathscr{K}_2\right] = 0$

QC3: $\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$

[These are the RFT forms, and assume $\mathbb{Z}_2$ symmetry]

Intermediate, unphysical scattering quantity

$\mathscr{K}_{\mathrm{df},3}$

Integral equations in infinite volume

Scattering amplitude

$\mathscr{M}_3$

QC2 $\longrightarrow$ QC3 [HS14]

$$\boxed{\det\left[F_{\rm PV}(E,\vec{P},L)^{-1}+\mathcal{K}_2(E^*)\right]=0} \rightarrow \boxed{\det\left[F_3(E,\vec{P},L)^{-1}+\mathcal{K}_{\rm df,3}(E^*)\right]=0}$$

- Total momentum (E, **P**)
- Matrix indices are $l,\ m$
- $F_{\rm PV}$ is a finite-volume geometric function
- $\mathcal{K}_2$ is an infinite-volume amplitude, which is real and smooth (no threshold cusps)
- It is related algebraically to $\mathcal{M}_2$:

$$\frac{1}{\mathcal{M}_2^{(\ell)}} \equiv \frac{1}{\mathcal{K}_2^{(\ell)}} - i\rho$$

- Total momentum (E, **P**)
- Matrix indices are $\boldsymbol{k},\ l,\ m$
- F_3 depends on geometric functions (F_{PV} and G) and also on K_2
  - F_3 is known if first solve QC2
- $\mathcal{K}_{\rm df,3}$ is an infinite-volume 3-particle amplitude, which is real and smooth
- It is cutoff dependent and thus unphysical
- It is related to $\mathcal{M}_3$ via integral equations [HS15]

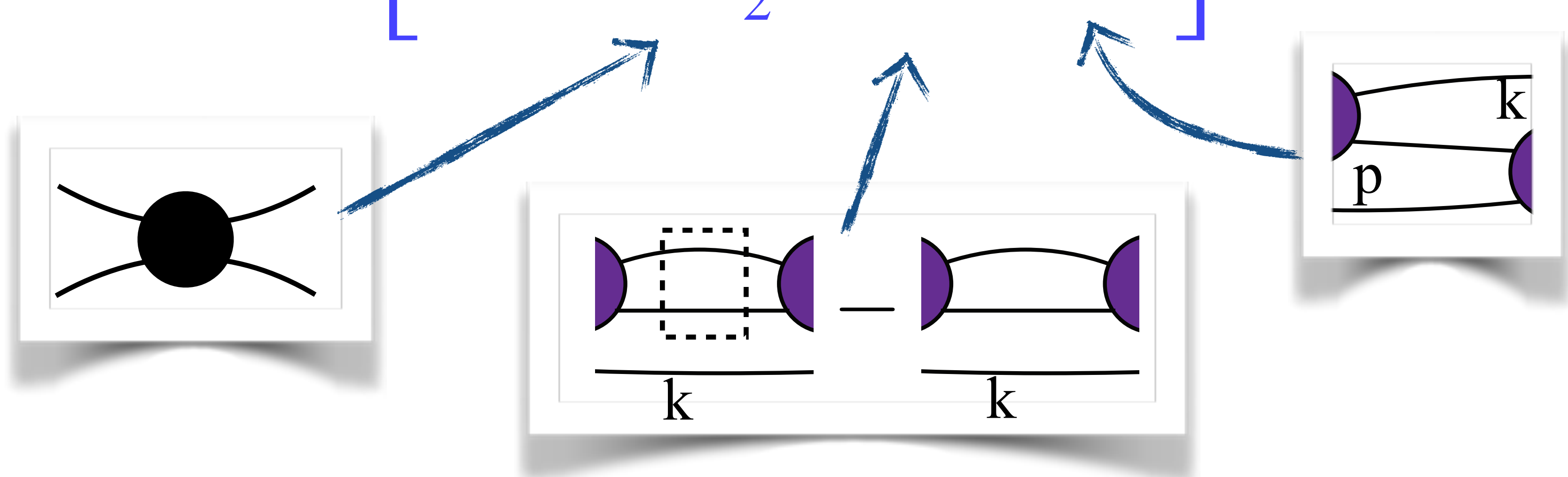
Further details of QC3

- All quantities are infinite-dimensional matrices with indices $\mathbf{k}\ell m$ describing 3 on-shell particles

[finite volume "spectator" momentum: **k**=2π**n**/L] x [2-particle CM angular momentum: *l,m*]

- F_3 contains two-particle interactions ($\mathscr{K}_2$) and kinematic functions (F & G)

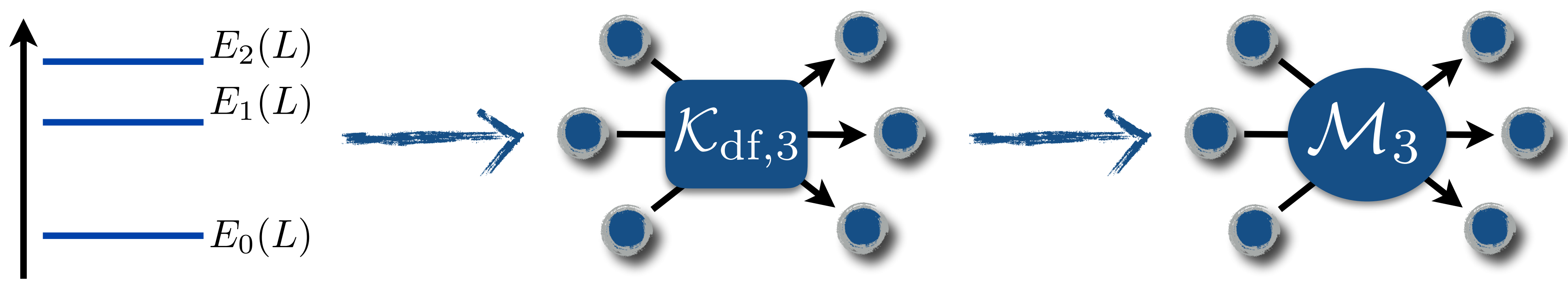
$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\frac{1}{\mathscr{K}_2^{-1}+F+G}F\right]$$



Status of RFT formalism

- Original work applied to scalars with $\mathbb{Z}_2$ symmetry & no subchannel resonances or 2-particle bound states (e.g. $3\pi^+$) [HS14, HS15]

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$



- Generalized PV prescription allows subchannel resonances & 2-particle bound states [BBHRS19]

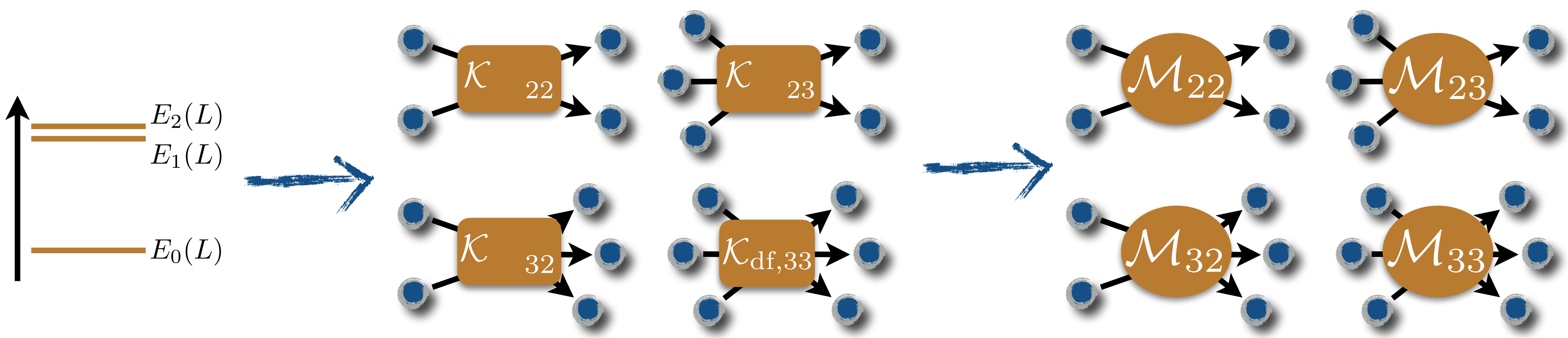
- Alternative more cumbersome approach given in [BHS19]

Status of RFT formalism

- [BHS17] removed G-parity constraint, allowing 2↔3 processes (step towards $N\pi \leftrightarrow N\pi\pi$)

F_2 appears in 2-particle quantization condition

$$\det\left[\begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix}^{-1} + \begin{pmatrix} \mathscr{K}_{22} & \mathscr{K}_{23} \\ \mathscr{K}_{32} & \mathscr{K}_{\mathrm{df},33} \end{pmatrix}\right] = 0$$



Status of RFT formalism

- [HRS20] generalized to distinguishable but degenerate particles (e.g. 3π with $I = 0,1,2$ in isosymmetric QCD)

$$\boxed{\det\left[1 - \mathbf{K}^{[I]}_{\mathrm{df},3}(E^\star)\,\mathbf{F}^{[I]}_3(E,\boldsymbol{P},L)\right] = 0}$$

$$\boxed{\mathbf{F}^{[I]}_3 \equiv \frac{\mathbf{F}^{[I]}}{3} + \mathbf{F}^{[I]}\frac{1}{1-\mathbf{M}^{[I]}_{2,L}\mathbf{G}^{[I]}}\mathbf{M}^{[I]}_{2,L}\mathbf{F}^{[I]} \qquad \mathbf{M}^{[I]}_{2,L} \equiv \frac{1}{\mathbf{K}^{[I]-1}_2 - \mathbf{F}^{[I]}}}$$

I	$\mathbf{F}^{[I]}$	$\mathbf{K}^{[I]}_2$	$\mathbf{G}^{[I]}$	
3	$\frac{iF}{2\omega L^3}$	$i[2\omega L^3]\mathcal{K}_{(\pi\pi)_2}$	$i\frac{1}{2\omega L^3}G$	e.g. $3\pi^+$
2	$\frac{iF}{2\omega L^3}\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$i[2\omega L^3]\begin{pmatrix}\mathcal{K}_{(\pi\pi)_2}&0\\0&\mathcal{K}_\rho\end{pmatrix}$	$i\frac{1}{2\omega L^3}G\begin{pmatrix}-\frac12&-\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&\frac12\end{pmatrix}$	
1	$\frac{iF}{2\omega L^3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$i[2\omega L^3]\begin{pmatrix}\mathcal{K}_{(\pi\pi)_2}&0&0\\0&\mathcal{K}_\rho&0\\0&0&\mathcal{K}_\sigma\end{pmatrix}$	$i\frac{1}{2\omega L^3}G\begin{pmatrix}\frac16&\frac{\sqrt{15}}{6}&\frac{\sqrt5}{3}\\\frac{\sqrt{15}}{6}&\frac12&-\frac{1}{\sqrt3}\\\frac{\sqrt5}{3}&-\frac{1}{\sqrt3}&\frac13\end{pmatrix}$	e.g. a_1
0	$\frac{iF}{2\omega L^3}$	$i[2\omega L^3]\mathcal{K}_\rho$	$-i\frac{1}{2\omega L^3}G$	e.g. ω, h_1

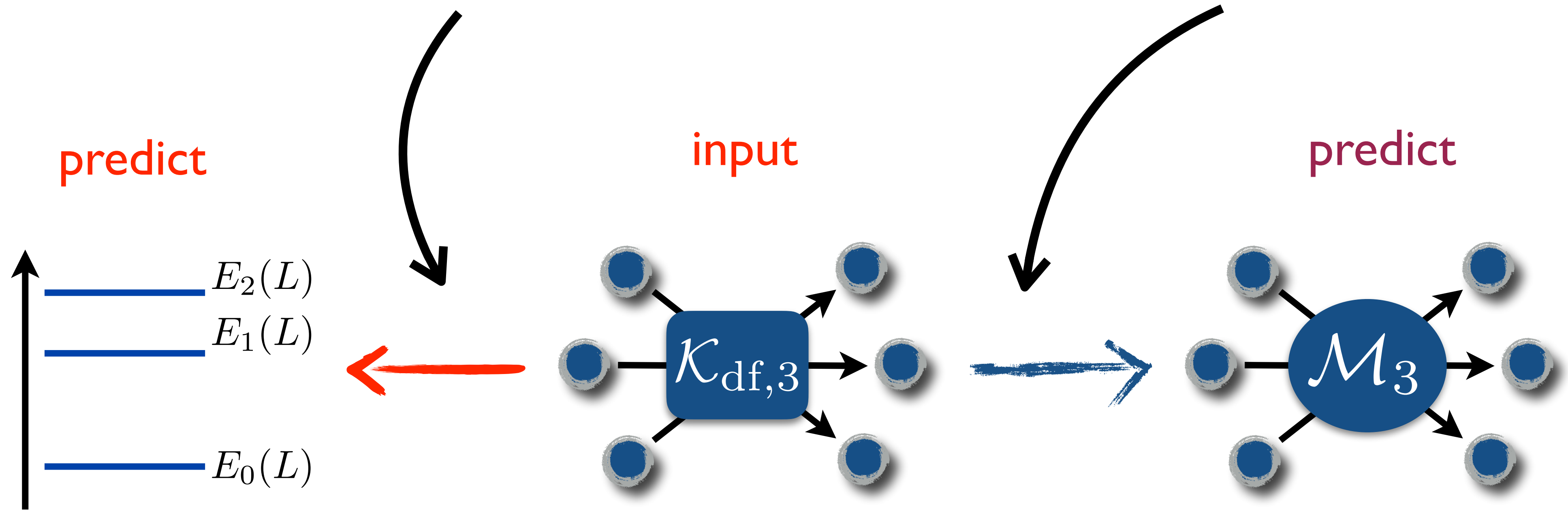
Examples of implementation of (RFT) QC3

Overview

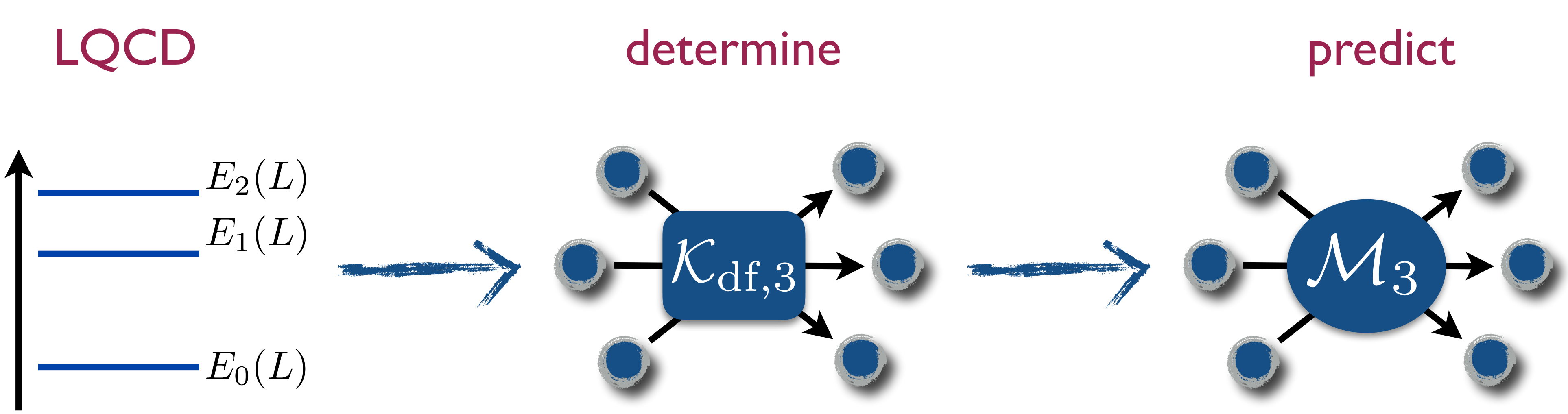
$$\det\left[F_3^{-1}+\mathscr{K}_{\mathrm{df},3}\right]=0$$

Integral equations

TOY MODELS:

