Equivalence of relativistic three-particle quantization conditions

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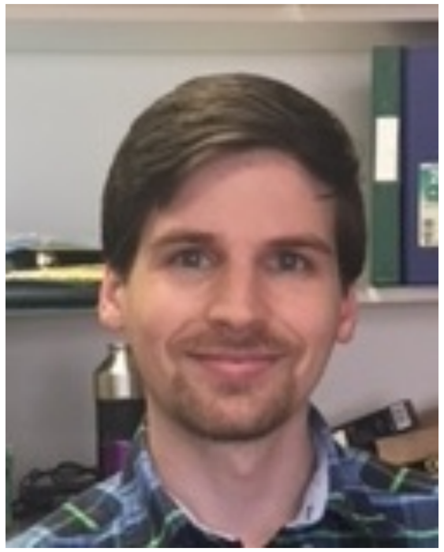
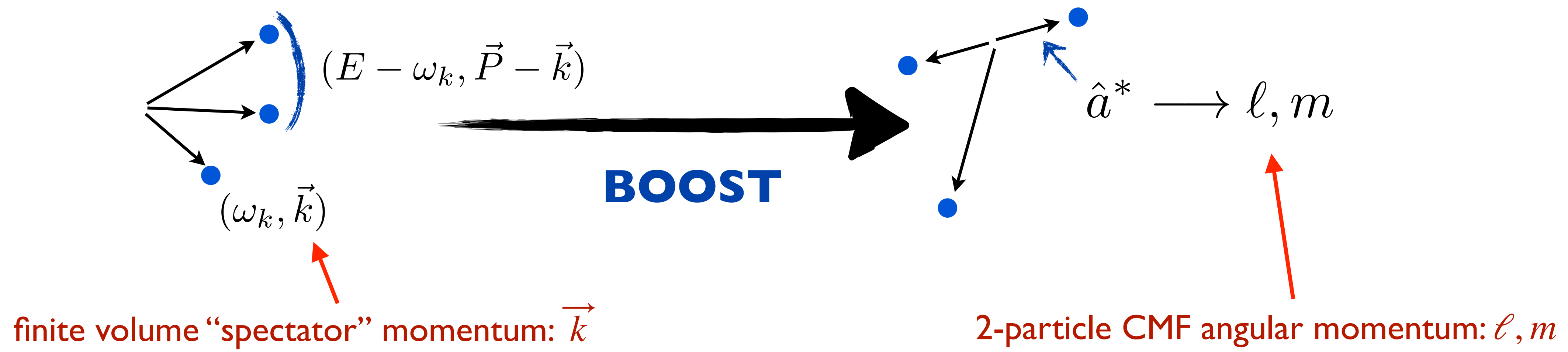
Based on work with Tyler Blanton:
arXiv:2007.16188 and
arXiv:2007.16190



Scope & Notation

- Identical spinless particles of mass m (e.g. $3\pi^+$)
- Z$_2$ symmetry — no $2 \to 3$ transitions
- All quantities in QC3 are infinite-dimensional matrices with indices $\{\overrightarrow{k}, \ell, m\}$ describing 3 on-shell particles with total energy-momentum $(E, \overrightarrow{P})$



e.g. $\left[\mathscr{K}^{(u,u)}_{\mathrm{df},3}\right]_{k\ell m;p\ell'm'}$

Relativistic QC3 landscape 2019

RFT = generic relativistic EFT

FVU = finite-volume unitarity

$\mathscr{K}_{\mathrm{df},3} \to \mathscr{M}_3$

Integral eqs.

Hansen & SS, 15

Equivalent representations of $\mathscr{M}_3$ (infinite volume)

Jackura, Dawid, Fernández-Ramírez, Mathieu, Mikhasenko, Pilloni, SS, Szczepaniak, 19

Unitary representation of $\mathscr{M}_3$ in terms of $\mathscr{R}^{(u,u)}$

Mai, Hu, Döring, Pilloni, Szczepaniak, 17; Jackura, Fernández-Ramírez, Mathieu, Mikhasenko, Nys, Pilloni, Saldaña, Sherril & Szepaniak, 18

$L \to \infty$

QC3 (all ℓ)

$$\det \left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3} \right] = 0$$

Hansen & SS, 14

QC3 ($\ell = 0$)

$$\det \left[\widetilde{H}_s + \frac{1}{2\omega L^3} \widetilde{C}_s^{(u,u)} \frac{1}{2\omega L^3} \right] = 0$$

