

Multiparticle scattering amplitudes from lattice QCD

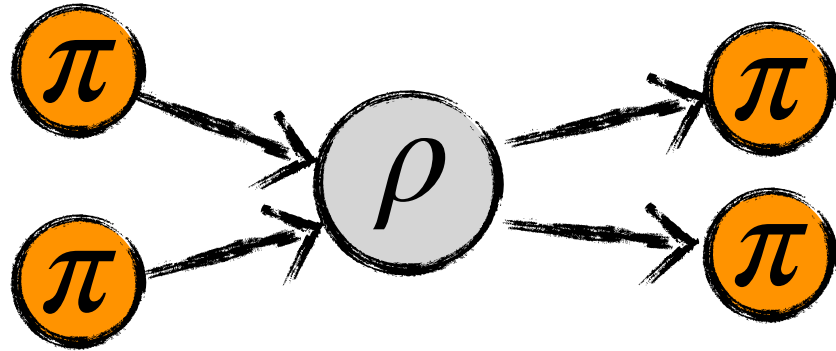


Steve Sharpe
University of Washington



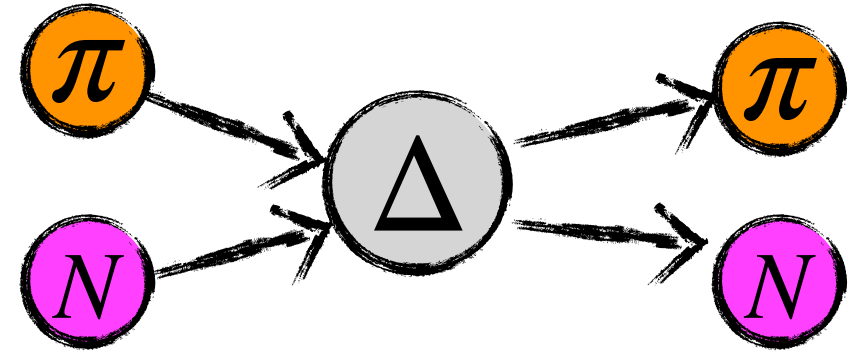
Overview: present status

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Physical quark masses

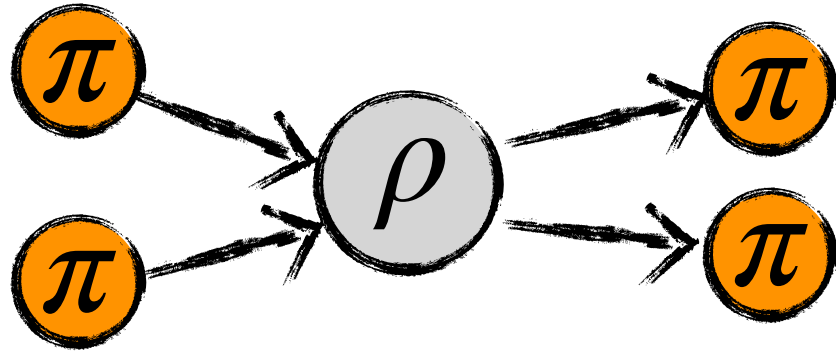
E.g.: Wang, Leinweber, Liu, Liu, Sun, Thomas, Wu, Xing, Yu,
2502.03700



$$M_\pi = 200 \text{ MeV}$$

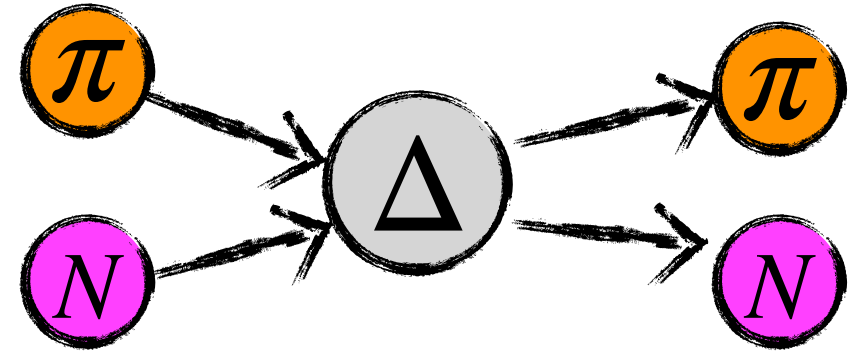
Bulava, Hanlon, Hörz, Morningstar, Nicholson, Romero-López
Skinner, Varnas, Walker-Loud, 2208.03867