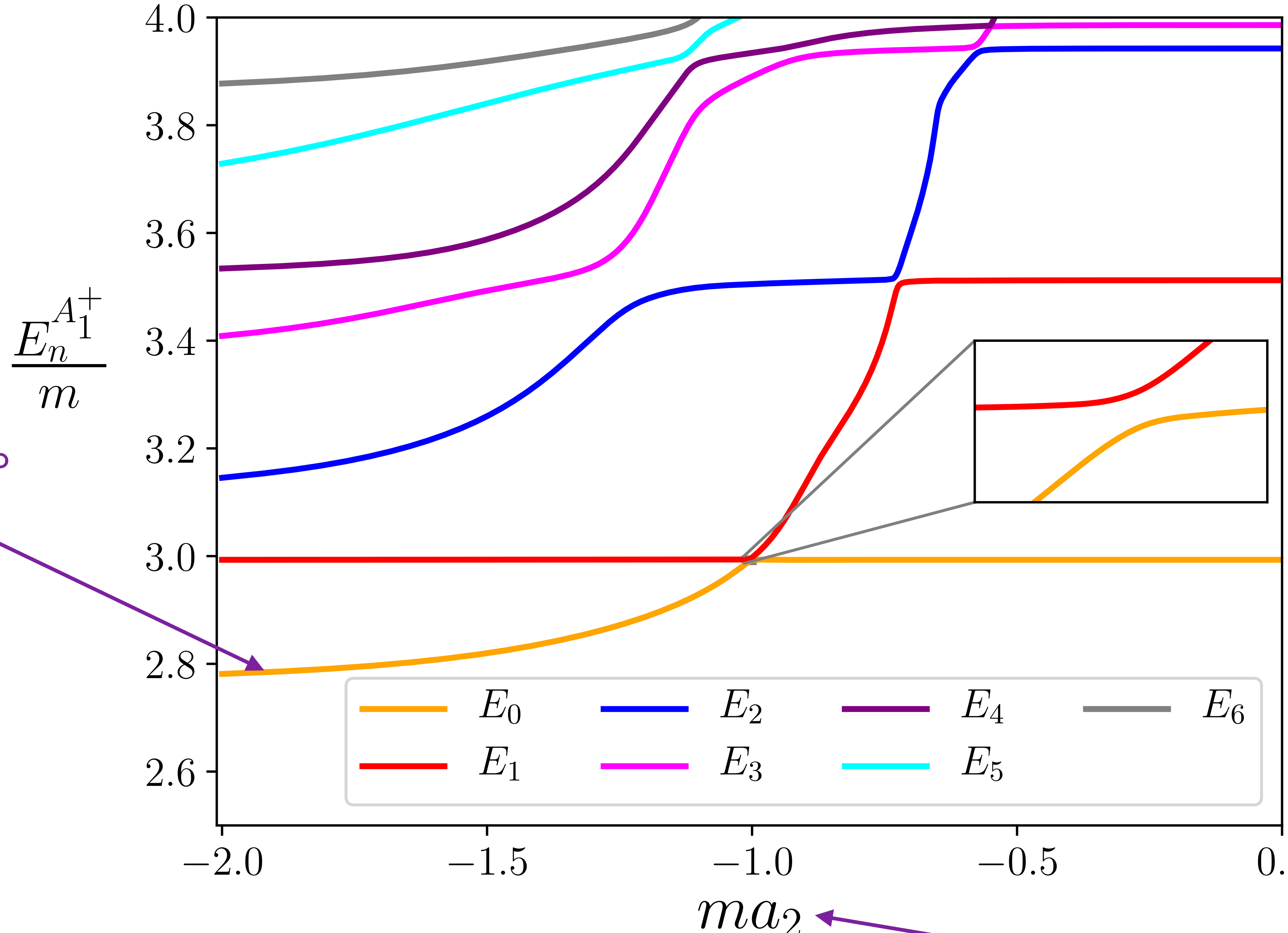


DREAM:



3-particle bound state from d-wave attraction

[BRS19]



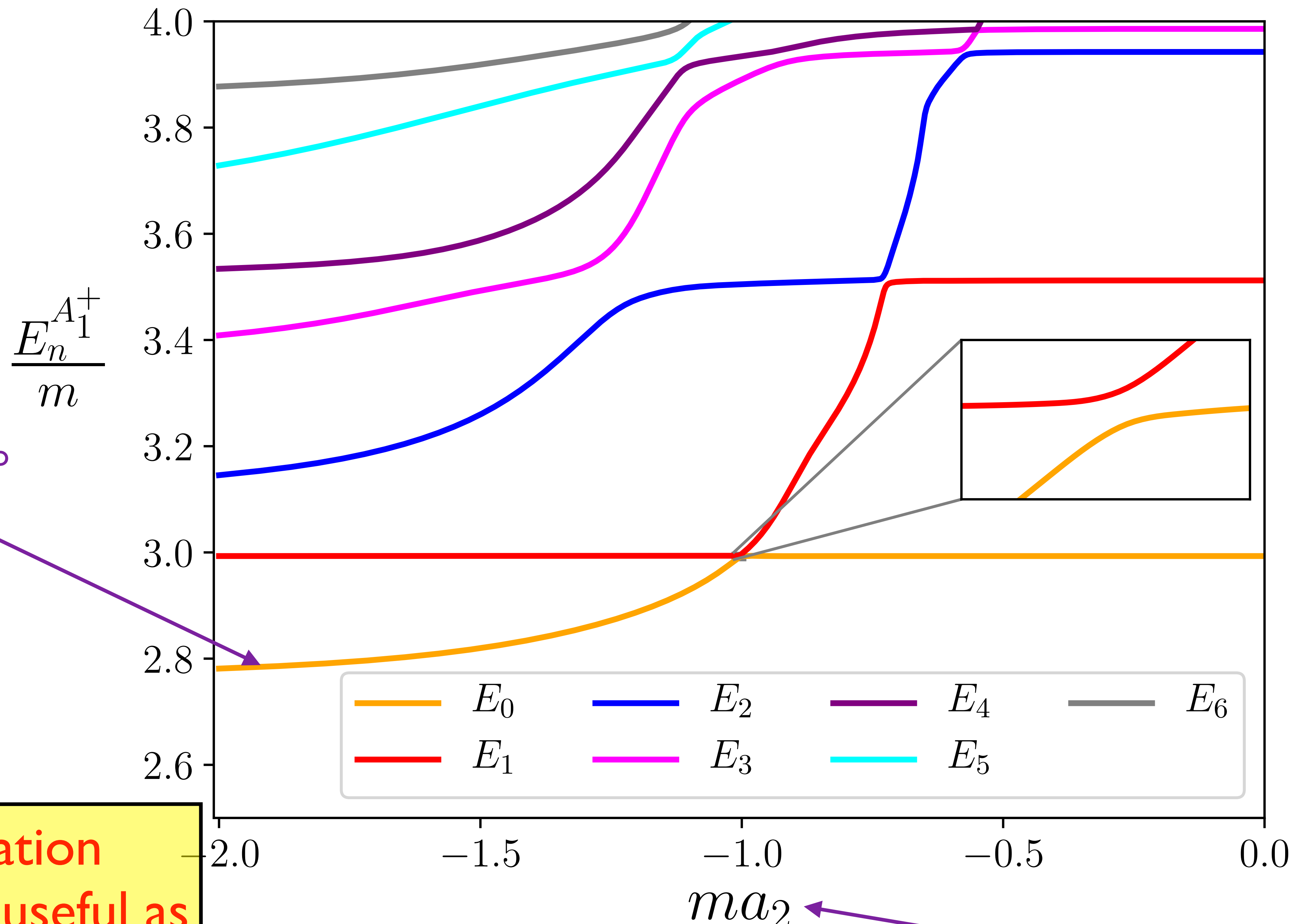
Trimer appears to remain bound as $L \to \infty$

$a_0,\ a_2 < 0 \Rightarrow$ no dimers

$$mL = 8.1,\ ma_0 = -0.1,\ r_0 = P_0 = \mathscr{K}_{df,3} = 0$$

3-particle bound state from d-wave attraction

[BRS19]



Trimer appears to remain bound as $L \to \infty$

$a_0,\ a_2 < 0 \ \Rightarrow$ no dimers

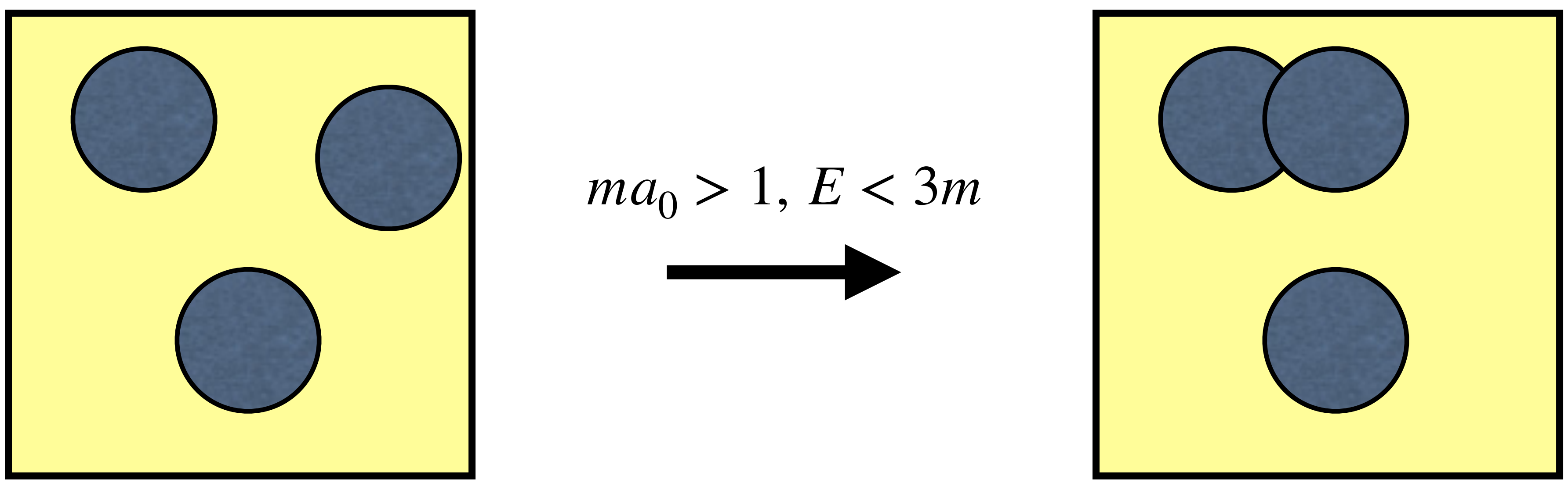
$$mL = 8.1,\ ma_0 = -0.1,\ r_0 = P_0 = \mathcal{K}_{df,3} = 0$$

Quantization condition is useful as tool for studying infinite-volume!

S-wave dimer properties vs a_0

[BBHRS19]

- Study 3 particles with only s-wave scattering length a_0 nonzero
  - Choose $ma_0 > 1$ so that there is a dimer (``deuteron")
  - Look at states with $E < 3m$: dimer+particle and, possibly, trimer ("triton")
  - Use QC2 applied to dimer+particle states to determine scattering amplitude (and, in particular, scattering length b_0)

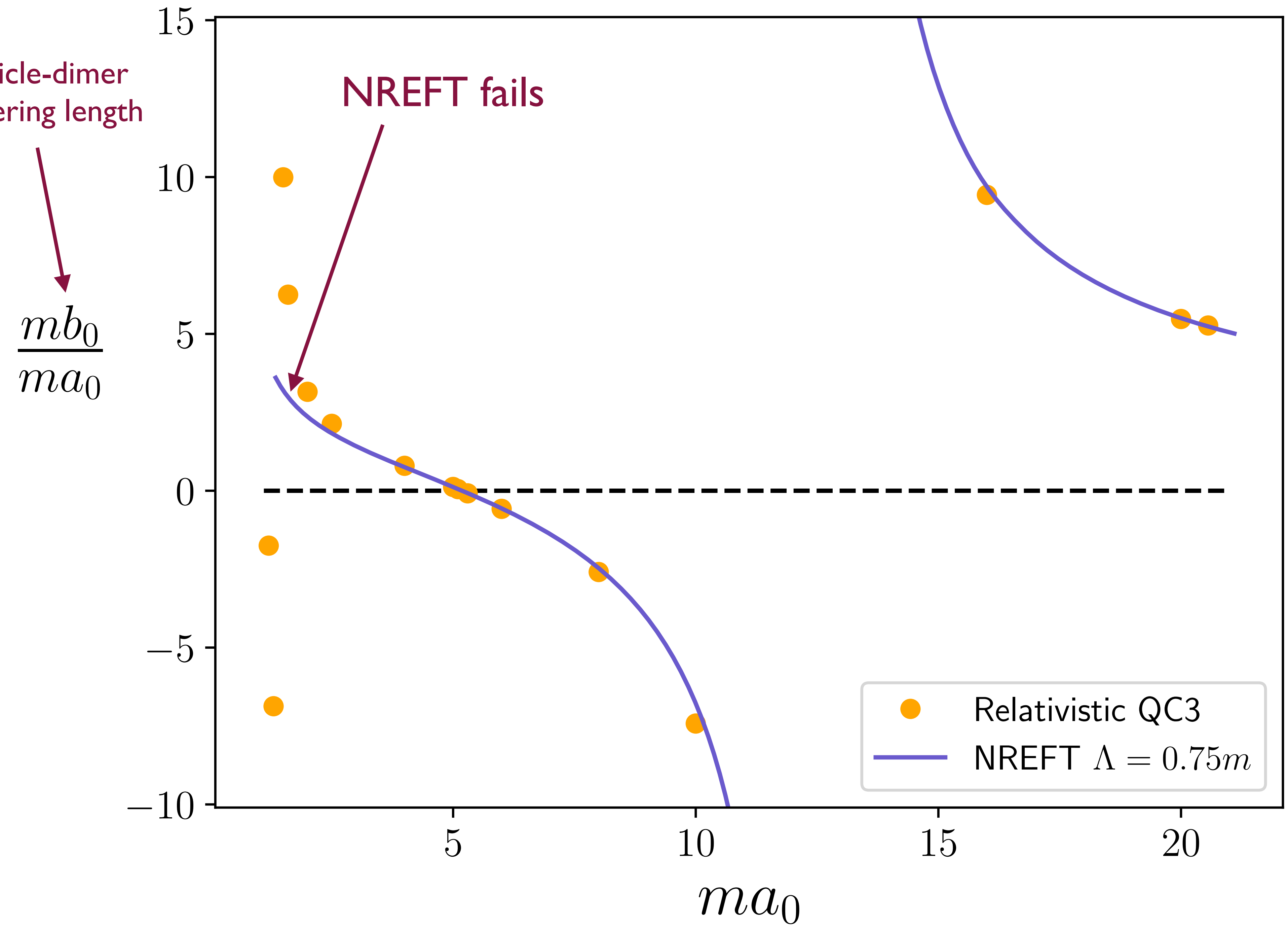


S-wave dimer properties vs a_0

Beginnings of Efimov series!

[BBHRS19]

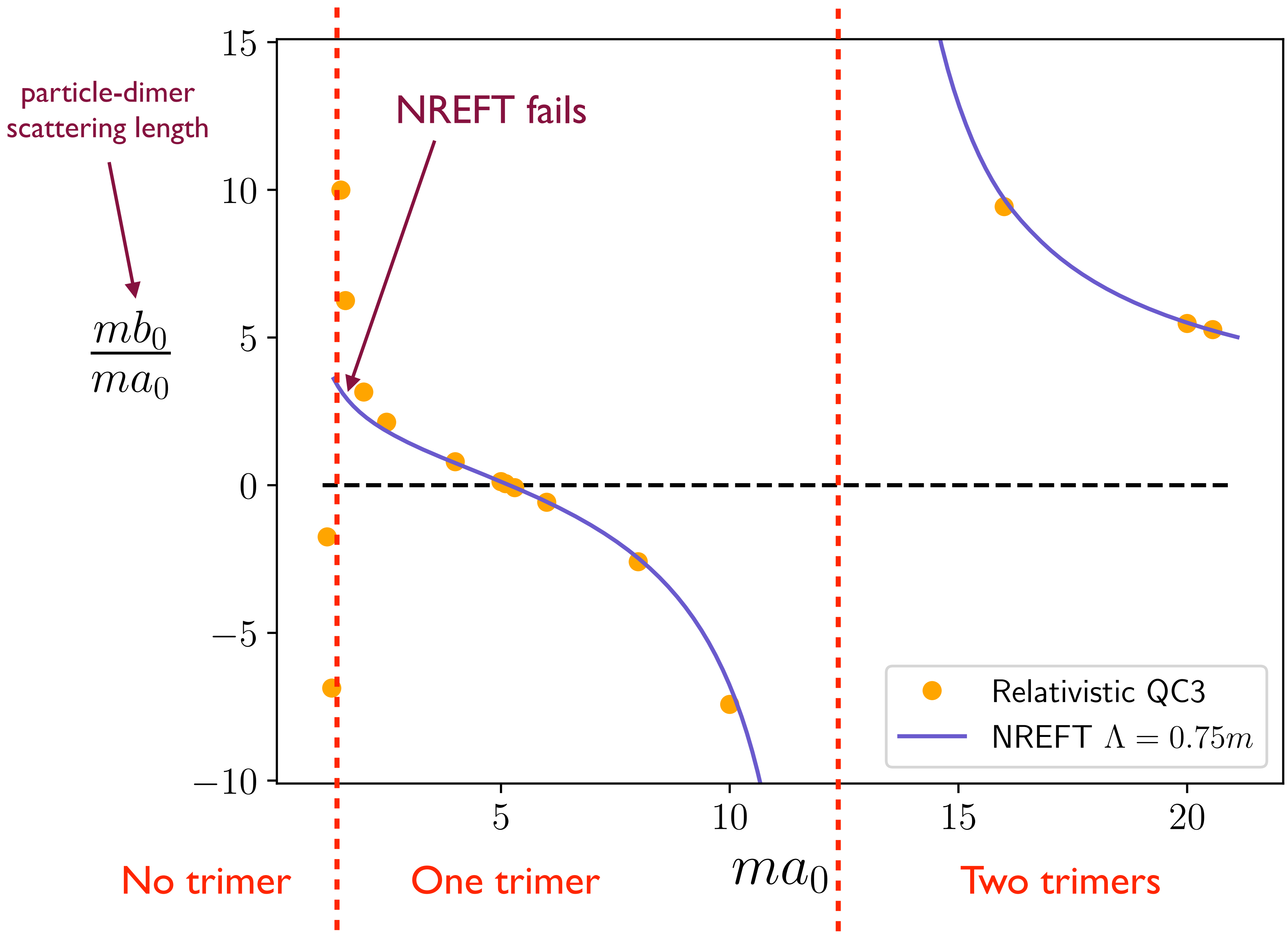
particle-dimer scattering length



S-wave dimer properties vs a_0

[BBHRS19]