Mai & Döring, 17

Form given in Hansen & SS, Review 19

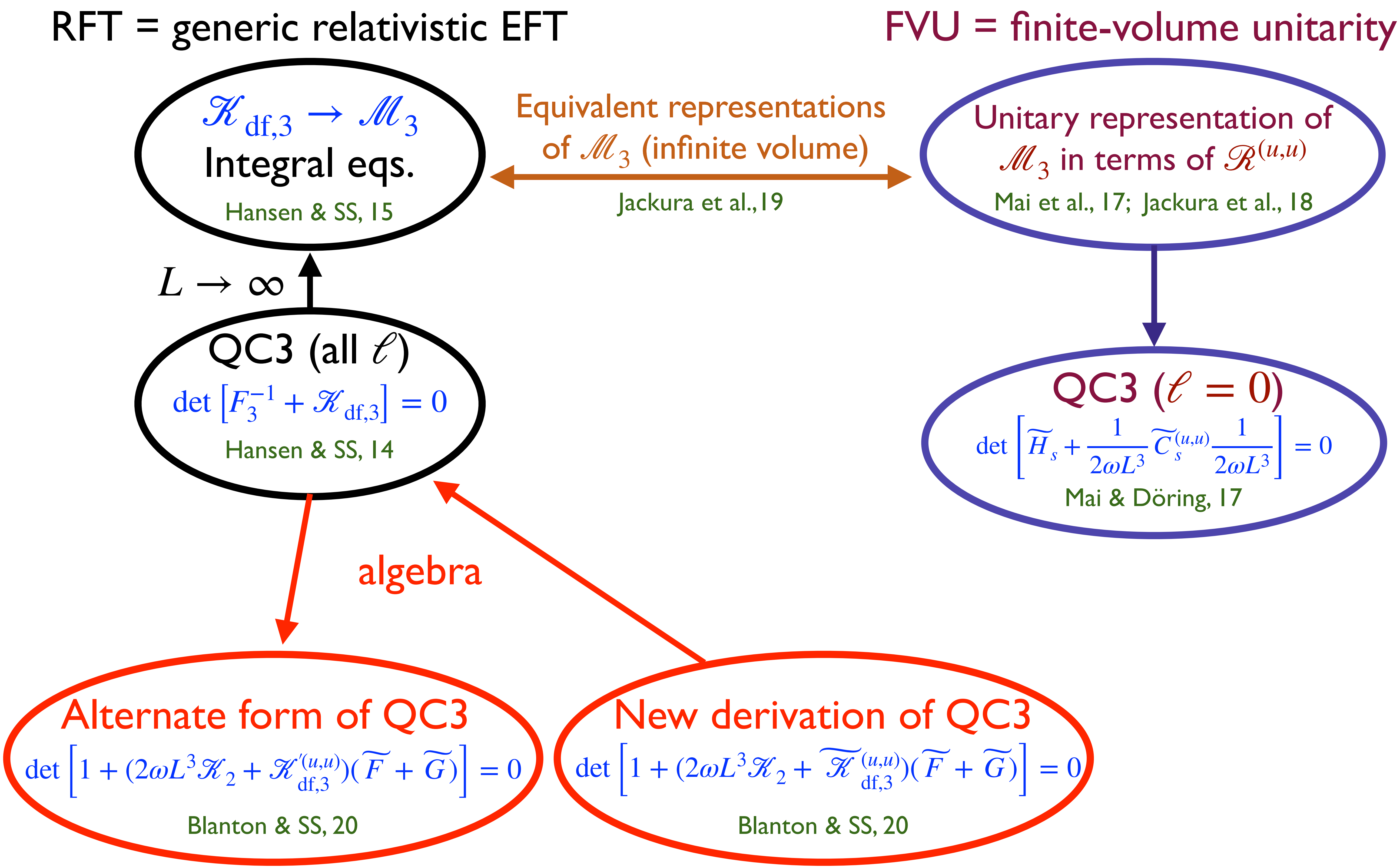
$$F_3 = \widetilde{F} \left[\frac{1}{3} - \frac{1}{\widetilde{H}} \widetilde{F} \right]$$

$$\widetilde{H} = \widetilde{F} + \widetilde{G} + 1/(2\omega L^3 \mathscr{K}_2)$$

Can show that $\widetilde{\mathscr{C}}_s^{(u,u)} = \mathscr{R}_s^{(u,u)}$

(Quantities are defined & full references are given in backup slides)