Overview: present status



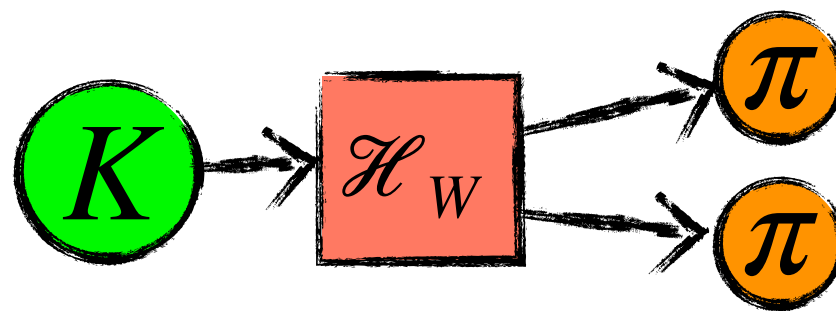
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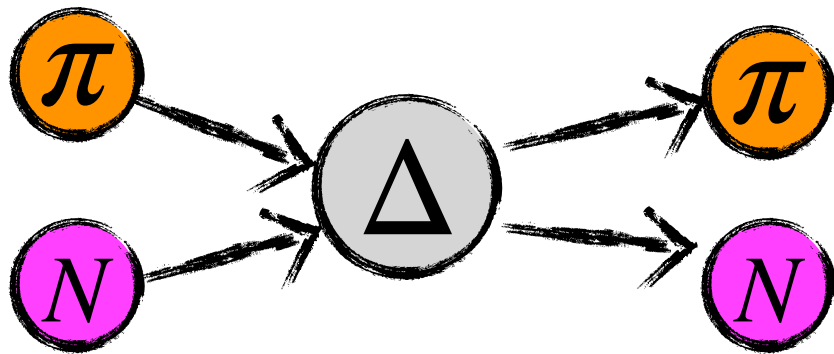


Almost “FLAG-ready”, i.e. fully controlled

Abbott et al. [RBC-UKQCD collaboration], 2004.09440

Δ resonance

Bulava, Hanlon, Hörz, Morningstar, Nicholson, Romero-López, Skinner, Varnas, Walker-Loud, 2208.03867