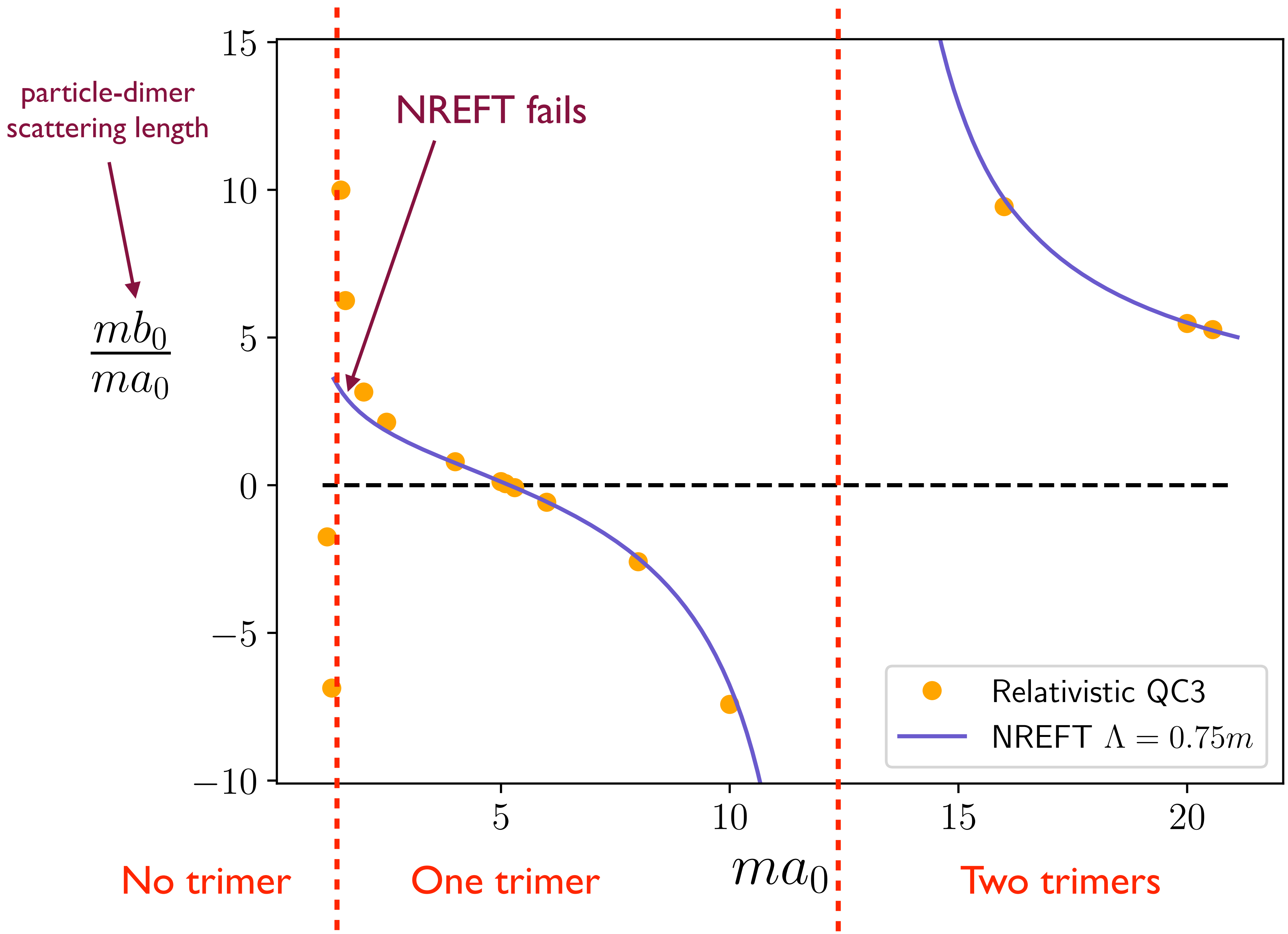
Beginnings of Efimov series!



S-wave dimer properties vs a_0

Beginnings of Efimov series!

[BBHRS19]

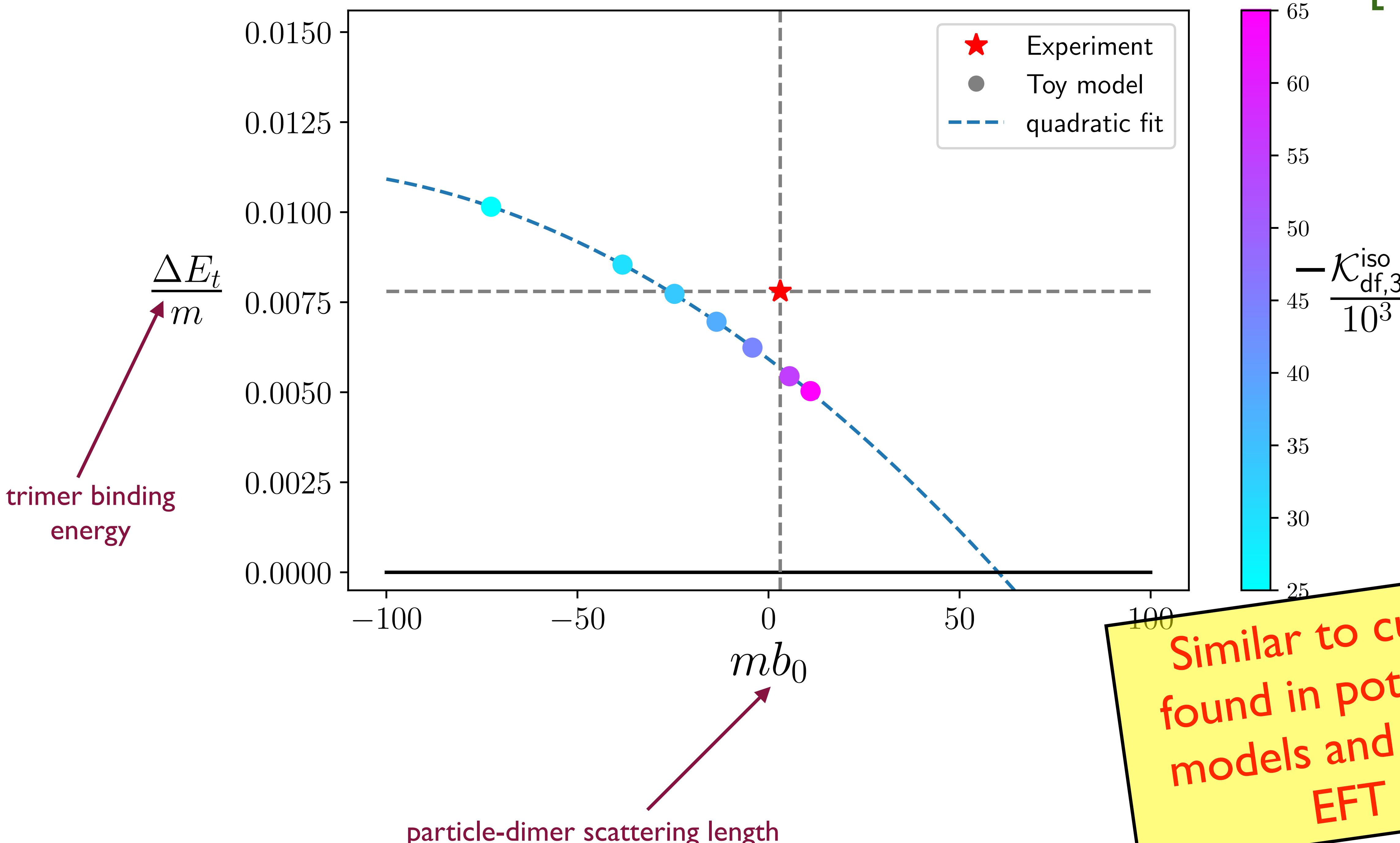


QC3 allows study of relativistic bound states

Phillips curve in toy N+D / Tritium system

Choose a_0 so that $m_{\mathrm{dimer}} : m = M_D : M$ and vary $\mathscr{K}_{\mathrm{df},3}$

[BBHRS19]



Similar to curve found in potential models and chiral EFT

First application to LQCD

Two- and three-pion finite-volume spectra at maximal isospin from lattice QCD

Ben Hörz*
Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

Andrew Hanlon†
Helmholtz-Institut Mainz, Johannes Gutenberg-Universität, 55099 Mainz, Germany
(Dated: May 13, 2019)

We present the three-pion spectrum with maximum isospin in a finite volume determined from lattice QCD, including, for the first time, excited states across various irreducible representations at zero and nonzero total momentum, in addition to the ground states in these channels. The required correlation functions, from which the spectrum is extracted, are computed using a newly implemented algorithm which reduces the number of operations, and hence speeds up the computation by more than an order of magnitude. The results for the $I = 3$ three-pion and the $I = 2$ two-pion spectrum are publicly available, including all correlations, and can be used to test the available three-particle finite-volume approaches to extracting three-pion interactions.

$2\pi^+$ (16 levels) and $3\pi^+$ (11 levels) spectra for $M_\pi = 200\,\mathrm{MeV}$

First application to LQCD

Two- and three-pion finite-volume spectra at maximal isospin from lattice QCD

Ben Hörz*
Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

Andrew Hanlon†
Helmholtz-Institut Mainz, Johannes Gutenberg-Universität, 55099 Mainz, Germany
(Dated: May 13, 2019)

We present the three-pion spectrum with maximum isospin in a finite volume determined from lattice QCD, including, for the first time, excited states across various irreducible representations at zero and nonzero total momentum, in addition to the ground states in these channels. The required correlation functions, from which the spectrum is extracted, are computed using a newly implemented algorithm which reduces the number of operations, and hence speeds up the computation by more than an order of magnitude. The results for the $I = 3$ three-pion and the $I = 2$ two-pion spectrum are publicly available, including all correlations, and can be used to test the available three-particle finite-volume approaches to extracting three-pion interactions.

$2\pi^+$ (16 levels) and $3\pi^+$ (11 levels) spectra for $M_\pi = 200\,\mathrm{MeV}$

- [BRS-PRL19] Simultaneous fit with QC2 and QC3
  - s-wave $\pi^+\pi^+$ interaction
  - Isotropic (momentum-independent) $\mathscr{K}_{\mathrm{df},3}$ with linear dependence on E^2_{CM}
- Predict $\mathscr{K}_{\mathrm{df},3}$ using leading order chiral perturbation theory

Global fit to $2\pi^+$ and $3\pi^+$ spectra

[BRS-PRL19]

Inelastic threshold

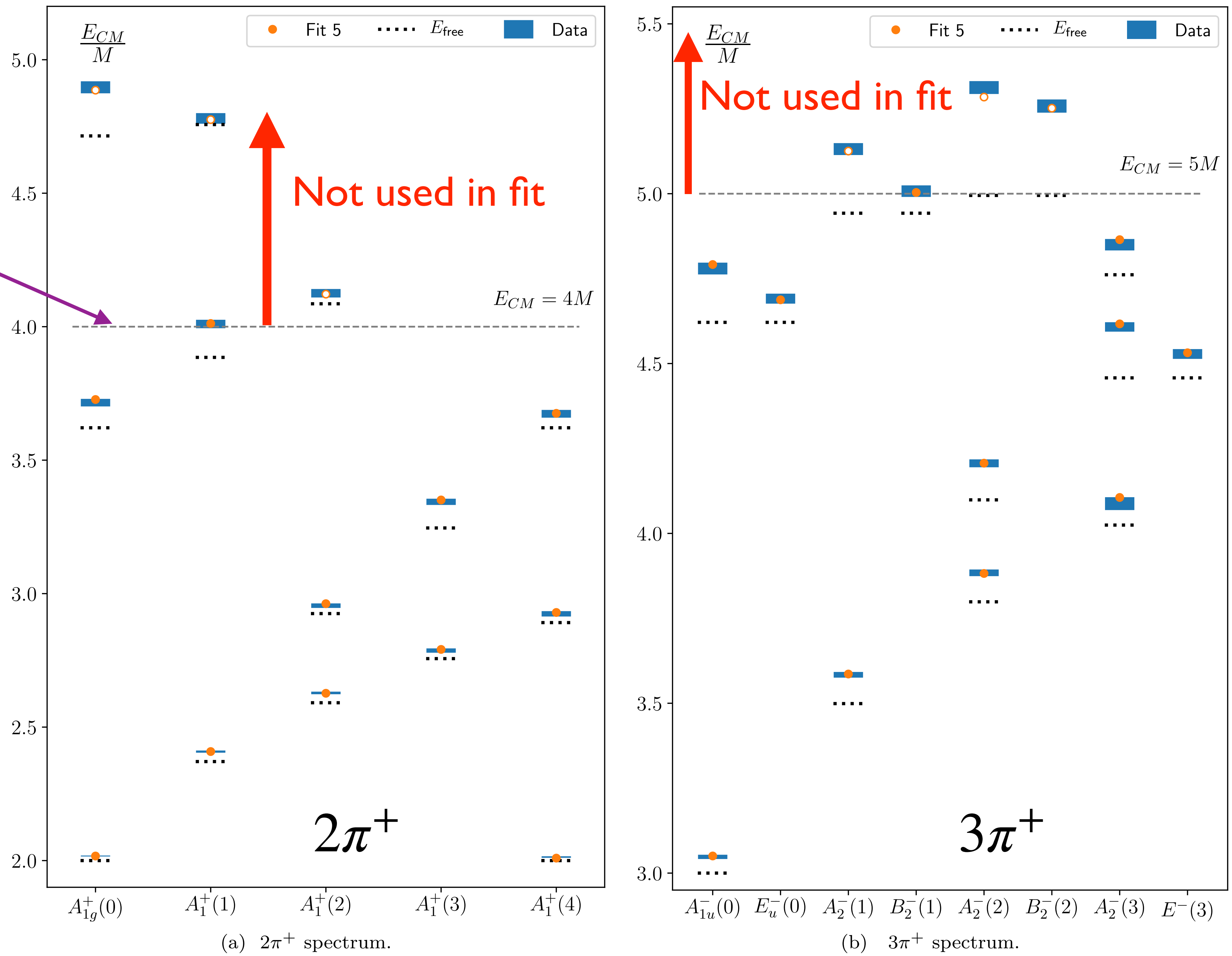


FIG. S3: Two- and three-pion spectra from Ref. [1] (blue) compared to the predictions from the global fit 5 (orange). Hollow orange points above the inelastic thresholds have not been included in the fit, but are shown for comparison. Dashed lines show the non-interacting energy levels.

Global fit

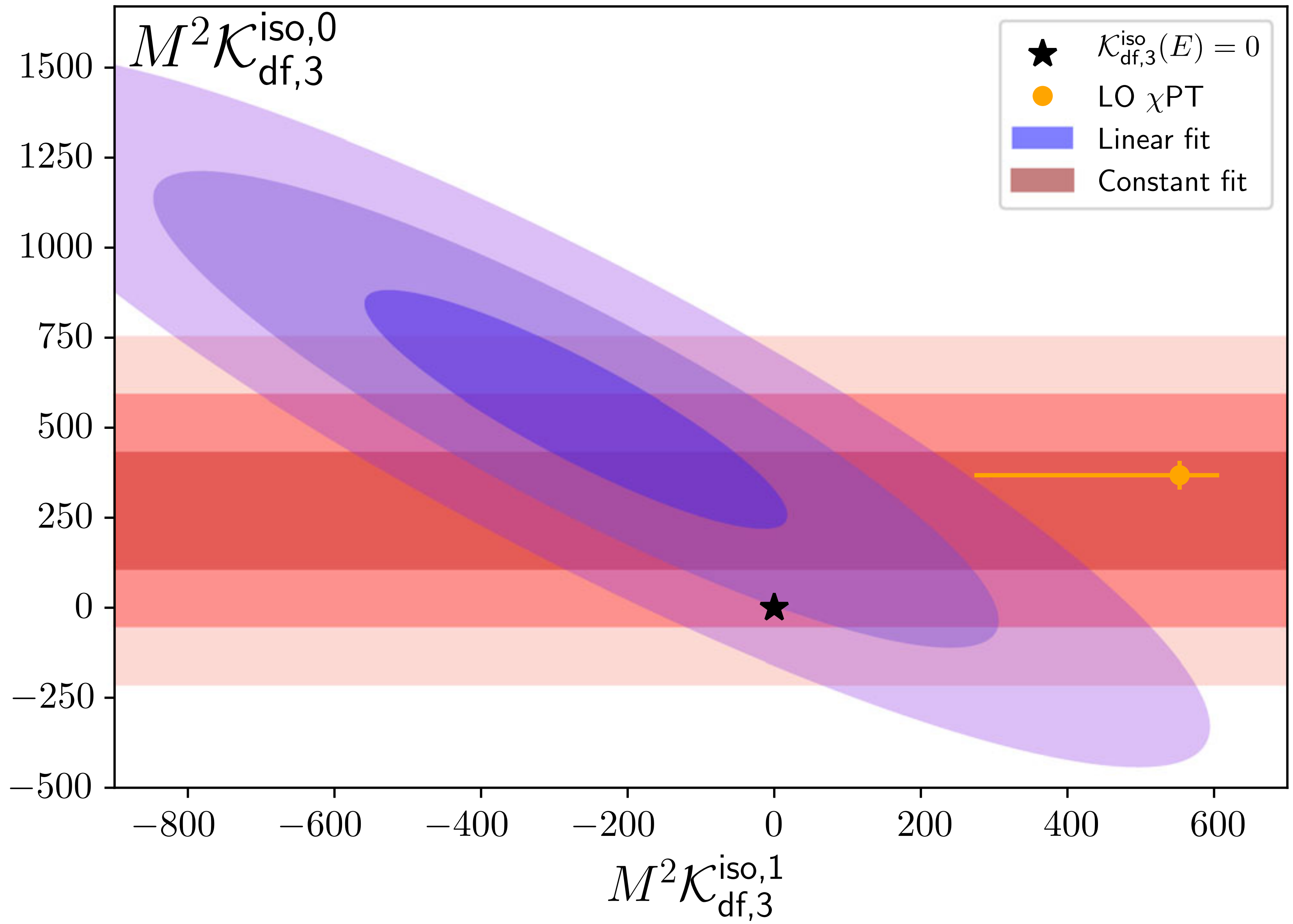
$$\frac{\chi^2}{\text{d.o.f.}} = \frac{26.04}{(22-4)}$$

Similar fit obtained using FVU QC3 [Mai et al, 19]

3-particle spectrum primarily determined by 2-particle interactions

Evidence for nonzero $\mathcal{K}_{\text{df},3}$

[BRS-PRL19]



Global fit

$$\frac{\chi^2}{\text{d.o.f.}} = \frac{26.04}{(22-4)}$$

$$\mathcal{K}_{\text{df},3} = \mathcal{K}^{\text{iso},0}_{\text{df},3} + \frac{E^2_{\text{CM}} - 9M^2}{9M^2}\mathcal{K}^{\text{iso},1}_{\text{df},3}$$

Alternative derivation and new form of QC3

[Blanton & SRS, arXiv:2007.16188] = [BS20a]

See also talk by Tyler Blanton at APLAT 2020:
https://conference-indico.kek.jp/event/113/contributions/2070/

Why a new RFT derivation?