Relativistic QC3 landscape 2020



Relativistic QC3 landscape 2020

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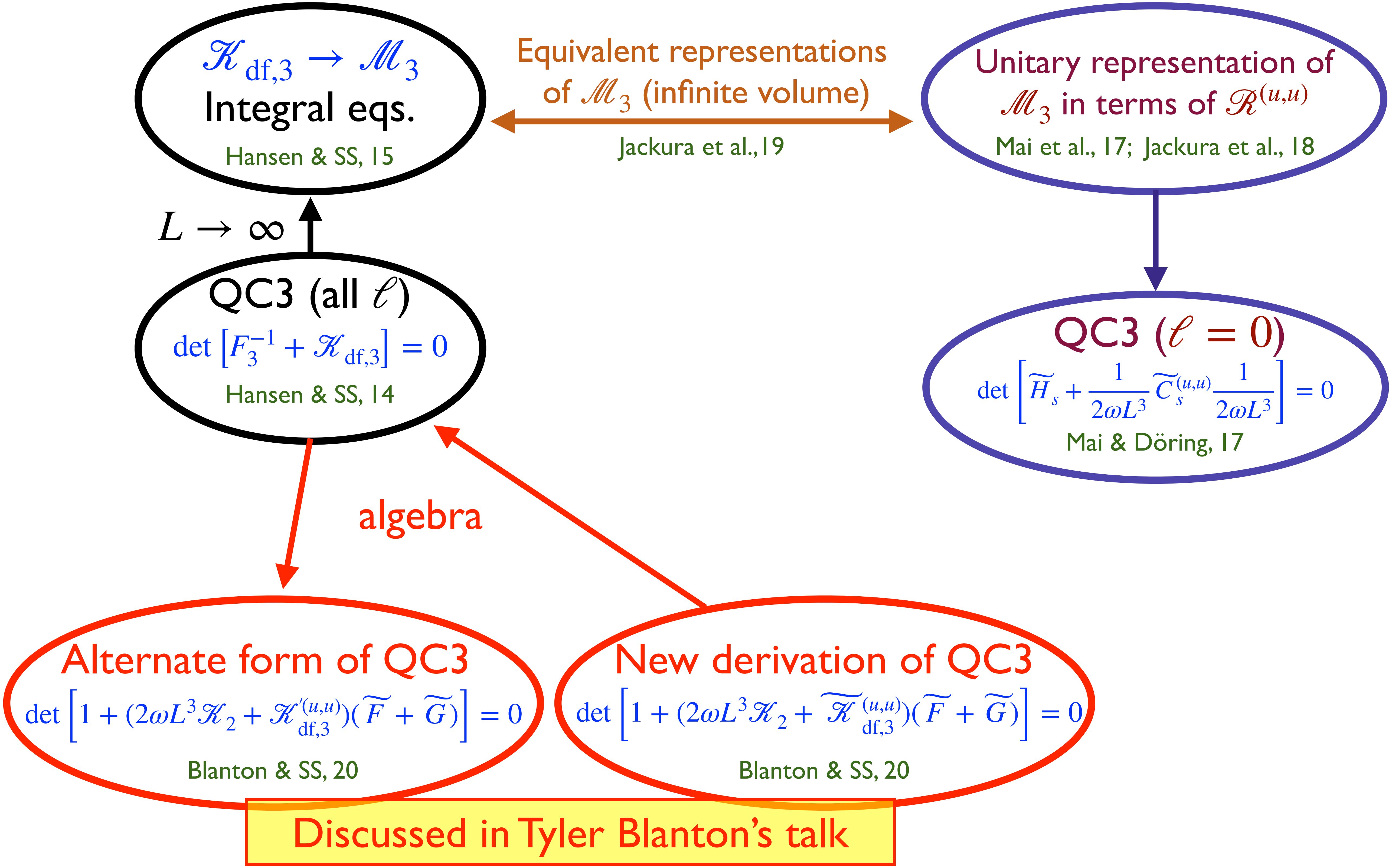
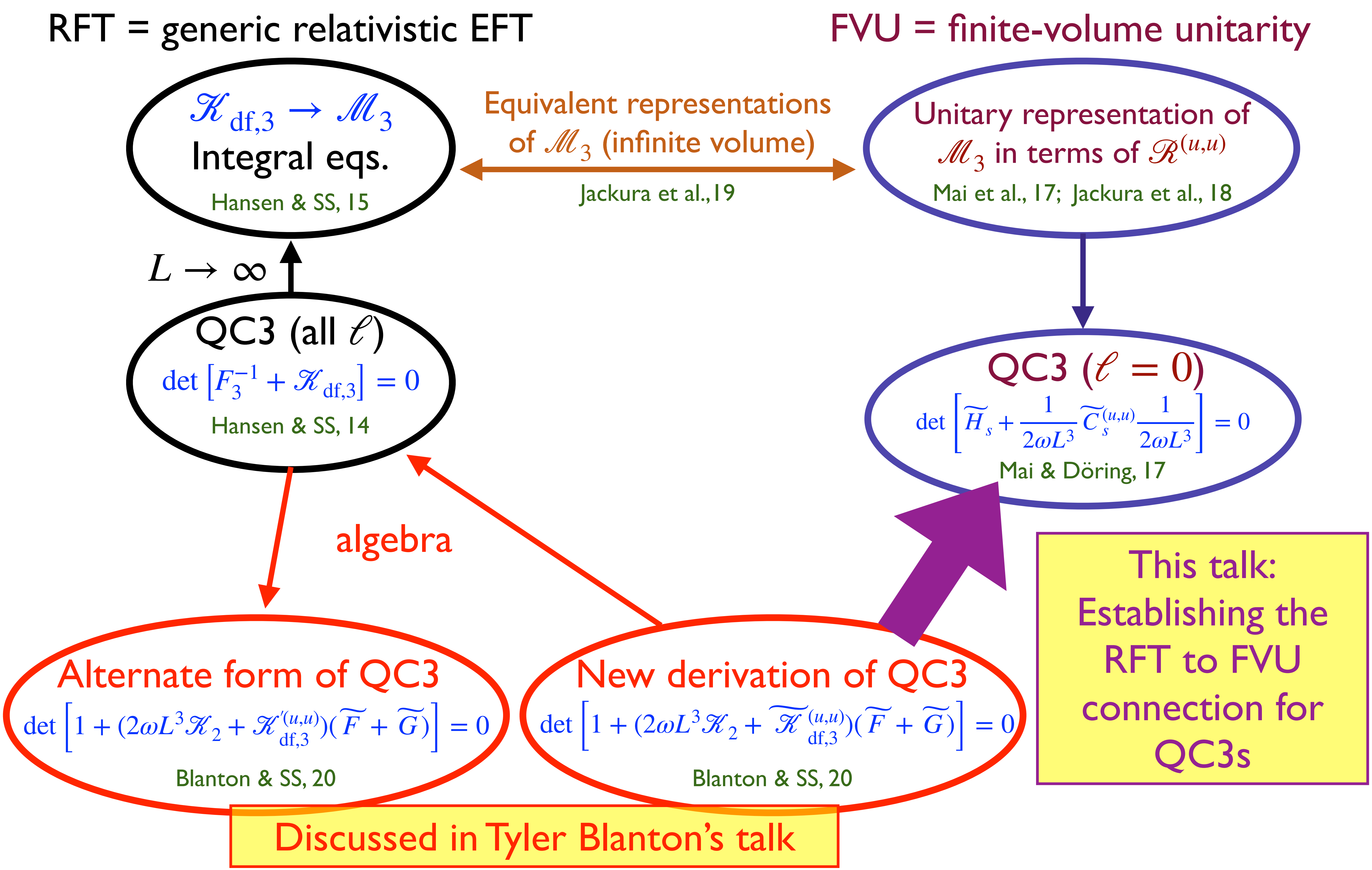
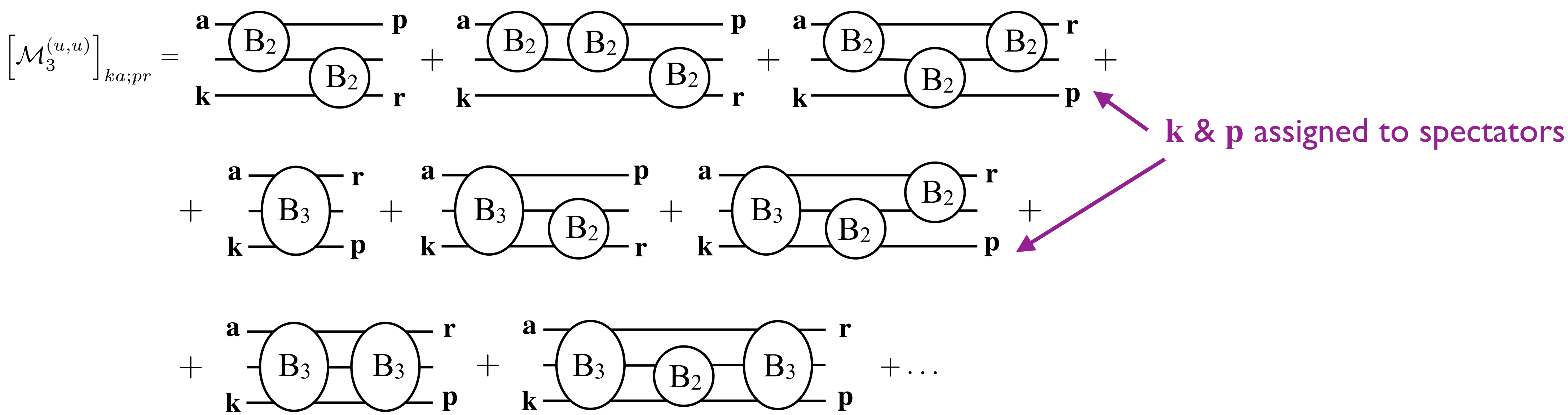