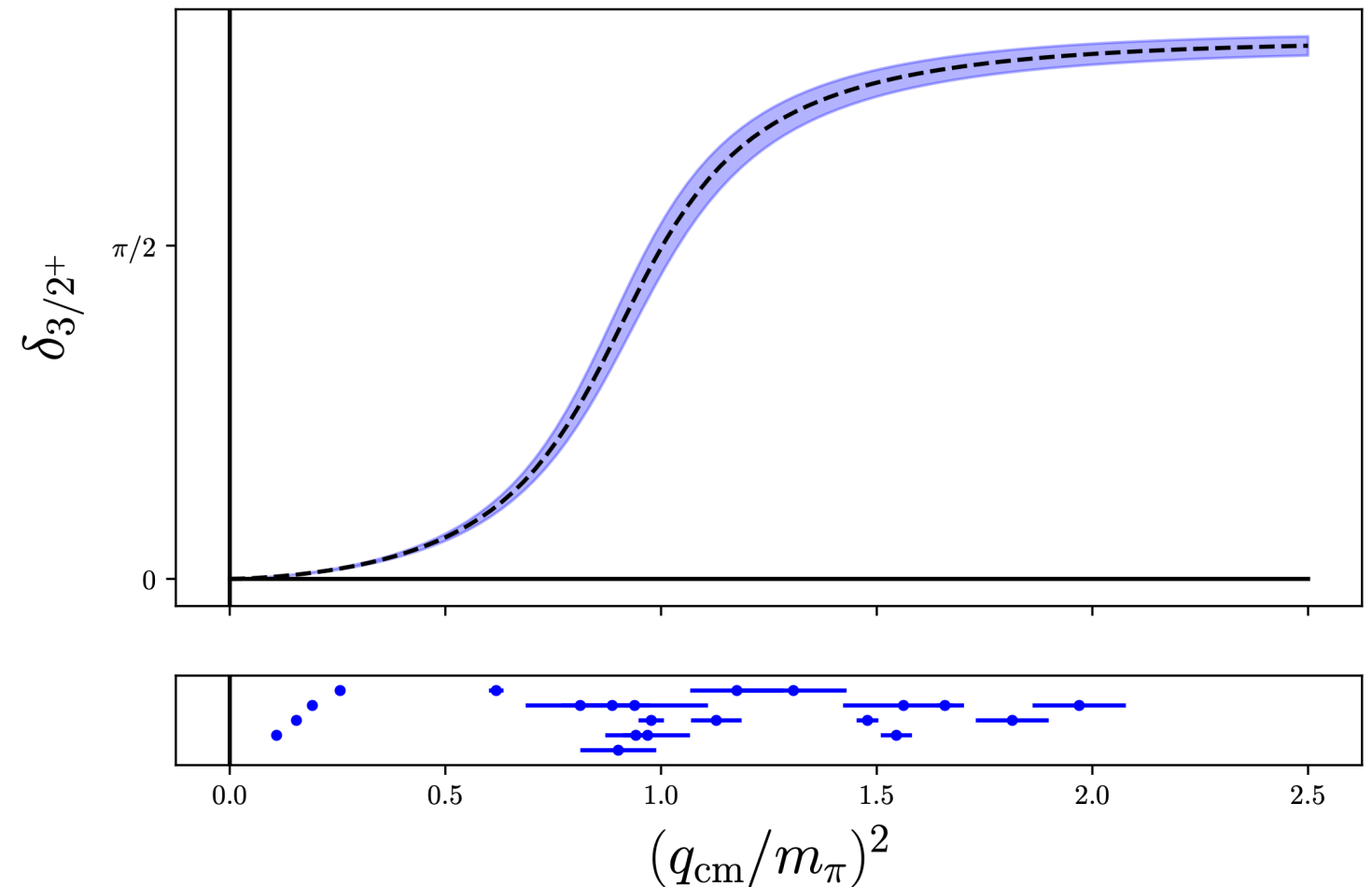
$$M_\pi \approx 200 \text{ MeV}$$

$$M_N \approx 950 \text{ MeV}$$

$$a = 0.063 \text{ fm}$$

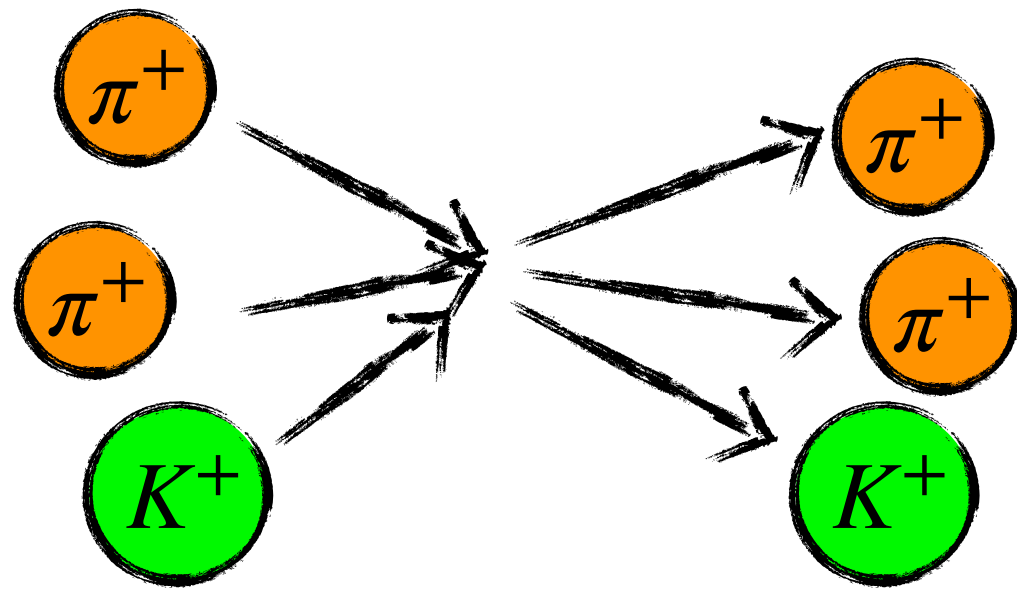
$$L^3 \times T = 64^3 \times 128$$

Clover fermions (CLS)