- To simplify generalization of QC3 to nondegenerate particles with spin (e.g. $N\pi\pi$), and to QC4, …
  - Original RFT derivation is long, complicated and does not give explicit results for all quantities (e.g. $\mathscr{K}_{\text{df},3}$)

[HS14]

RELATIVISTIC, MODEL-INDEPENDENT, THREE- … PHYSICAL REVIEW D **90,** 116003 (2014)

enter the analysis. It thus seems that some finite-volume states have been lost. In fact, all but one of the free states are present once one takes into account that the equality of all N^2 elements of the truncated $\mathcal{K}_{\text{df},3}$ will not be exact. This is shown in a particular example in Appendix C.[12]

IV. DERIVATION

In this section we present a derivation of the quantization condition described in the previous section. Following Ref. [17], we obtain the spectrum from the poles in the finite-volume Minkowski-space correlator[13]

$$C_L(E,\vec P) \equiv \int_L d^4x e^{i(Ex^0-\vec P\cdot\vec x)}\langle 0|\mathrm{T}\sigma(x)\sigma^\dagger(0)|0\rangle. \quad (40)$$

Here T indicates time-ordering and $\sigma(x)$ is an interpolating field coupling to states with an odd number of particles. The Fourier transform, implemented via an integral over the finite spatial volume, restricts the states to have total energy E and momentum $\vec P = 2\pi\vec n_P/L$.

The simplest choice for $\sigma(x)$ is a one-particle interpolating field, $\phi(x)$, since in the interacting theory this will couple to states with any odd number of particles. In a simulation, however, it is advantageous to use a choice with larger overlap to the three-particle states of interest. An example is

$$\sigma(x) = \int_L d^4y d^4z f(y,z)\phi(x)\phi(x+y)\phi(x+z), \quad (41)$$

with f being a smooth function with period L in all directions.

At fixed $\{L,\vec n_P\}$,[14] the spectrum of our theory is the set of CM frame energies E_j^*, $j=1,2,\cdots$ for which $C_L(E_j,\vec P)$ has a pole, with $E_j = (E_j^{*2}+\vec P^2)^{1/2}$. Our goal is thus to include all contributions to C_L which fall at most like a power of $1/L$, and determine the pole structure. In the previous section we summarized the main result of this work, but made no reference to the correlator in doing so. The connection is given by the following identity, the demonstration of which is the task of this section:

$$C_L(E,\vec P) = C_\infty(E,\vec P) + iA'\frac{1}{1+F_3\mathcal{K}_{\text{df},3}}F_3A. \quad (42)$$

This result is valid up to terms exponentially suppressed in the volume, terms which we will discard implicitly throughout this section. The quantities $A' \equiv A'_{k',\ell',m'}$ and $A \equiv A_{k,\ell,m}$ are, respectively, row and column vectors in [finite-volume momentum] × [two-particle angular momentum] space. Since A and A' do not enter the quantization condition, we have not given their definitions above. Indeed, we think it most useful to introduce their definitions as they emerge in our all-orders summation. We have also introduced C_∞, which is an infinite-volume correlator whose definition we will also build up over the following subsections.

A key technical issue in the derivation is the need to use a nonstandard pole prescription when defining momentum integrals in infinite-volume Feynman diagrams. This is at the root of the complications in defining A', A, and C_∞. Despite these complications, these are infinite-volume quantities that we do not expect to lead to poles in C_L.[15] It follows that, at fixed $\{L,\vec n_P\}$, C_L diverges at all energies for which the matrix between A and A' has a divergent eigenvalue. In addition, as long as $\mathcal{K}_{\text{df},3}$ is nonzero, diverging eigenvalues of F_3 leave the finite-volume correlator finite. The spectrum is therefore given by energies for which $[1+F_3\mathcal{K}_{\text{df},3}]$ has a vanishing eigenvalue, which is the quantization condition quoted above.

The demonstration of Eq. (42) proceeds by an all-orders analysis of the Feynman diagrams building up the correlator. As we accommodate any scalar field theory (assuming only a $\mathbb{Z}_2$ symmetry), Feynman diagrams consist of any number of even-legged vertices, as well as one each of the interpolating fields σ and $\sigma^\dagger$, connected by propagators. The finite-volume condition enters here only through the prescription of summing (rather than integrating) the spatial components of all loop momenta, i.e.

$$\frac{1}{L^3}\sum_{\vec q=2\pi\vec n/L}\int\frac{dq^0}{2\pi} \quad \text{over all } \vec n\in\mathbb{Z}^3. \quad (43)$$

We now introduce the crucial observation that makes our derivation possible: Power-law finite-volume effects only enter through on shell intermediate states. This motivates a reorganization of the sum of diagrams into a skeleton expansion that keeps all on shell intermediate states explicit, while grouping off shell states into Bethe-Salpeter kernels. Heuristically, the importance of on shell intermediate states can be understood by noting that on shell particles can travel arbitrarily far, and are thus

[12]A similar issue arises with the two-particle quantization condition when one truncates the angular-momentum expansion. The lost states involving higher angular momenta are recovered if one reintroduces the higher partial-wave amplitudes but with infinitesimal strength. The quantization condition then has solutions corresponding to free two-particle states projected onto states in appropriate irreps (irreducible representations) of the finite-volume symmetry group.

[13]Minkowski time turns out to be convenient for our analysis, even though numerical lattice determinations of the spectrum work in Euclidean time. The point is that the finite-volume spectrum is the same, however it is determined.

[14]It is more natural to think in terms of $\{L,\vec n_P\}$ rather than $\{L,\vec P\}$, since $\vec n_P$ is quantized whereas $\vec P$ varies with L.

[15]We discuss this point, following the derivation, at the end of this section.

116003-9 **9**

MAXWELL T. HANSEN AND STEPHEN R. SHARPE PHYSICAL REVIEW D **90,** 116003 (2014)

$$C_{L,2} = C_{\infty,2} + iA'_2\frac{1}{1+F\mathcal{K}_2}FA_2. \quad (252)$$

The subscripts "2" on A, A', and C indicate that these are the two-particle end caps and correlator, while F is defined in Eq. (22) (although here we drop the spectator-momentum argument).

What we now show is that there are poles in A_2, A'_2, and $C_{\infty,2}$, but these cancel in $C_{L,2}$. To see this we use the freedom to arbitrarily choose the interpolating functions σ and $\sigma^\dagger$ without affecting the position of poles in $C_{L,2}$. Specifically, we set both σ and $\sigma^\dagger$ equal to the two-particle Bethe-Salpeter kernel iB_2, which, we recall, is a smooth nonsingular function. One then finds that

$$C_{\infty,2} = i\mathcal{K}_2 - iB_2 \quad \text{and} \quad A_2 = A'_2 = i\mathcal{K}_2. \quad (253)$$

Inserting these results into Eq. (252) we find that (for this choice of end caps)

$$\begin{aligned} C_{L,2} &= -iB_2 + i\mathcal{K}_2 + i\mathcal{K}_2\frac{1}{1-iFi\mathcal{K}_2}iFi\mathcal{K}_2 \\ &= -iB_2 + \frac{i}{\mathcal{K}_2^{-1}+F}. \end{aligned} \quad (254)$$

From Eqs. (253) and (254) we draw two conclusions. First, A_2, A'_2, and $C_{\infty,2}$ have poles whenever $\mathcal{K}_2$ diverges. Such poles occur, for a given angular momentum, when $\delta_\ell = \pi/2 \bmod \pi$. Thus, using the $\widetilde{\text{PV}}$ prescription, there are, in general, poles in A_2, A'_2, and $C_{\infty,2}$. Second, these poles cancel in $C_{L,2}$, as shown by the second form in Eq. (254), which is clearly finite when $\mathcal{K}_2$ diverges.

We suspect that a similar result holds for the three-particle analysis, but have not yet been able to demonstrate this. Thus, in the three-particle case we must rely for now on the intuitive argument given above.

V. CONCLUSIONS AND OUTLOOK

In this work we have presented and derived a three-particle quantization condition relating the finite-volume spectrum to two-to-two and three-to-three infinite-volume scattering quantities. This condition separates the dependence on the volume into kinematic quantities, as was achieved previously for two particles.

There are two new features of the result compared to the two-particle case. First, the three-particle scattering quantity entering the quantization condition has the physical on shell divergences removed. The resulting divergence-free quantity is thus spatially localized. This is crucial for any practical application of the formalism since it allows for the partial-wave expansion to be truncated. Indeed, it is difficult to imagine a quantization condition involving the three-particle scattering amplitude itself, given that the latter is divergent for certain physical momenta.

The second feature is that the three-particle scattering quantity is nonstandard—it is not simply related to the (divergence-free part) of the physical scattering amplitude. This is because it is defined using the $\widetilde{\text{PV}}$ pole prescription, and also because of the decorations explained in Sec. IV E. We strongly suspect, however, that a relation to the physical amplitude exists. In particular, we know from Ref. [13] that the finite-volume spectrum in a nonrelativistic theory can be determined solely in terms of physical amplitudes, and the same is true in the approximations adopted in Ref. [14]. We are actively investigating this issue.

The three-particle quantization condition involves a determinant over a larger space than that required for two particles. Nevertheless, as explained in Secs. III, because the three-particle quantity that enters has a uniformly convergent partial-wave expansion, one can make a consistent truncation of the quantization condition so that it involves only a finite number of parameters. This opens the way to practical application of the formalism.

We have provided in this paper two mild consistency checks on the formalism—that it correctly reproduces the known results if one particle is noninteracting (see Sec. IV A), and that the number of solutions to the quantization condition in the isotropic approximation is as expected (see Appendix C). We have also worked out a more detailed check by comparing our result close to the three-particle threshold $E^*\approx 3m$ to those obtained using nonrelativistic quantum mechanics [27,28]. Here one has an expansion in powers of $1/L$, and we have checked that the results agree for the first four nontrivial orders. This provides, in particular, a nontrivial check of the form of F_3, Eq. (19), and allows us to relate $\mathcal{K}_{\text{df},3}$ to physical quantities in the nonrelativistic limit. We will present this analysis separately [29].

Two other issues are deferred to future work. First, we would like to understand in detail the relation of our formalism and quantization condition to those obtained in Refs. [13,14]. Second, we plan to test the formalism using simple models for the scattering amplitudes, in order to ascertain how best to use it in practice.

ACKNOWLEDGMENTS

We thank Raúl Briceño, Zohreh Davoudi, and Akaki Rusetsky for discussions. This work was supported in part by the U.S. Department of Energy Grants No. DE-FG02-96ER40956 and No. DE-SC0011637. M. T. H. was supported in part by the Fermilab Fellowship in Theoretical Physics. Fermilab is operated by Fermi Research Alliance, LLC, under Contract No. DE-AC02-07CH11359 with the United States Department of Energy.

APPENDIX A: SUM-MINUS-INTEGRAL IDENTITY

In this appendix we derive the sum-minus-integral identity that plays a central role in the main text. This identity is

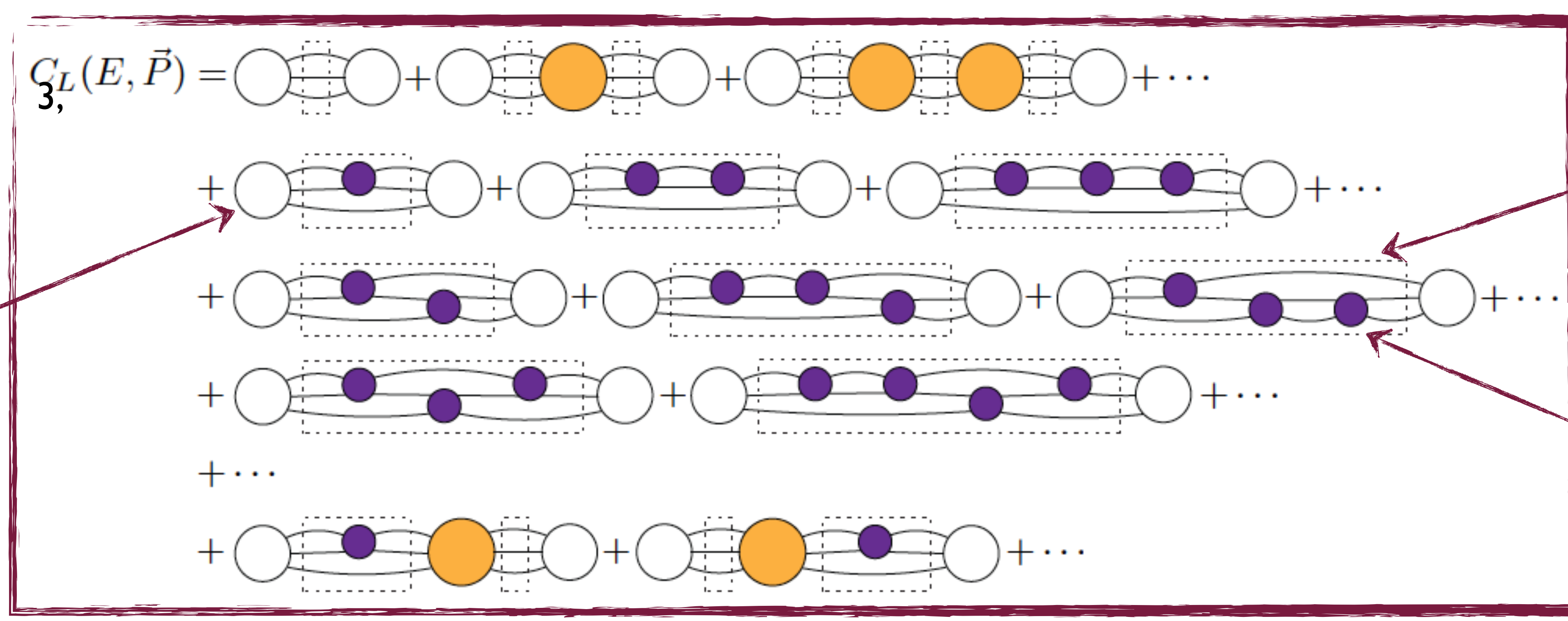
116003-40 **40**

Comparison of strategies

- Both consider generic EFT for identical particles with a $\mathbb{Z}_2$ symmetry
  - Work diagrammatically to all orders in PT; no power-counting needed
- Both analyze three-particle, finite-volume correlator for $E_{\text{CM}} < 5M$