Relativistic QC3 landscape 2020

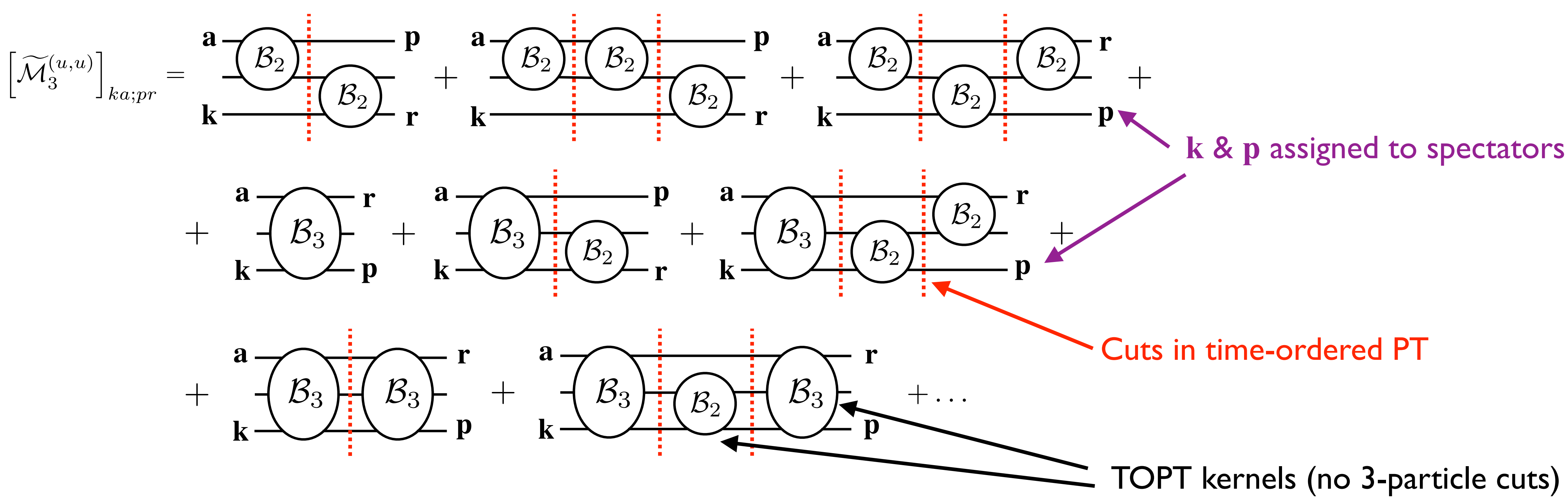


Definitions of asymmetric kernels

- In original RFT approach (using Feynman diagrams & Bethe-Salpeter kernels)

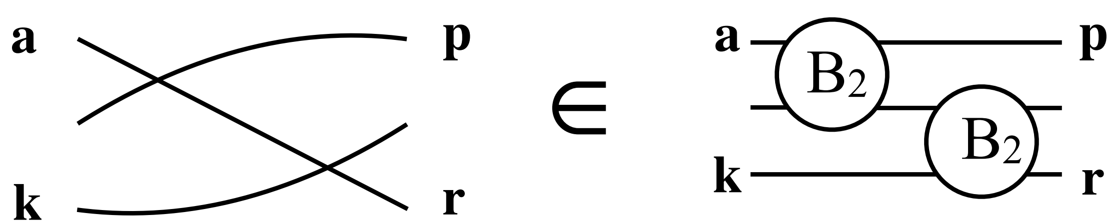


- In our approach (using time-ordered perturbation theory)



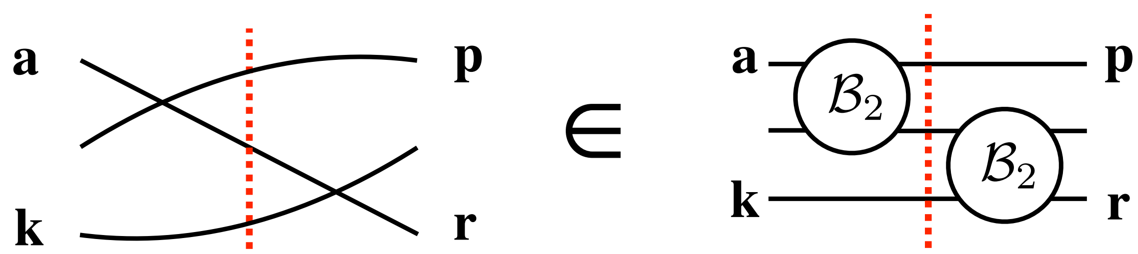
Asymmetric kernels differ!

- Consider a particular Feynman diagram

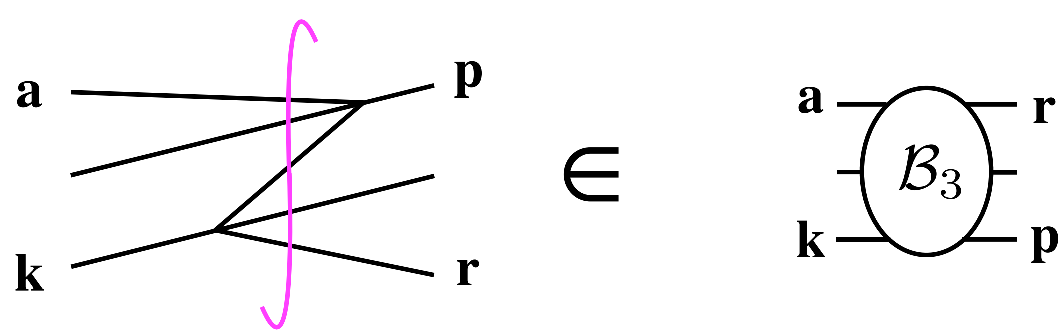


Asymmetric

- In TOPT the two time orderings are put into different terms—one being symmetrized