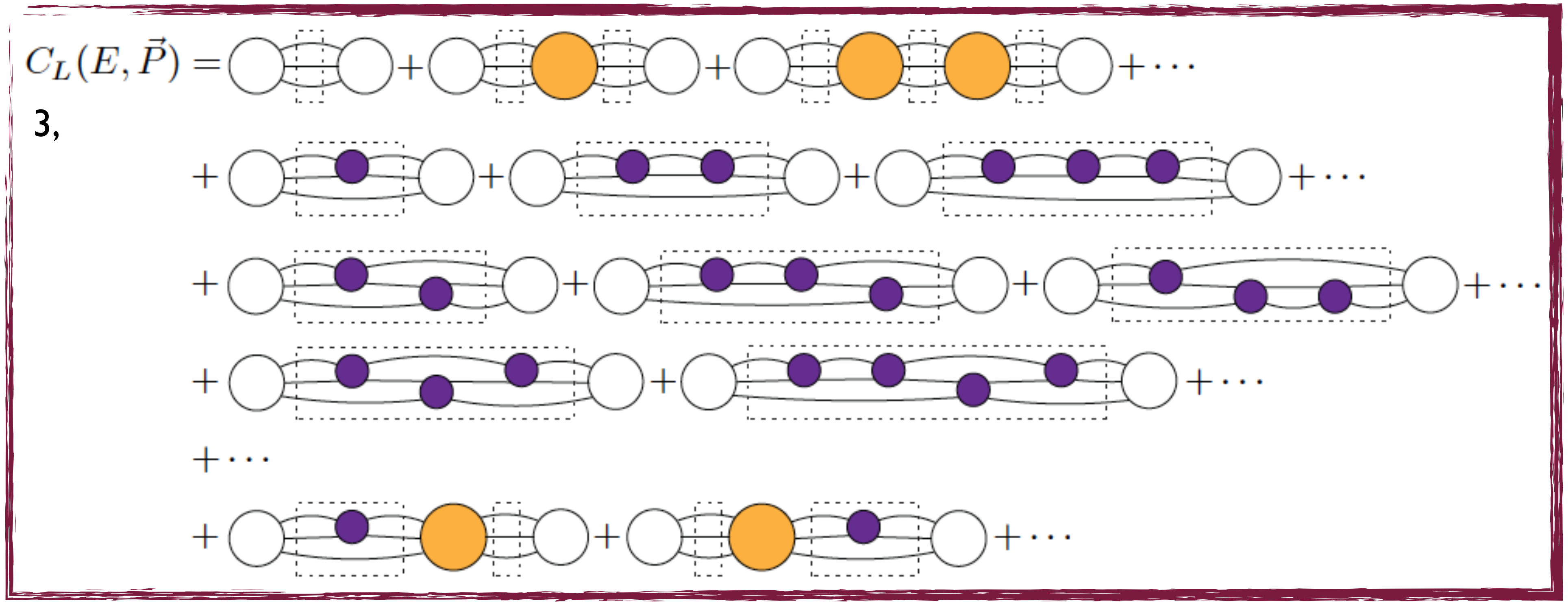
$$C_{3,L}(E,\vec{P}) = \int_L d^4x \; e^{i(Ex^0 - \vec{P}\cdot x)} \, \langle 0 | \mathrm{T}\sigma(x)\sigma^\dagger(0)|0\rangle$$

- Both use time-ordered PT (TOPT) to argue that only 3-particle cuts lead to singularities and thus power-law volume dependence; all others can be integrated
- [HS14] use skeleton expansion in terms of Feynman diagrams



Comparison of strategies: old

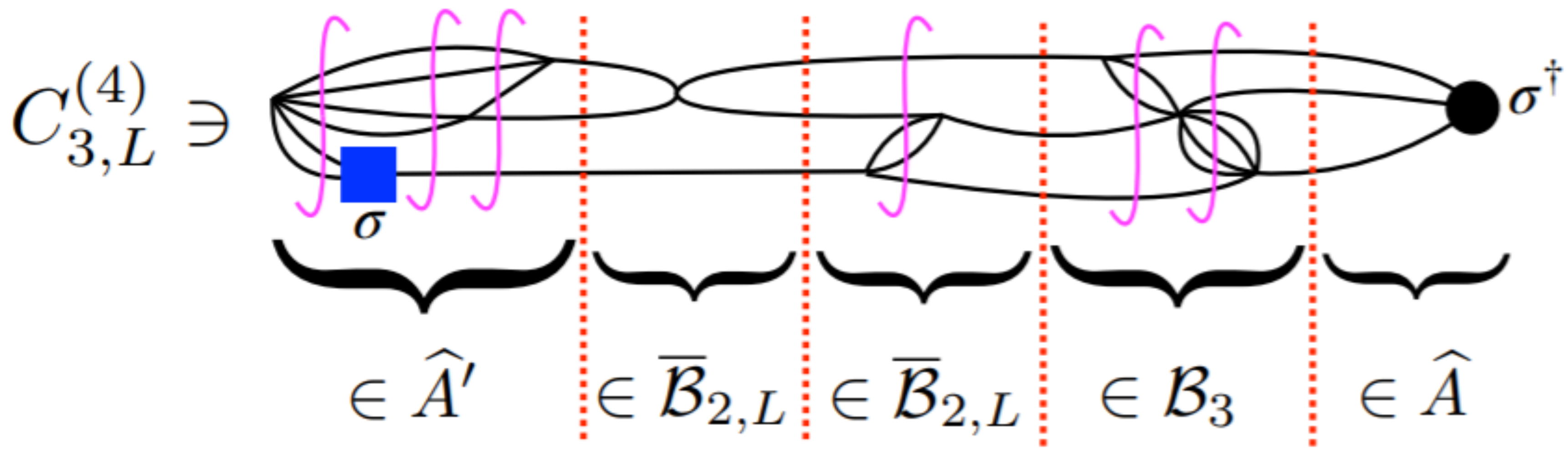
- [HS14] use skeleton expansion in terms of Feynman diagrams



- Replace sums with integral (PV regulated) + (sum-integral) wherever possible
- Reshuffle contributions of antiparticle poles at 3-particle cuts—which are nonsingular—into infinite-volume quantities
- Expend great effort recombining terms so that final expression is written in terms of a $\mathcal{K}_{\rm df,3}$ that is symmetric under particle exchange
- Define $\mathcal{K}_{\rm df,3}$ constructively, rather than explicitly

Comparison of strategies: new

- [BS20a] use skeleton expansion in terms of TOPT diagrams, ordered by the number of ``relevant cuts” (3-particle cuts), e.g.



- Loops with only irrelevant cuts can be integrated; build up TOPT kernels $\widehat{A}'$, $\overline{\mathscr{B}}_{2,L} = 2\omega L^3 \mathscr{B}_2$, $\mathscr{B}_3$, $\widehat{A}$
    - The infinite-volume objects $\widehat{A}'$, $\mathscr{B}_2$, $\mathscr{B}_3$, $\widehat{A}$ contain Feynman-diagram antiparticle contributions to 3-particle cuts
    - $\overline{\mathscr{B}}_{2,L}$ is intrinsically asymmetric, since it picks out a ``spectator”

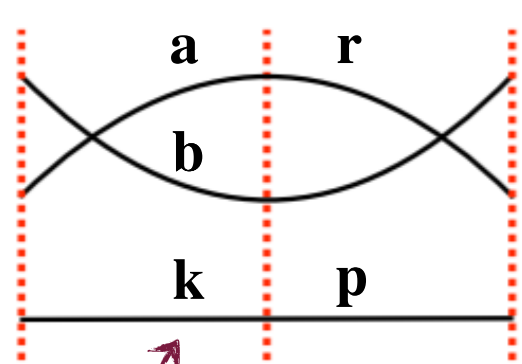
Comparison of strategies: new

- Two types of relevant cuts: F- and G-like

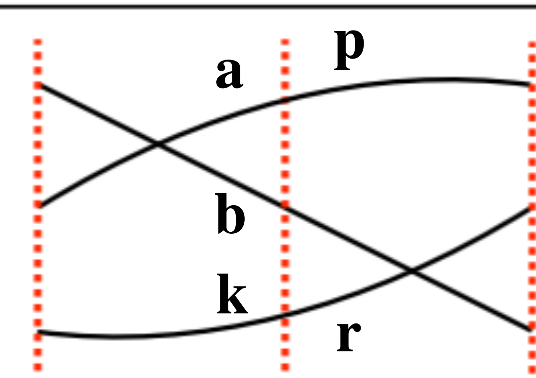
$$[iD_F]_{ka;pr} \equiv \delta_{kp}\delta_{ar}\frac{iD_{ka}}{2!}\,, \qquad [iD_G]_{ka;pr} \equiv \delta_{kr}\delta_{ap} iD_{kp}\,,$$

$$iD_{ka} \equiv \frac{1}{2\omega_k L^3}\frac{i}{2\omega_b(E-\omega_k-\omega_a-\omega_b)}\frac{1}{2\omega_a L^3}$$

$\overline{\mathcal{B}}_{2,L} D_F \overline{\mathcal{B}}_{2,L} \ni$



$\overline{\mathcal{B}}_{2,L} D_G \overline{\mathcal{B}}_{2,L} \ni$



Kernels depend on two 3-momenta and are off shell

Comparison of strategies: new

- Do all orders summation before dealing with momentum sums in relevant cuts

$$
\begin{aligned}
C_{3,L}^{(1)}(E,\vec{P}) &= \widehat{A}' i(D_F + D_G)\widehat{A} \\
C_{3,L}^{(2)}(E,\vec{P}) &= \widehat{A}' i(D_F + D_G) i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)\widehat{A} \\
C_{3,L}^{(3)}(E,\vec{P}) &= \widehat{A}' i(D_F + D_G) i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G) i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)\widehat{A} \\
&\vdots \\
C_{3,L}^{(n)}(E,\vec{P}) &= \widehat{A}' i(D_F + D_G)\left[i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)\right]^{n-1}\widehat{A}
\end{aligned}
$$

$$
\Rightarrow \boxed{C_{3,L}(E,\vec{P}) = C_{3,\infty}^{(0)}(E,\vec{P}) + \widehat{A}' i(D_F + D_G)\frac{1}{1 - i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)}\widehat{A}}
$$

- Simple, explicit expression!
- However, $\widehat{A}'$, $\mathscr{B}_2$, $\mathscr{B}_3$, $\widehat{A}$ are off shell

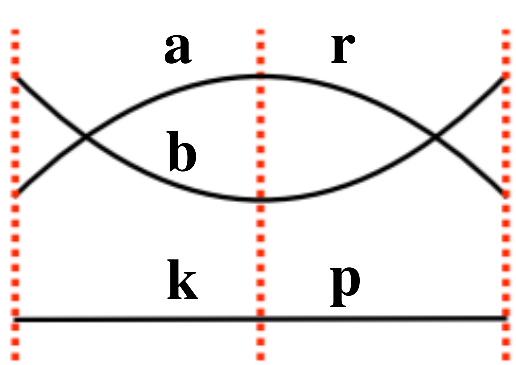
Comparison of strategies: new

- Do on-shell projection of F- and G-cuts in parallel (using methods from [HS14])

$$[iD_F]_{ka;pr} \equiv \delta_{kp}\delta_{ar}\frac{iD_{ka}}{2!}\,, \qquad [iD_G]_{ka;pr} \equiv \delta_{kr}\delta_{ap} iD_{kp}\,,$$

$$iD_{ka} \equiv \frac{1}{2\omega_k L^3}\frac{i}{2\omega_b(E-\omega_k-\omega_a-\omega_b)}\frac{1}{2\omega_a L^3}$$

$\overline{\mathcal{B}}_{2,L} D_F \overline{\mathcal{B}}_{2,L} \ni$

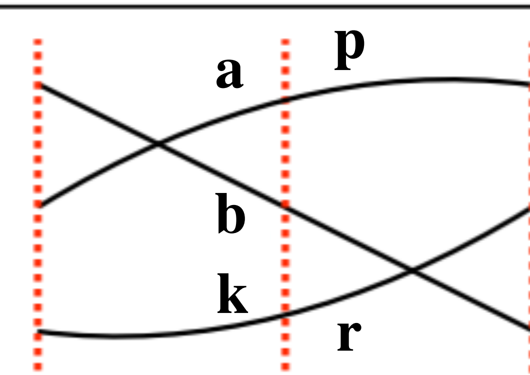


$$\frac{1}{L^6}\sum_k \sum_a = \int_k \mathrm{PV}\int_a + \frac{1}{L^3}\sum_k \underbrace{\left[\frac{1}{L^3}\sum_a - \mathrm{PV}\int_a\right]}_{\sim \widetilde{F}}$$

$$\Rightarrow D_F = \widetilde{I}_F + \widetilde{F}$$

Integral operator that sews kernels together

$\overline{\mathcal{B}}_{2,L} D_G \overline{\mathcal{B}}_{2,L} \ni$



- Decompose adjacent kernels into on-shell part and residue
- Residue cancels pole & leads to integral operator

$$\Rightarrow D_G = \widetilde{G} + \delta\widetilde{G}$$

Integral operator that sews kernels together

Project adjacent kernels on shell: $\{\mathbf{k}, \mathbf{a}\} \rightarrow \{\mathbf{k}\ell m\}$

Comparison of strategies: new

- Implement on shell projection without symmetrizing K matrices

$$C_{3,L}(E,\vec{P}) = C_{3,\infty}^{(0)}(E,\vec{P}) + \widehat{A}' i(\widetilde{F} + \widetilde{G} + \widetilde{\mathcal{I}}_F + \delta\widetilde{G}) \frac{1}{1 - i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(\widetilde{F} + \widetilde{G} + \widetilde{\mathcal{I}}_F + \delta\widetilde{G})} \widehat{A}$$

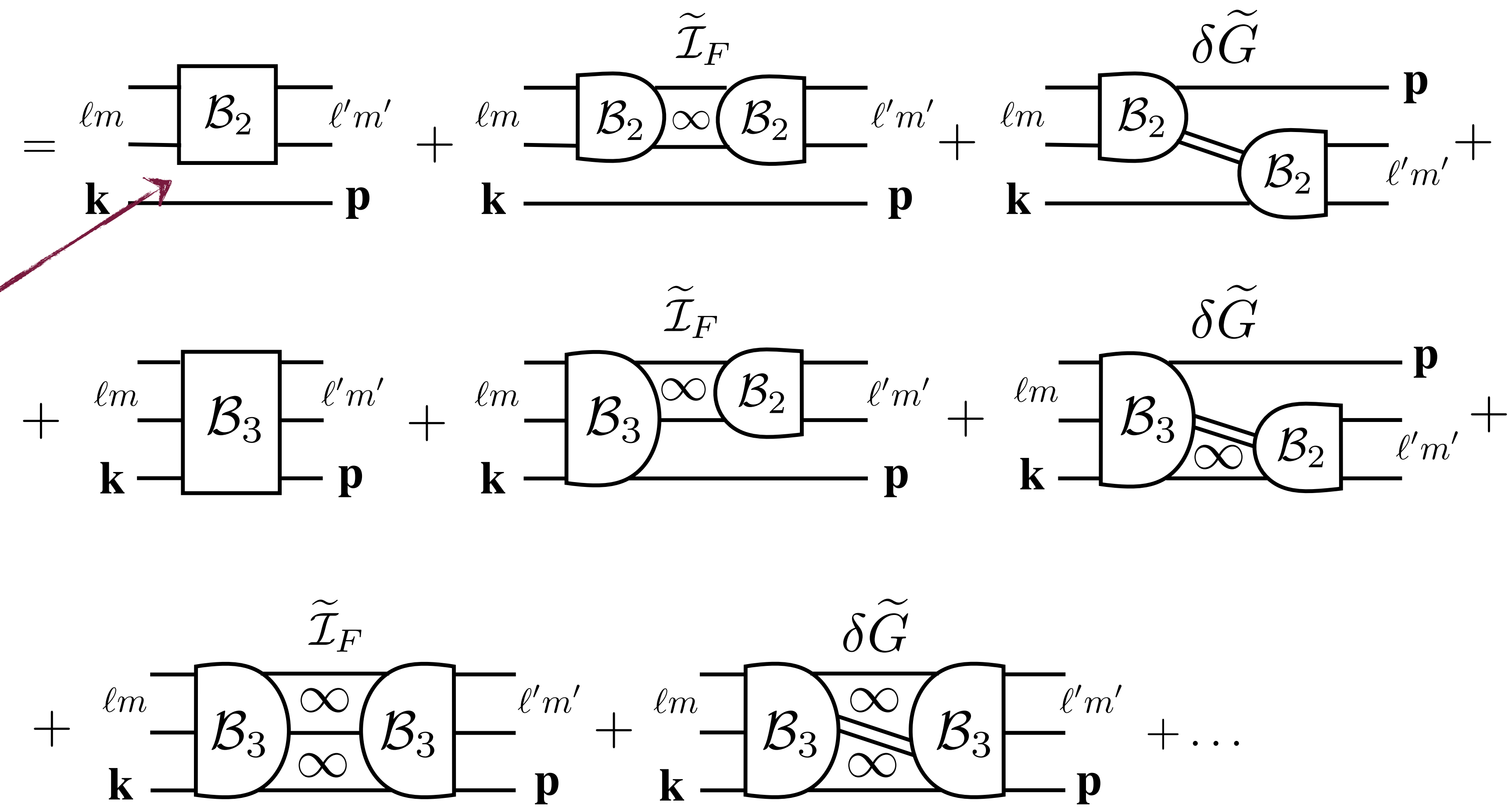
$$= \widetilde{C}_{3,\infty}(E,\vec{P}) + \widetilde{A}'^{(u)} i(\widetilde{F} + \widetilde{G}) \frac{1}{1 - i\left(2\omega L^3 \mathcal{K}_2 + \widetilde{\mathcal{K}}_{\mathrm{df},3}^{(u,u)}\right) i(\widetilde{F} + \widetilde{G})} \widetilde{A}^{(u)}$$

where $$i\left(2\omega L^3 \mathcal{K}_2 + \widetilde{\mathcal{K}}_{\mathrm{df},3}^{(u,u)}\right) = \frac{1}{1 - i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(\widetilde{\mathcal{I}}_F + \delta\widetilde{G})} i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3)$$

- Simplicity of expression is due to combining 2- and 3-particle K matrices
- $\widetilde{A}'^{(u)}$, $\mathcal{K}_2$, $\widetilde{\mathcal{K}}_{\mathrm{df},3}^{(u,u)}$, & $\widetilde{A}^{(u)}$ are on-shell, infinite-volume quantities
- "(u)" & "(u,u)" indicate asymmetry due to factors of $\overline{\mathcal{B}}_{2,L}$

Meaning of asymmetry

$$2\omega L^3 \mathscr{K}_2 + \widetilde{\mathscr{K}}^{(u,u)}_{\mathrm{df},3} =$$



On-shell kernels shown by flat ends

- Momenta $\mathbf{k}$, $\mathbf{p}$ spectate if external interaction involves two particles

New form of QC3

$$C_{3,L} - \widetilde{C}_{3,\infty} = \widetilde{A}'^{(u)} i(\widetilde{F} + \widetilde{G}) \frac{1}{1 - i\left(2\omega L^3 \mathcal{K}_2 + \widetilde{\mathcal{K}}_{\mathrm{df},3}^{(u,u)}\right) i(\widetilde{F} + \widetilde{G})} \widetilde{A}^{(u)}$$

- Spectrum determined by poles in $C_{3,L}(E, \mathbf{P})$

$$\Rightarrow \quad \boxed{\det\left[1 + \left(2\omega L^3 \mathcal{K}_2 + \widetilde{\mathcal{K}}_{\mathrm{df},3}^{(u,u)}\right)\left(\widetilde{F} + \widetilde{G}\right)\right] = 0}$$

- $\widetilde{\mathscr{K}}_{df,3}^{(u,u)}$ related to $\mathscr{M}_3$ by known integral equations

What has been gained?

[HS14, HS15]

$$\det[1 + F_3 \mathscr{K}_{\text{df},3}] = 0$$

$$F_3 = \widetilde{F}\left[\frac{1}{3} - \frac{1}{1/(2\omega L^3 \mathcal{K}_2) + \widetilde{F} + \widetilde{G}}\widetilde{F}\right]$$

- ☐ Complicated derivation, hard to generalize
- ☐ Implicit, constructive definitions
- ☑ $\mathscr{K}_{\text{df},3}$ is Lorentz invariant
- ☑ $\mathscr{K}_{\text{df},3}$ is symmetric under particle exchange, so easier to parametrize

[BS20a]

$$\det[1 + (2\omega L^3 \mathscr{K}_2 + \widetilde{\mathscr{K}}^{(u,u)}_{\text{df},3})(\widetilde{F} + \widetilde{G})] = 0$$

- ☑ Greatly simplified derivation, easy to generalize
- ☑ Explicit expressions for all quantities
- ☑ Clean separation of infinite- and finite-volume quantities
- ☐ $\widetilde{\mathscr{K}}^{(u,u)}_{\text{df},3}$ is not Lorentz invariant (because we used TOPT)
- ☐ Asymmetry of $\widetilde{\mathscr{K}}^{(u,u)}_{\text{df},3}$ implies that description requires additional parameters

Best of both worlds?

$$\det[1 + F_3 \mathcal{K}_{\rm df,3}] = 0$$

$$F_3 = \widetilde{F}\left[\frac{1}{3} - \frac{1}{1/(2\omega L^3 \mathcal{K}_2) + \widetilde{F} + \widetilde{G}}\widetilde{F}\right]$$

can asymmetrize to new form

$$\det[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}'^{(u,u)}_{\rm df,3})(\widetilde{F} + \widetilde{G})] = 0$$

- $\mathcal{K}'^{(u,u)}_{\rm df,3}$ given by an implicit definition
- $\mathcal{K}'^{(u,u)}_{\rm df,3}$ is Lorentz invariant
- However, $\mathcal{K}'^{(u,u)}_{\rm df,3} \neq \widetilde{\mathcal{K}}^{(u,u)}_{\rm df,3}$, due to ambiguity in asymmetrization

$$\det[1 + (2\omega L^3 \mathcal{K}_2 + \widetilde{\mathcal{K}}^{(u,u)}_{\rm df,3})(\widetilde{F} + \widetilde{G})] = 0$$

can symmetrize to original form

$$\det[1 + F_3 \widetilde{\mathcal{K}}'_{\rm df,3}] = 0$$

- $\widetilde{\mathcal{K}}'_{\rm df,3}$ obtained from $\widetilde{\mathcal{K}}^{(u,u)}_{\rm df,3}$ by solving an integral equation and symmetrizing
- Can show that $\widetilde{\mathcal{K}}'_{\rm df,3} = \mathcal{K}_{\rm df,3}$ so obtain exactly the original [HS14] QC3

Equivalence of relativistic QC3s

[Blanton & SRS, arXiv:2007.16190]

Relativistic QC3 landscape

RFT = generic relativistic EFT

FVU = finite-volume unitarity

$\mathcal{K}_{\mathrm{df},3} \to \mathcal{M}_3$
Integral eqs.
[HS15]

Equivalent representations of $\mathcal{M}_3$ (infinite volume)

Jackura, Dawid, Fernández-Ramírez, Mathieu, Mikhasenko, Pilloni, SRS, Szczepaniak, 19

Unitary representation of $\mathcal{M}_3$ in terms of $\mathcal{R}^{(u,u)}$
Mai et al., 17; Jackura et al., 18

$L \to \infty$

QC3 (all ℓ)

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

[HS14]

QC3 ($\ell = 0$)

$$\det\left[\widetilde{H}_s - \frac{1}{2\omega L^3}\widetilde{C}_s^{(u,u)}\frac{1}{2\omega L^3}\right] = 0$$

Mai & Döring, 17

"algebra"

$$\widetilde{H}_s = \widetilde{F}_s + \widetilde{G}_s + 1/(2\omega L^3 \mathcal{K}_2^s)$$

$$\widetilde{C}_s^{(u,u)} = \mathcal{R}_s^{(u,u)}$$

[HSREV19]

Alternate form of QC3

$$\det\left[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}'^{(u,u)}_{\mathrm{df},3})(\widetilde{F} + \widetilde{G})\right] = 0$$

[BS20a]

Relativistic QC3 landscape

RFT = generic relativistic EFT

FVU = finite-volume unitarity

$\mathcal{K}_{\mathrm{df},3} \to \mathcal{M}_3$

Integral eqs.

[HS15]

Equivalent representations of $\mathcal{M}_3$ (infinite volume)

Jackura, Dawid, Fernández-Ramírez, Mathieu, Mikhasenko, Pilloni, SRS, Szczepaniak, 19

Unitary representation of $\mathcal{M}_3$ in terms of $\mathcal{R}^{(u,u)}$

Mai et al., 17; Jackura et al., 18

$L \to \infty$

QC3 (all ℓ)

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

[HS14]

QC3 ($\ell = 0$)

$$\det\left[\widetilde{H}_s - \frac{1}{2\omega L^3}\widetilde{C}_s^{(u,u)}\frac{1}{2\omega L^3}\right] = 0$$

Mai & Döring, 17

Form from [HSREV19]

"algebra"

Is there a relation?

$$\widetilde{H}_s = \widetilde{F}_s + \widetilde{G}_s + 1/(2\omega L^3 \mathcal{K}_2^s)$$

$$\widetilde{C}_s^{(u,u)} = \mathcal{R}_s^{(u,u)}$$

Alternate form of QC3

$$\det\left[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{\mathrm{df},3}^{\prime(u,u)})(\widetilde{F} + \widetilde{G})\right] = 0$$

[BS20a]

Relativistic QC3 landscape

RFT = generic relativistic EFT

FVU = finite-volume unitarity

$\mathcal{K}_{\mathrm{df},3} \to \mathcal{M}_3$

Integral eqs.

[HS15]

Equivalent representations of $\mathcal{M}_3$ (infinite volume)

Jackura, Dawid, Fernández-Ramírez, Mathieu, Mikhasenko, Pilloni, SRS, Szczepaniak, 19

Unitary representation of $\mathcal{M}_3$ in terms of $\mathcal{R}^{(u,u)}$

Mai et al., 17; Jackura et al., 18

$L \to \infty$

QC3 (all ℓ)

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

[HS14]

QC3 ($\ell = 0$)

$$\det\left[\widetilde{H}_s - \frac{1}{2\omega L^3}\mathcal{R}_s^{(u,u)}\frac{1}{2\omega L^3}\right] = 0$$

Mai & Döring, 17

Form from [HSREV19]

"algebra"

$$\widetilde{H} = \widetilde{F} + \widetilde{G} + 1/(2\omega L^3 \mathcal{K}_2)$$

Alternate form of QC3

$$\det\left[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}_{\mathrm{df},3}^{\prime(u,u)})(\widetilde{F} + \widetilde{G})\right] = 0$$

[BS20a]

"algebra"

Alternate form of QC3

$$\det\left[\widetilde{H} - \frac{1}{2\omega L^3}\mathcal{R}^{(u,u)}\frac{1}{2\omega L^3}\right] = 0$$

[BS20b]

Relativistic QC3 landscape

RFT = generic relativistic EFT

FVU = finite-volume unitarity

$\mathcal{K}_{\mathrm{df},3} \to \mathcal{M}_3$

Integral eqs.

[HS15]

Equivalent representations of $\mathcal{M}_3$ (infinite volume)

Jackura, Dawid, Fernández-Ramírez, Mathieu, Mikhasenko, Pilloni, SRS, Szczepaniak, 19

Unitary representation of $\mathcal{M}_3$ in terms of $\mathcal{R}^{(u,u)}$

Mai et al., 17; Jackura et al., 18

$L \to \infty$

QC3 (all ℓ)

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

[HS14]

- Provides derivation of FVU QC3
- Generalizes result to all ℓ
- Shows equivalence of approaches

QC3 ($\ell = 0$)

$$\det\left[\widetilde{H}_s - \frac{1}{2\omega L^3}\mathcal{R}_s^{(u,u)}\frac{1}{2\omega L^3}\right] = 0$$

Mai & Döring, 17

"algebra"

Alternate form of QC3

$$\det\left[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}'^{(u,u)}_{\mathrm{df},3})(\widetilde{F} + \widetilde{G})\right] = 0$$

[BS20a]

Alternate form of QC3

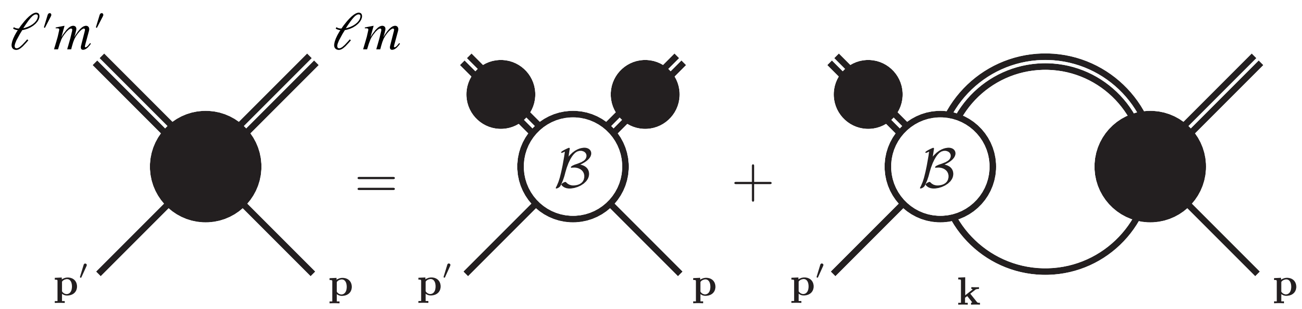
$$\det\left[\widetilde{H} - \frac{1}{2\omega L^3}\mathcal{R}^{(u,u)}\frac{1}{2\omega L^3}\right] = 0$$

[BS20b]

Step 1: equate forms for $\mathscr{M}_3^{(u,u)}$

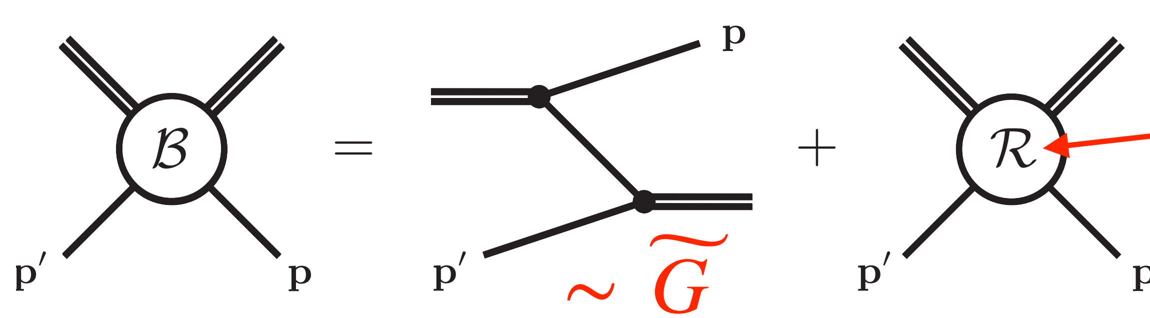
- R-matrix form for asymmetric, infinite-volume three-particle amplitudes

$$\mathcal{M}_3^{\mathcal{R},(u,u)} =$$



Formula from Mai et al., 17 & Jackura et al., 18

Figures from Jackura et al., 19



We call it $\mathcal{R}^{(u,u)}$ to emphasize its asymmetry

- If symmetrize, get unitary $\mathscr{M}_3$
- Asymmetry not defined in terms of diagrams (Feynman or TOPT)
- If $\mathscr{R}^{(u,u)}$ is Lorentz invariant, then so is $\mathscr{M}_3^{\mathscr{R},(u,u)}$

Step 1: equate forms for $\mathscr{M}_3^{(u,u)}$

- RFT form for Feynman-diagram-based asymmetric amplitude [BS20a]

$$\mathscr{M}_3^{(u,u)} = \lim_{L\to\infty} \mathscr{M}_{3,L}^{(u,u)}$$

$$\mathscr{M}_{3,L}^{(u,u)} = \mathscr{M}_{\mathrm{df},3,L}^{(u,u)} + \mathscr{D}_L^{(u,u)}$$

$$\mathcal{M}_{\mathrm{df},3,L}^{(u,u)} = \frac{1}{1+\overline{\mathcal{K}}_{2,L}(\widetilde{F}+\widetilde{G})}\mathcal{K}'^{(u,u)}_{\mathrm{df},3}\frac{1}{1+(\widetilde{F}+\widetilde{G})\frac{1}{1+\overline{\mathcal{K}}_{2,L}(\widetilde{F}+\widetilde{G})}\mathcal{K}'^{(u,u)}_{\mathrm{df},3}}\frac{1}{1+(\widetilde{F}+\widetilde{G})\overline{\mathcal{K}}_{2,L}}$$

$$\mathcal{D}_L^{(u,u)} = -\overline{\mathcal{M}}_{2,L}\widetilde{G}\overline{\mathcal{M}}_{2,L}\frac{1}{1+\widetilde{G}\overline{\mathcal{M}}_{2,L}}$$

- If symmetrize, get unitary $\mathscr{M}_3$
- Asymmetry defined in terms of Feynman diagrams
- $\mathscr{K}'^{(u,u)}_{\mathrm{df},3}$ is Lorentz invariant

Step 1: equate forms for $\mathscr{M}_3^{(u,u)}$

- RFT form for Feynman-diagram-based asymmetric amplitude [BS20a]

$$\mathscr{M}_3^{(u,u)} = \lim_{L\to\infty} \mathscr{M}_{3,L}^{(u,u)}$$

$$\mathscr{M}_{3,L}^{(u,u)} = \mathscr{M}_{\text{df},3,L}^{(u,u)} + \mathscr{D}_L^{(u,u)}$$

$$\mathcal{M}_{\text{df},3,L}^{(u,u)} = \frac{1}{1+\overline{\mathcal{K}}_{2,L}(\widetilde{F}+\widetilde{G})}\mathcal{K}_{\text{df},3}^{\prime(u,u)}\frac{1}{1+(\widetilde{F}+\widetilde{G})\frac{1}{1+\overline{\mathcal{K}}_{2,L}(\widetilde{F}+\widetilde{G})}\mathcal{K}_{\text{df},3}^{\prime(u,u)}}\frac{1}{1+(\widetilde{F}+\widetilde{G})\overline{\mathcal{K}}_{2,L}}$$

$$\mathcal{D}_L^{(u,u)} = -\overline{\mathcal{M}}_{2,L}\widetilde{G}\overline{\mathcal{M}}_{2,L}\frac{1}{1+\widetilde{G}\overline{\mathcal{M}}_{2,L}}$$

  - If symmetrize, get unitary $\mathscr{M}_3$
  - Asymmetry defined in terms of Feynman diagrams
  - $\mathscr{K}_{\text{df},3}^{\prime\,(u,u)}$ is Lorentz invariant

Equating $\mathscr{M}_3^{\mathscr{R},(u,u)}$ and $\mathscr{M}_3^{(u,u)}$ gives integral equation relating $\mathscr{R}^{(u,u)}$ and $\mathscr{K}_{\text{df},3}^{\prime\,(u,u)}$