Asymmetric



Symmetrized

- Thus $\mathcal{M}_3^{(u,u)} \neq \widetilde{\mathcal{M}}_3^{(u,u)}$, although both symmetrize to $\mathcal{M}_3$

Asymmetric kernels ⇒ redundancy

- E.g., asymmetric form of QC3 holds with (at least) two different kernels

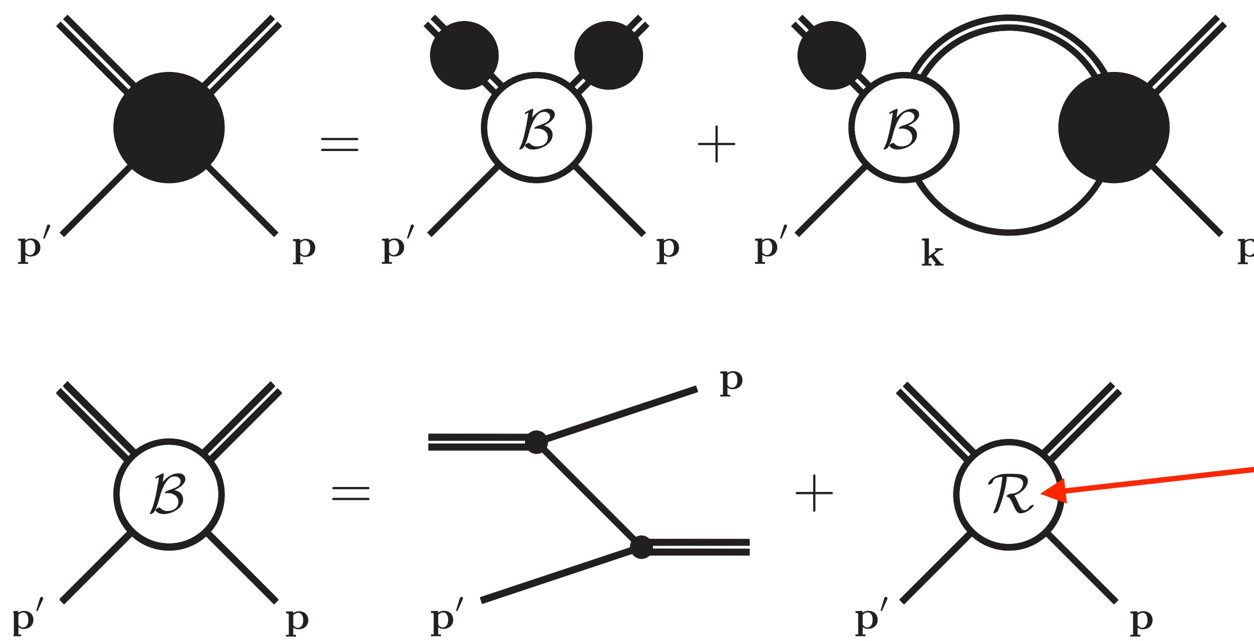
$$\det\left[1 + (2\omega L^3 \mathscr{K}_2 + \mathscr{K}'^{(u,u)}_{\mathrm{df},3})(\widetilde{F} + \widetilde{G})\right] = 0$$

$$\det\left[1 + (2\omega L^3 \mathscr{K}_2 + \widetilde{\mathscr{K}}^{(u,u)}_{\mathrm{df},3})(\widetilde{F} + \widetilde{G})\right] = 0$$

Blanton & SS, 20

- R matrix representation of $\mathscr{M}_3^{(u,u)}$ holds for all choices of asymmetry

$$\mathcal{M}_3^{\mathcal{R},(u,u)} =$$



Formula from Mai et al., 17 & Jackura et al., 18

Figures from Jackura et al., 19

We call it $\mathcal{R}^{(u,u)}$ to emphasize its asymmetry

Can set this equal to either $\mathscr{M}_3^{(u,u)}$ or $\widetilde{\mathscr{M}}_3^{(u,u)}$: leads to different, equally valid, $\mathscr{R}^{(u,u)}$

Key steps in derivation

1. Rewrite asymmetric QC3

$$\det\left[1+(2\omega L^3\mathscr{K}_2+\widetilde{\mathscr{K}}_{\rm df,3}^{(u,u)})(\widetilde{F}+\widetilde{G})\right]=0 \quad\Rightarrow\quad \det\left[\widetilde{H}-X^{(u,u)}\right]=0$$

$$\widetilde{H}=\widetilde{F}+\widetilde{G}+\overline{\mathscr{K}}_{2,L}^{-1} \qquad\qquad \overline{\mathscr{K}}_{2,L}=(2\omega L^3)\widetilde{\mathscr{K}}_2$$

$$X^{(u,u)}=\overline{\mathscr{K}}_{2,L}^{-1}\widetilde{\mathscr{K}}_{\rm df,3}^{(u,u)}\overline{\mathscr{K}}_{2,L}^{-1}\frac{1}{1+\widetilde{\mathscr{K}}_{\rm df,3}^{(u,u)}\overline{\mathscr{K}}_{2,L}^{-1}}$$