Step 2: rewrite asymmetric QC3

- Algebraic manipulations

$$\det\left[1+(2\omega L^3\mathcal{K}_2+\mathcal{K}'^{(u,u)}_{\rm df,3})(\widetilde{F}+\widetilde{G})\right]=0 \quad\Rightarrow\quad \det\left[\widetilde{H}-X^{(u,u)}\right]=0$$

$$\widetilde{H}=\widetilde{F}+\widetilde{G}+\overline{\mathcal{K}}^{-1}_{2,L} \qquad\qquad \overline{\mathcal{K}}_{2,L}=(2\omega L^3)\mathcal{K}_2$$

$$X^{(u,u)}=\overline{\mathcal{K}}^{-1}_{2,L}\mathcal{K}'^{(u,u)}_{\rm df,3}\overline{\mathcal{K}}^{-1}_{2,L}\frac{1}{1+\mathcal{K}'^{(u,u)}_{\rm df,3}\overline{\mathcal{K}}^{-1}_{2,L}}$$

Step 3: combine

$$\det\left[\widetilde{H} - X^{(u,u)}\right] = 0 \qquad X^{(u,u)} = \overline{\mathcal{K}}_{2,L}^{-1}\mathcal{K}'^{(u,u)}_{\mathrm{df},3}\overline{\mathcal{K}}_{2,L}^{-1}\frac{1}{1+\mathcal{K}'^{(u,u)}_{\mathrm{df},3}\overline{\mathcal{K}}_{2,L}^{-1}}$$

- Using integral equation relating $\mathcal{R}^{(u,u)}$ and $\mathcal{K}'^{(u,u)}_{\mathrm{df},3}$, can show that

$$\left[(2\omega L^3)X^{(u,u)}(2\omega L^3)\right]_{k\ell m;p\ell'm'} = \left[\mathcal{R}^{(u,u)}\right]_{k\ell m;p\ell'm'} + \mathcal{O}(e^{-mL})$$

- Substituting gives claimed result

$$\det\left[\widetilde{H} - (2\omega L^3)^{-1}\mathcal{R}^{(u,u)}(2\omega L^3)^{-1}\right] = 0$$

  - Derivation valid only if use smooth cutoff function, and appropriate form for $\widetilde{G}$
  - We expect that, if we take the NR limit, we will obtain the NREFT form of the QC3 generalized to all ℓ

Conclusions & Outlook

Summary

- Formalism is ready to use for phenomenologically interesting case of 3 pions with all allowed isospins
  - Also provides numerical method for studying properties of relativistic 2- and 3-particle bound states
- First comparisons with LQCD data for $3\pi^+$ show evidence for 3-particle quasilocal interaction $\mathscr{K}_{\text{df},3}$
  - Several similar studies have recently appeared: [Culver et al., 1911.09047 (GWU), Fischer et al., 2008.03035 (ETMC), Hansen et al., 2009.04931 (Hadspec)]
- Simplified derivation of QC3 using TOPT
- Equivalence of RFT and FVU forms demonstrated, and FVU form generalized

Relative merits of forms of QC3

$$\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$

- Symmetric three-particle K matrix
- Threshold expansion requires fewer parameters
- Little intuition in presence of three-particle resonances
- $\mathscr{K}_{\mathrm{df},3}$ depends on PV pole prescription

$$\det\left[\widetilde{F} + \widetilde{G} + \frac{1}{2\omega L^3 \mathcal{K}_2} - \frac{1}{2\omega L^3}\mathcal{R}^{(u,u)}\frac{1}{2\omega L^3}\right] = 0$$

- Asymmetric three-particle R matrix
- Threshold expansion requires more (redundant) parameters
- Some experience and intuition from JPAC studies of fitting amplitudes to experimental data
- $\mathscr{R}^{(u,u)}$ independent of pole prescription

To-do list for QC3s

- Generalize formalism to broaden applications
  - Nondegenerate particles, e.g. $K^+K^+\pi^+$ [Pang et al., 2008.13014 (NREFT)]
  - Spin for, e.g., $N\pi\pi$
  - Determination of Lellouch-Lüscher factors to allow application to K→3π etc
- Develop physics-based parametrizations of $\mathcal{K}_{\rm df,3}$ to describe resonances
  - Need to learn how to solve integral equations relating $K_{df,3}$ to M_3 above threshold [Jackura, INT talk, 8/20; Hansen et al., 2009.04931]
- Understand appearance of unphysical solutions (wrong residue) for some values of parameters—observed in [BHS18; BRS19]
  - May be due to truncation, or due to exponentially suppressed effects, or both
- Move on to QC4 !?

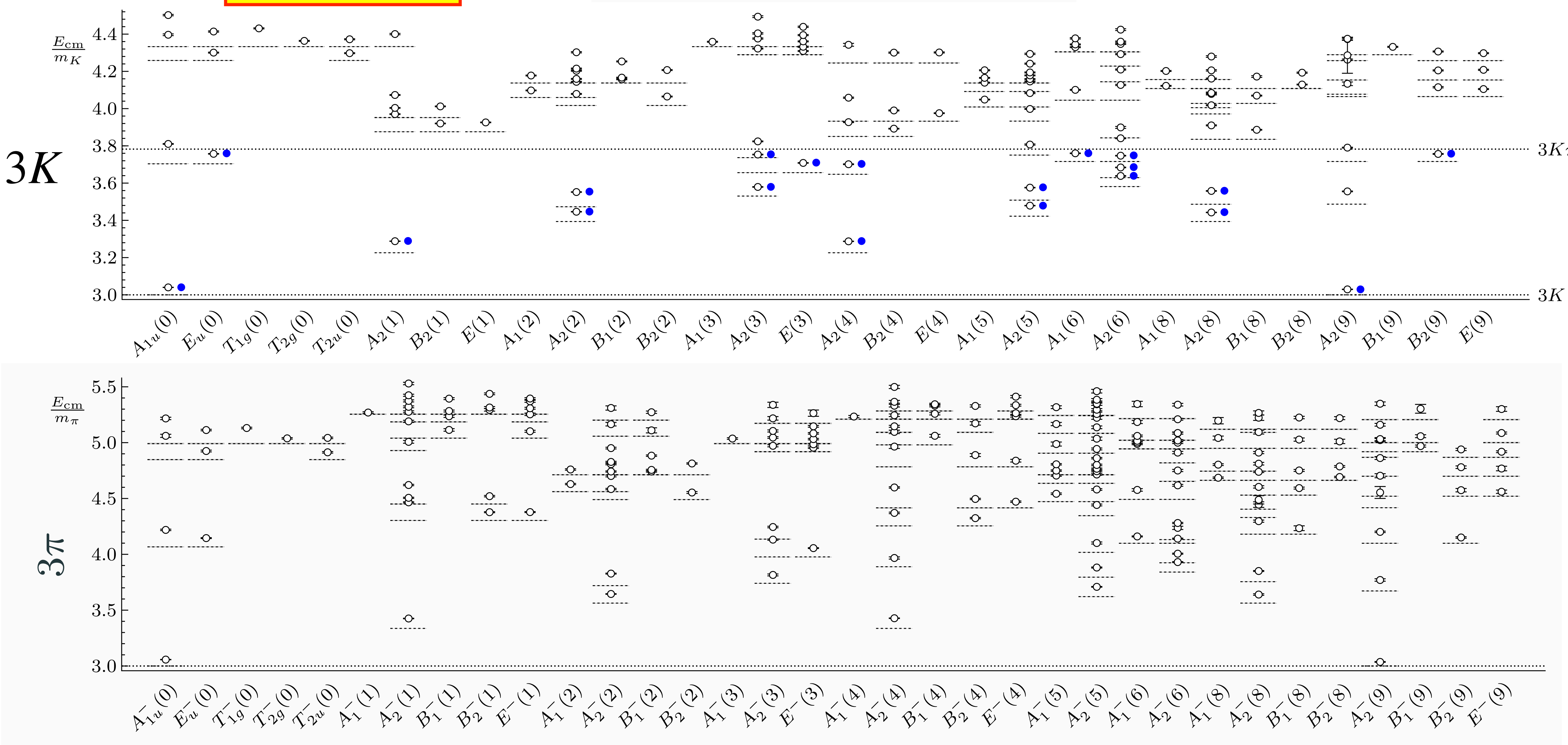
A taste of the future?

[Blanton, Hanlon, Hörz, Romero-López, SRS, in progress]

- Many precise $3\pi^+$ & $3K^+$ levels in all irreps in 8 frames for three choices of quark masses
- Should allow extraction of s- and d-wave parameters of 2- and 3-particle interactions

Preliminary

$m_\pi = 345\,\mathrm{MeV},\ m_K = 440\,\mathrm{MeV}$



Thank you!
Questions?

Backup slides

F_3 collects 2-particle interactions

$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\frac{1}{\mathcal{K}_2^{-1} + F + G}F\right]$$

- F & G are known geometrical functions, containing cutoff function H

$$F_{p\ell'm';k\ell m} = \delta_{pk}\ H(\vec{k})\ F_{\mathrm{PV},\ell'm';\ell m}(E-\omega_k, \vec{P}-\vec{k}, L)$$

$$G_{p\ell'm';k\ell m} = \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell'm'}(\hat{k}^*)H(\vec{p})H(\vec{k})Y^*_{\ell m}(\hat{p}^*)}{(P-k-p)^2-m^2}\left(\frac{p^*}{q_k^*}\right)^{\ell}\frac{1}{2\omega_k L^3}$$

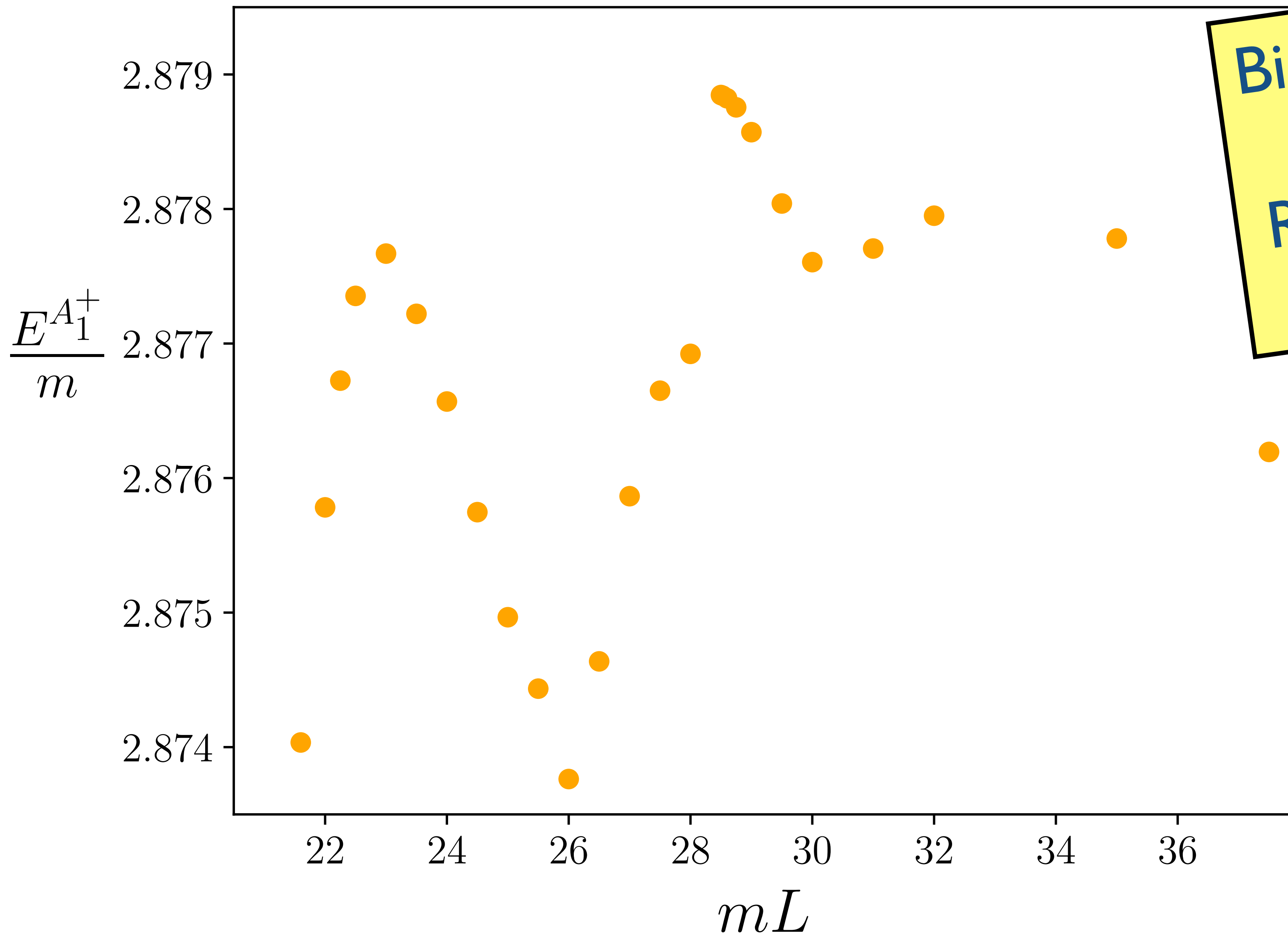
Relativistic form introduced in [BHS17]

$$F_{\mathrm{PV};\ell'm';\ell m}(E,\vec{P},L) = \frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \mathrm{PV}\int\frac{d^3k}{(2\pi)^3}\right)\frac{\mathcal{Y}_{\ell'm'}(\vec{k}^*)\mathcal{Y}^*_{\ell m}(\vec{k}^*)\,h(\vec{k})}{2\omega_k 2\omega_{P-k}(E-\omega_k-\omega_{P-k})}$$

Relativistic form equivalent up to exponentially-suppressed terms

$$\mathcal{Y}_{\ell m}(\vec{k}^*) = \sqrt{4\pi}\left(\frac{k^*}{q^*}\right)^{\ell} Y_{\ell m}(\hat{k}^*)$$

Evidence for trimer bound by a_2



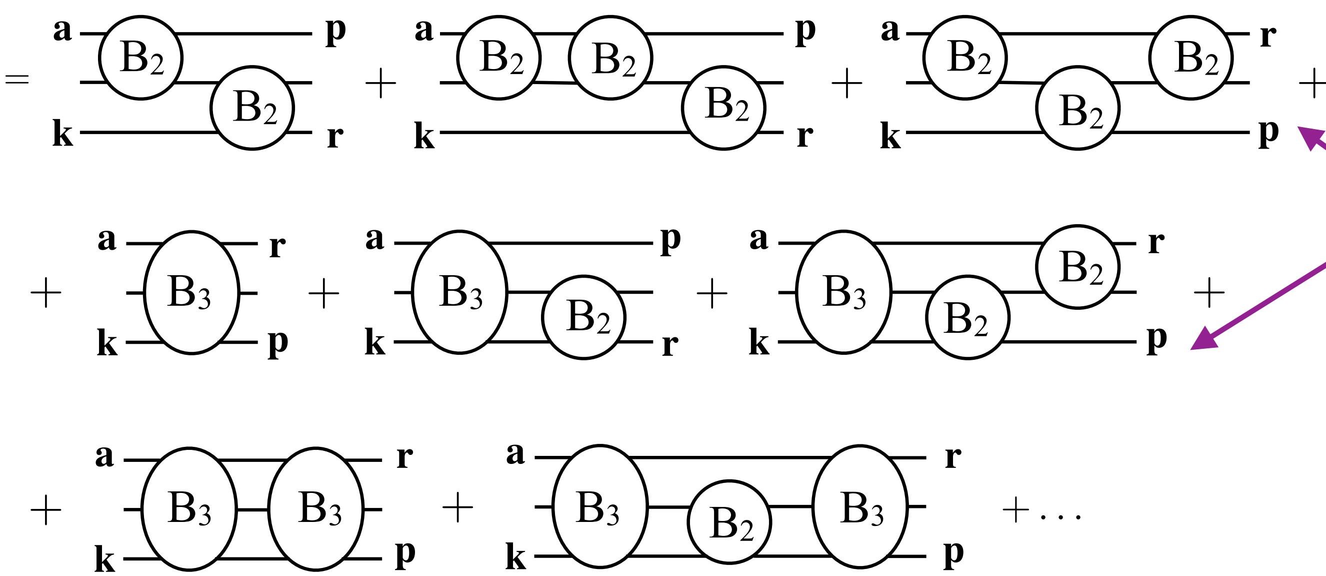
Binding caused by d-wave attraction! Relevant for atomic physics?

$$ma_0 = -0.1,\ ma_2 = -1.3,\ r_0 = P_0 = \mathscr{K}_{df,3} = 0$$

Definitions of asymmetric kernels

- In original RFT approach (using Feynman diagrams & Bethe-Salpeter kernels)

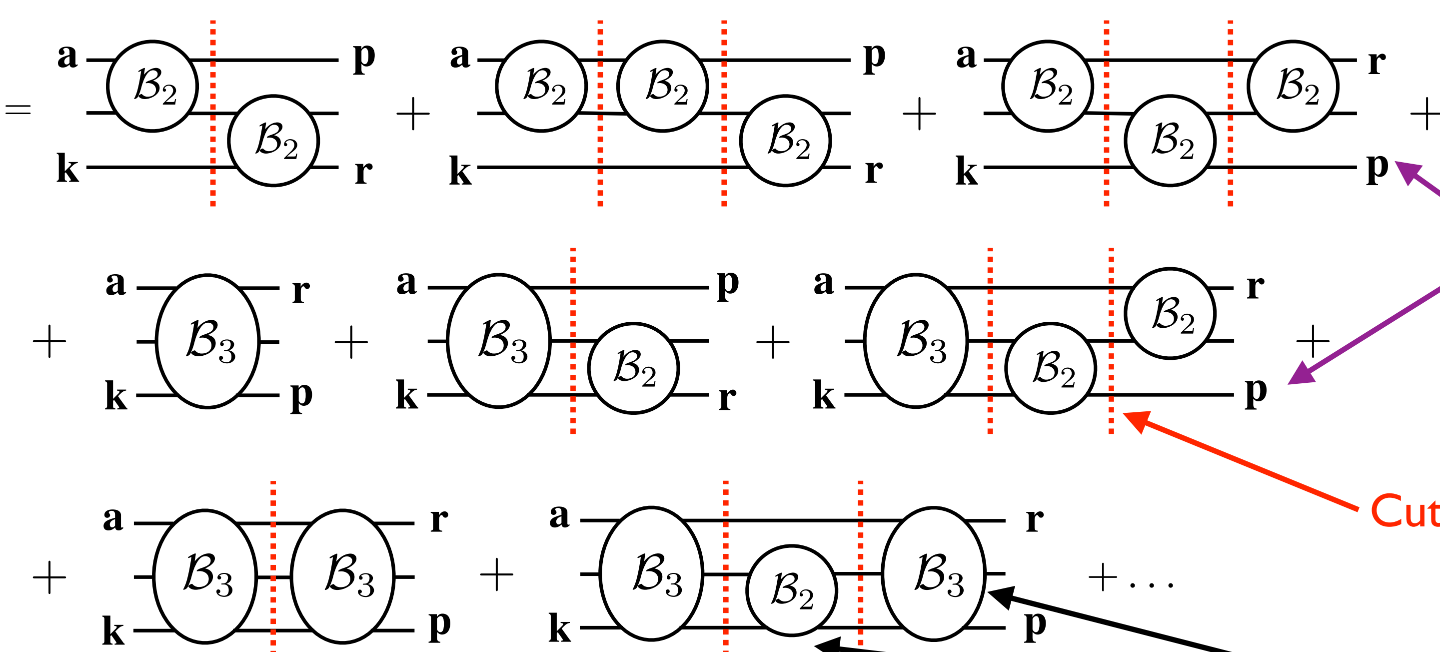
$$\left[\mathcal{M}_3^{(u,u)}\right]_{ka;pr} =$$



k & **p** assigned to spectators

- In our approach (using time-ordered perturbation theory)

$$\left[\widetilde{\mathcal{M}}_3^{(u,u)}\right]_{ka;pr} =$$



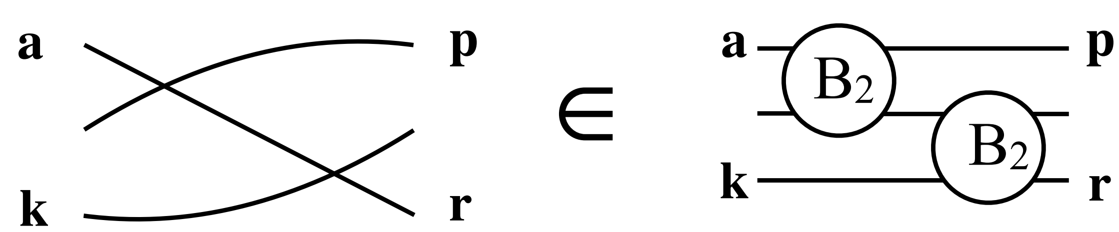
k & **p** assigned to spectators

Cuts in time-ordered PT

TOPT kernels (no 3-particle cuts)

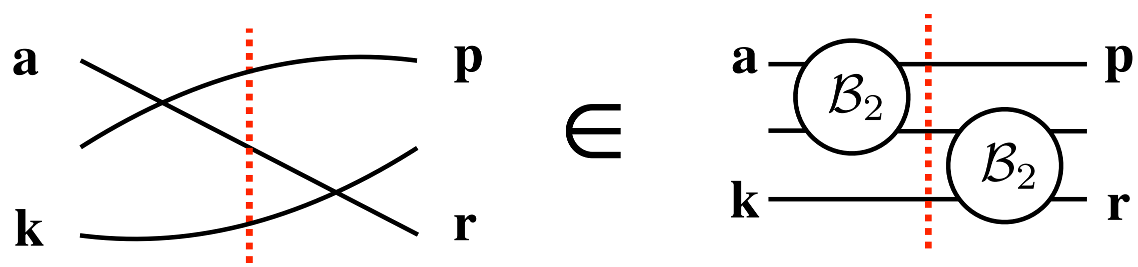
Asymmetric kernels differ!

- Consider a particular Feynman diagram

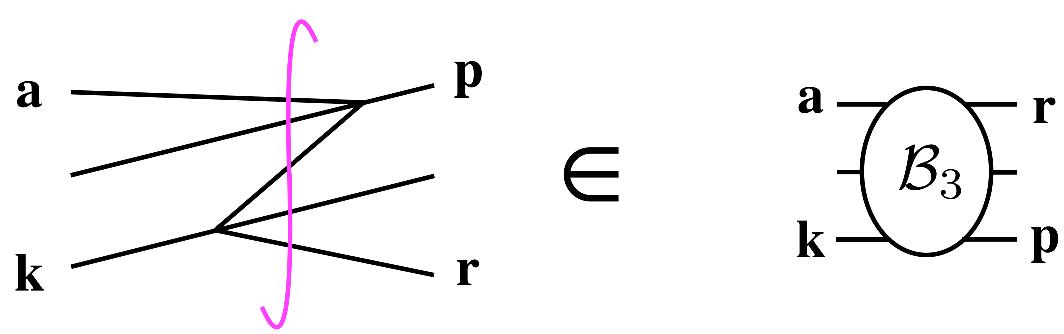


Asymmetric

- In TOPT the two time orderings are put into different terms—one being symmetrized



Asymmetric



Symmetrized

- Thus $\mathcal{M}_3^{(u,u)} \neq \widetilde{\mathcal{M}}_3^{(u,u)}$, although both symmetrize to $\mathcal{M}_3$

Asymmetric kernels ⇒ redundancy

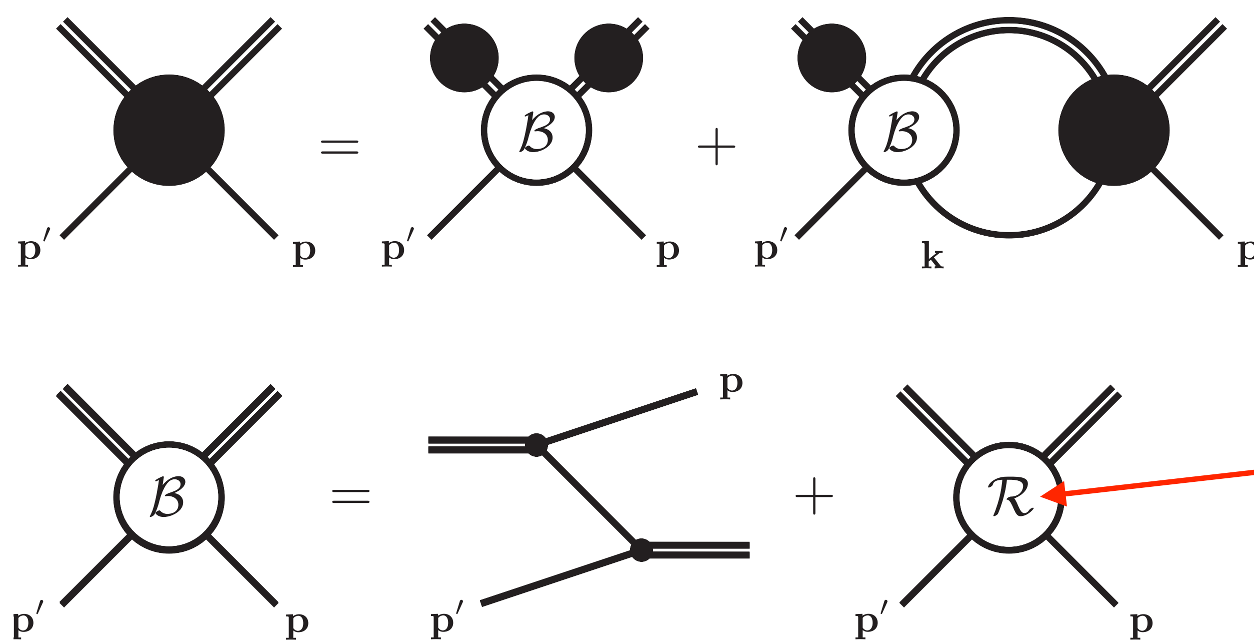
- E.g., asymmetric form of QC3 holds with (at least) two different kernels

$$\det\left[1 + (2\omega L^3 \mathcal{K}_2 + \mathcal{K}'^{(u,u)}_{\mathrm{df},3})(\widetilde{F} + \widetilde{G})\right] = 0$$

$$\det\left[1 + (2\omega L^3 \mathcal{K}_2 + \widetilde{\mathcal{K}}^{(u,u)}_{\mathrm{df},3})(\widetilde{F} + \widetilde{G})\right] = 0$$

Blanton & SS, 20

- R matrix representation of $\mathcal{M}_3^{(u,u)}$ holds for all choices of asymmetry



Formula from Mai et al., 17 & Jackura et al., 18

Figures from Jackura et al., 19

We call it $\mathcal{R}^{(u,u)}$ to emphasize its asymmetry

Can set this equal to either $\mathcal{M}_3^{(u,u)}$ or $\widetilde{\mathcal{M}}_3^{(u,u)}$: leads to different, equally valid, $\mathcal{R}^{(u,u)}$

Sketch of derivation of 3-particle quantization condition

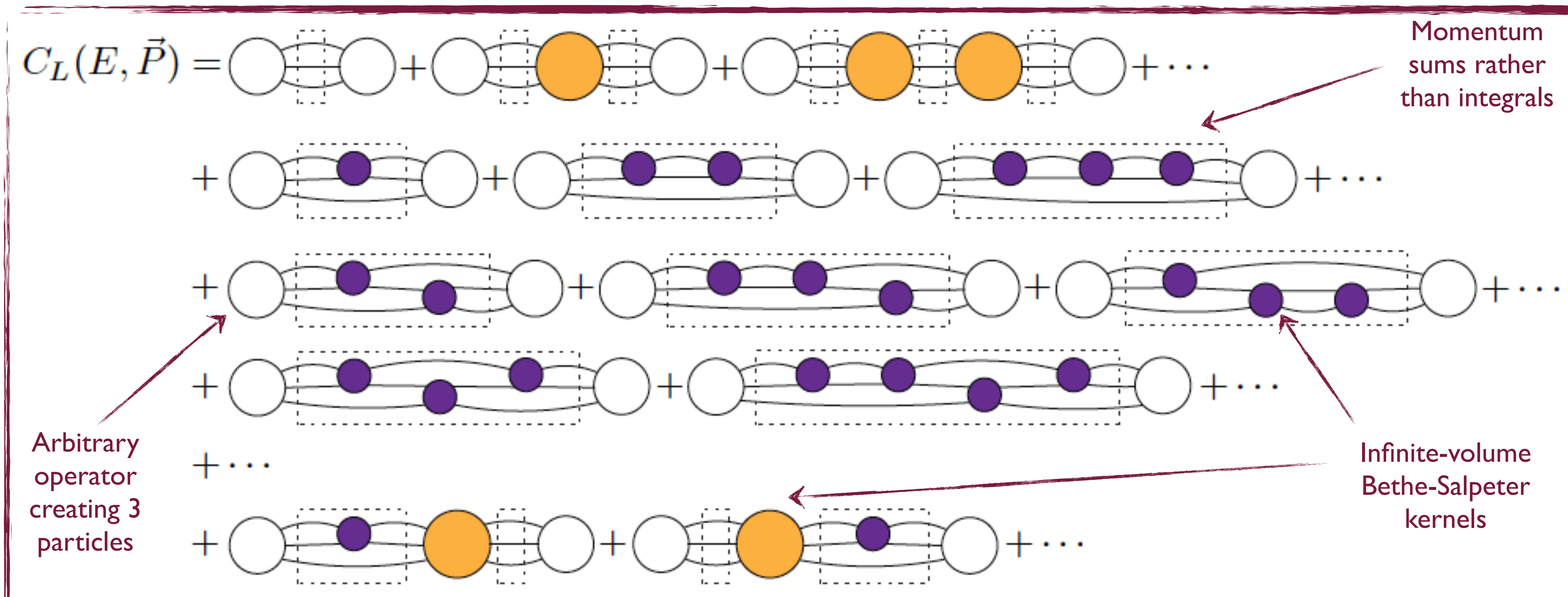
[Hansen & SRS, arXiv:1408.5933 & 1504.04248]

Derivation

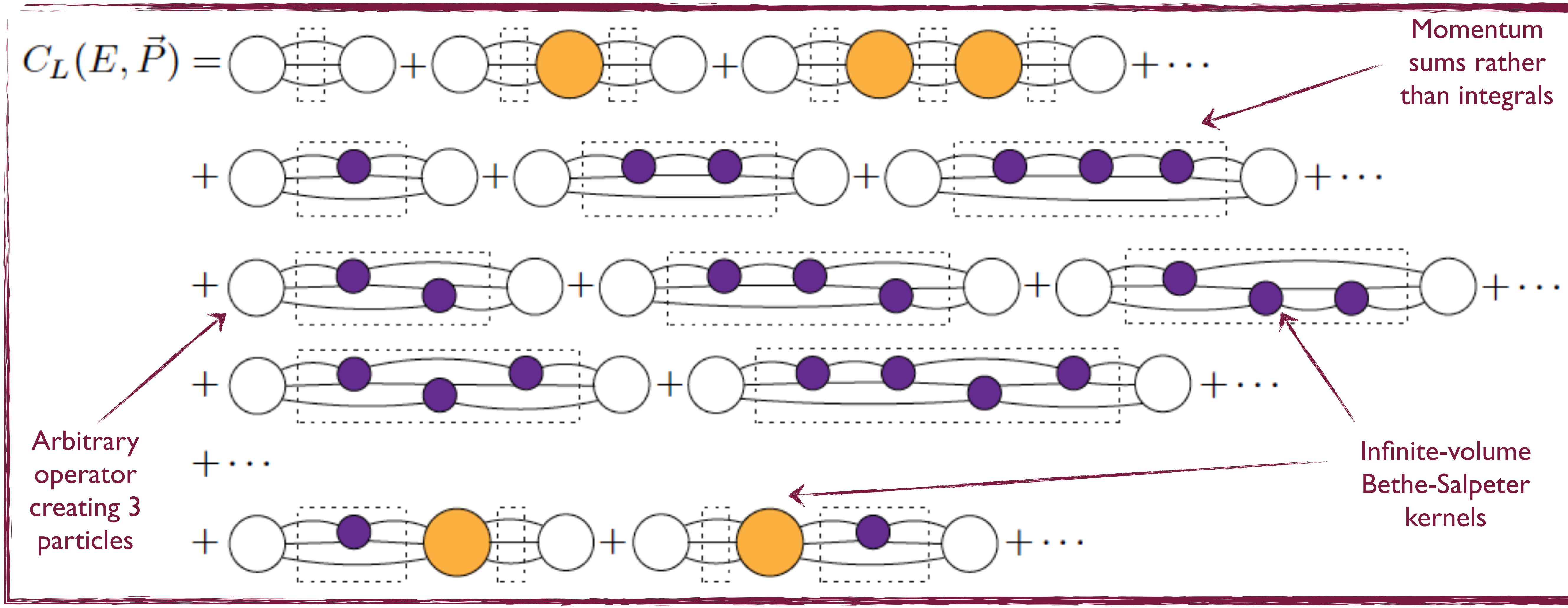
(1)

- Generic relativistic EFT, working to all orders
  - Do not need a power-counting scheme
  - To simplify analysis: impose a global Z_2 symmetry (G parity) & consider identical scalars
- Obtain spectrum from poles in finite-volume correlator
  - Consider $E_{CM} < 5m$ so on-shell states involve only 3 particles

$C_L(E, \vec{P}) =$



Derivation



(2)

- Replace sums with integrals plus sum-integral differences to extent possible
  - If summand has pole or cusp then difference ~$1/L^n$ and must keep (Lüscher zeta function)
  - If summand is smooth then difference ~ exp(-mL) and drop
- Avoid cusps by using PV prescription—leads to generalized 3-particle K matrix
- Subtract above-threshold divergences of 3-particle K matrix—leads to $K_{df,3}$

Derivation

(3)

- Reorganize, resum, … to separate infinite-volume on-shell relativistically-invariant non-singular scattering quantities (K_2, $K_{df,3}$) from known finite-volume functions (F [Lüscher zeta function] & G [“switch function”])

$$\Rightarrow \quad \det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$

Derivation

(4)

- Relate $K_{df,3}$ to M_3 by taking infinite-volume limit of finite-volume scattering amplitude
  - Leads to infinite-volume integral equations involving M_2 & cut-off function H
  - Can formally invert equations to show that $K_{df,3}$ (while unphysical) is relativistically invariant and has same properties under discrete symmetries (P,T) as M_3

Involve only M_2 and G
so "known"

$$i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3} \frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}} \mathcal{R}_L\right]$$

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon} i\mathcal{M}_{L,3\to3}$$

Sums over k go over
to integrals with iε pole prescription