2. Equate $\mathscr{M}_3^{\mathscr{R},(u,u)}$ to $\widetilde{\mathscr{M}}_3^{(u,u)}$ to determine the relation between $\mathscr{R}^{(u,u)}$ and $\widetilde{\mathscr{K}}_{\rm df,3}^{(u,u)}$, leading to

$$\left[(2\omega L^3)X^{(u,u)}(2\omega L^3)\right]_{k\ell m;p\ell' m'}=\left[\mathcal{R}^{(u,u)}\right]_{k\ell m;p\ell' m'}+\mathcal{O}(e^{-mL})$$

3. Combine these results

$$\det\left[\widetilde{H}-(2\omega L^3)^{-1}\mathcal{R}^{(u,u)}(2\omega L^3)^{-1}\right]=0$$

Conclusions & Outlook

- RFT quantization condition<u>s</u> can be rewritten in terms of R matrix

$$\det\left[\widetilde{F} + \widetilde{G} + \frac{1}{2\omega L^3 \mathcal{K}_2} - \frac{1}{2\omega L^3}\mathcal{R}^{(u,u)}\frac{1}{2\omega L^3}\right] = 0$$

- Holds for both choices of $\mathscr{R}^{(u,u)}$—we conjecture it holds for family of redundant choices
- Provides generalization of FVU quantization condition $\det\left[\widetilde{H}_s + \frac{1}{2\omega L^3}\widetilde{C}_s^{(u,u)}\frac{1}{2\omega L^3}\right] = 0$ to all ℓ
- Derivation requires use of smooth cutoff function, and barrier factors in $\widetilde{G}$
- We expect that by taking the NR limit we would obtain the generalization of the NREFT form of QC3 to all ℓ

Conclusions & Outlook

- Which form of QC3 is the most useful in practical applications?

$$\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$

- ○ Symmetric three-particle K matrix
- ☑ Threshold expansion requires fewer parameters
- ✖ Little intuition in presence of three-particle resonances
- ✖ $\mathscr{K}_{\mathrm{df},3}$ depends on PV pole prescription

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- ○ Asymmetric three-particle R matrix
- ✖ Threshold expansion requires more (redundant) parameters
- ☑ Considerable experience and intuition from JPAC studies of fitting amplitudes to experimental data
- ☑ $\mathscr{R}^{(u,u)}$ independent of pole prescription

Conclusions & Outlook

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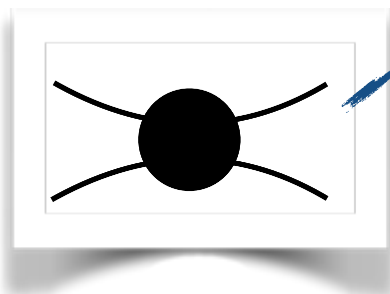
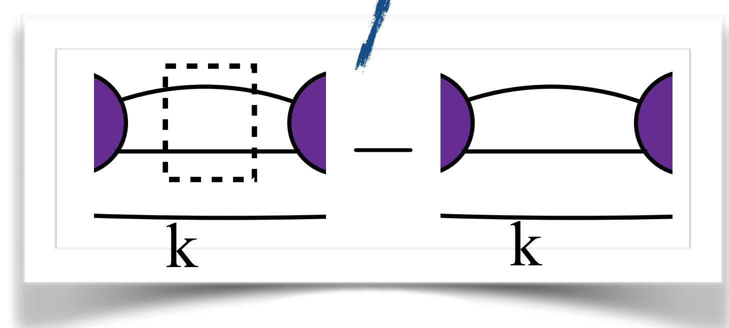
Thank you. Questions?

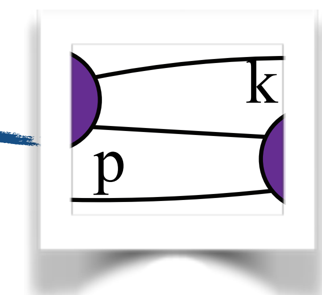
Backup slides

F_3 collects 2-particle interactions

$$F_3 = \left[\frac{\widetilde{F}}{3} - \widetilde{F} \frac{1}{(2\omega L^3 \mathscr{K}_2)^{-1} + \widetilde{F} + \widetilde{G}} \widetilde{F} \right]$$







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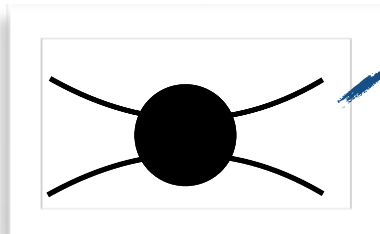
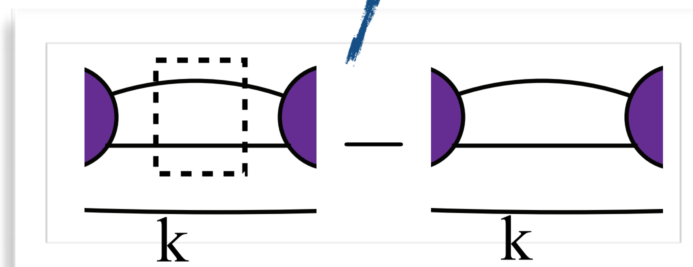
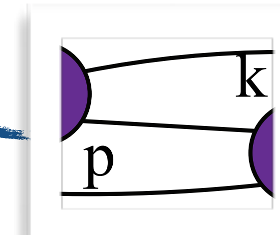
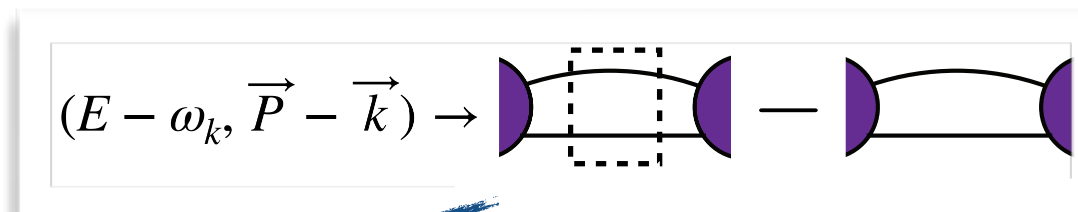






- F & G are known geometrical functions, containing cutoff function H



$$\widetilde{F}_{p\ell'm';k\ell m} = \frac{1}{2\omega_k L^3}\delta_{pk}\; H(\vec{k})\; F_{\mathrm{PV},\ell'm';\ell m}(E - \omega_k, \vec{P} - \vec{k}, L)$$

$$\widetilde{G}_{p\ell'm';k\ell m} = \frac{1}{2\omega_p L^3}\left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell'm'}(\hat{k}^*) H(\vec{p}) H(\vec{k}) Y^*_{\ell m}(\hat{p}^*)}{(P-k-p)^2 - m^2}\left(\frac{p^*}{q_k^*}\right)^{\ell}\frac{1}{2\omega_k L^3}$$

3-particle papers: RFT approach

Max Hansen & SRS:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]

Raúl Briceño, Max Hansen & SRS:

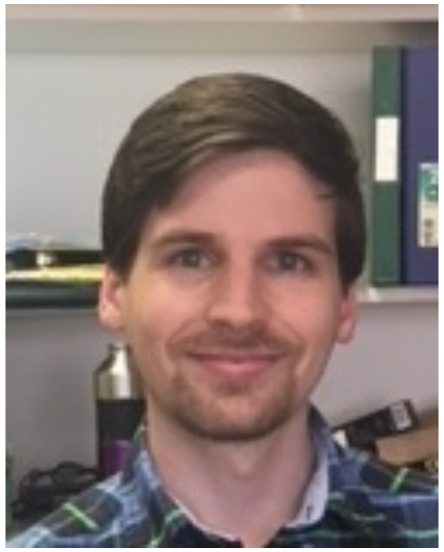
"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]

"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

SRS

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory," arXiv:1707.04279 (PRD) [SPT17]



Tyler Blanton, Fernando Romero-López & SRS:

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]

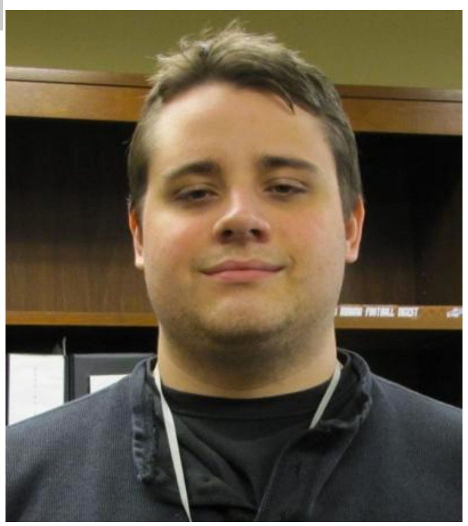
"I=3 three-pion scattering amplitude from lattice QCD," arXiv:1909.02973 (PRL)

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP)

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)



Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

"Generalizing the relativistic quantization condition to include all three-pion isospin channels", arXiv:2003.10974 (JHEP)

Alternate 3-particle approaches

★ **NREFT approach**

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, 1706.07700, JHEP & 1707.02176, JHEP [Formalism & examples]
- M. Döring et al., 1802.03362, PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111, PRD [large volume expansion for excited levels]

★ **Finite-volume unitarity (FVU) approach**

- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of M_3 involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749, PRD [applying FVU approach to $3\pi^+$ spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]

★ **HALQCD approach**

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]