Overview: present frontier

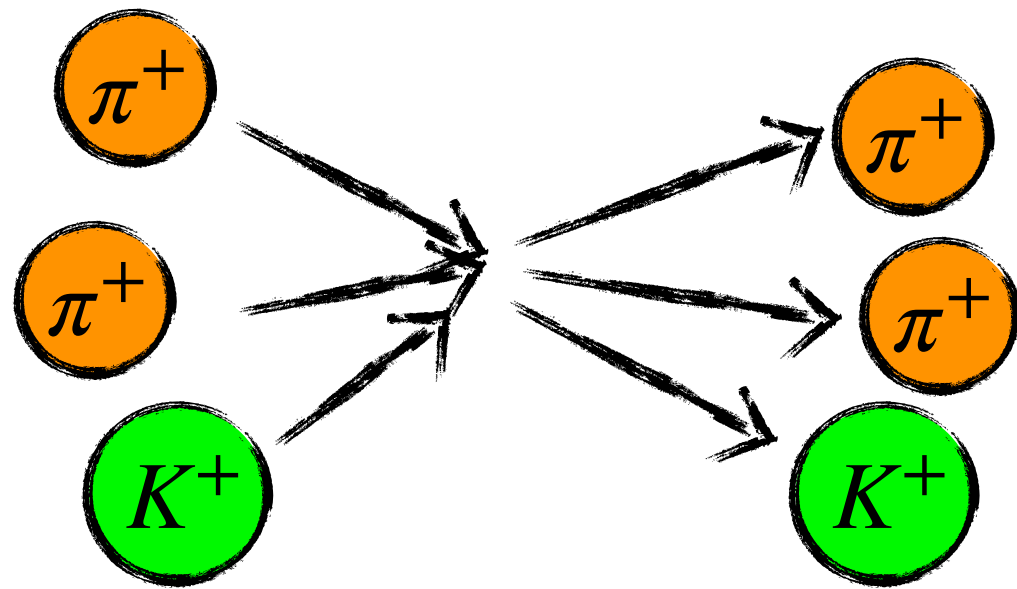
Overview: present frontier



Physical quark masses

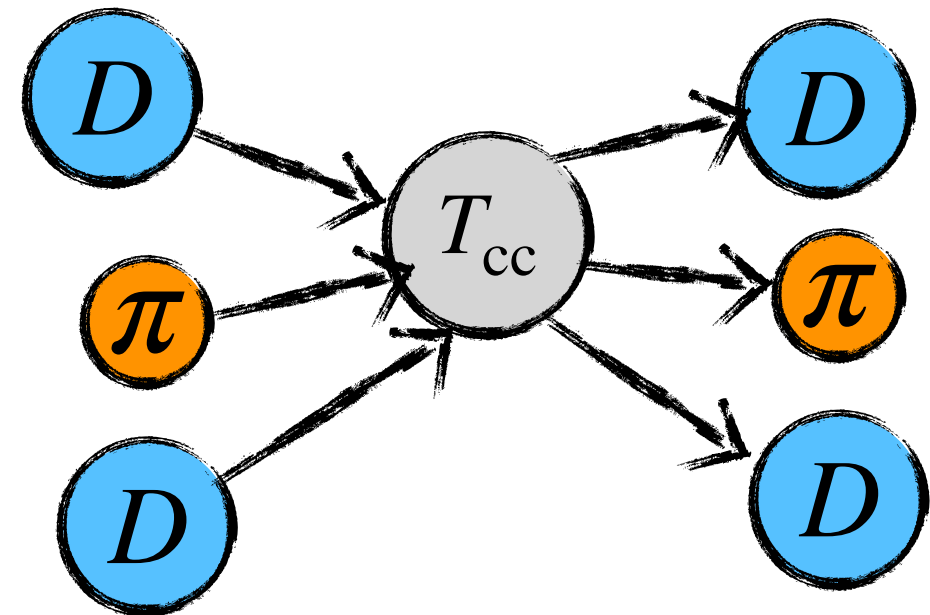
Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López,
SRS, Skinner, [2502.14348](#) & [2502.17976](#)

Overview: present frontier



Physical quark masses

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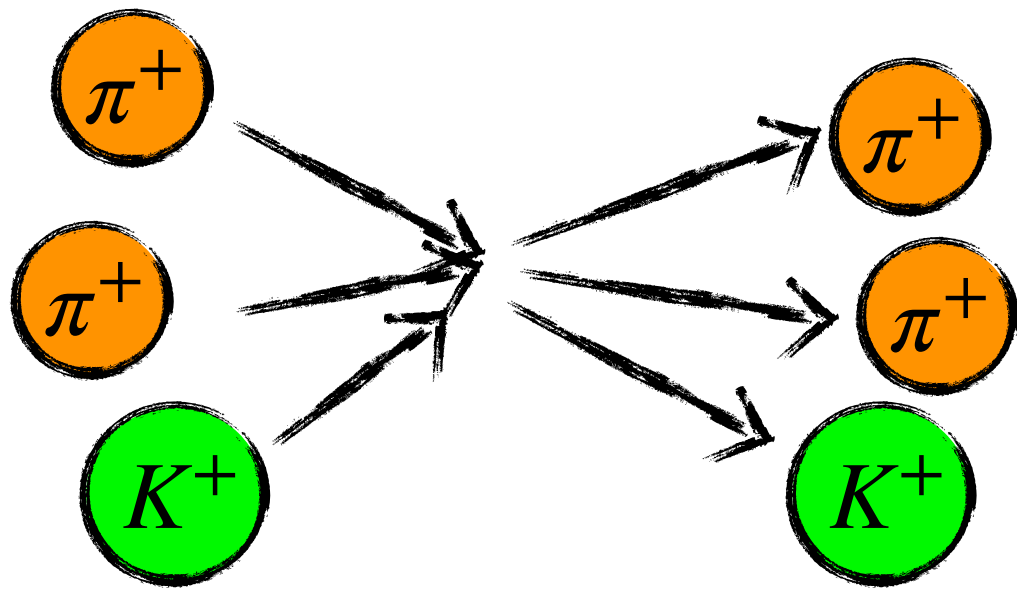
$M_\pi = 200 \text{ MeV}$

Padmanath, Prelovsek, [2202.10110](#)

...

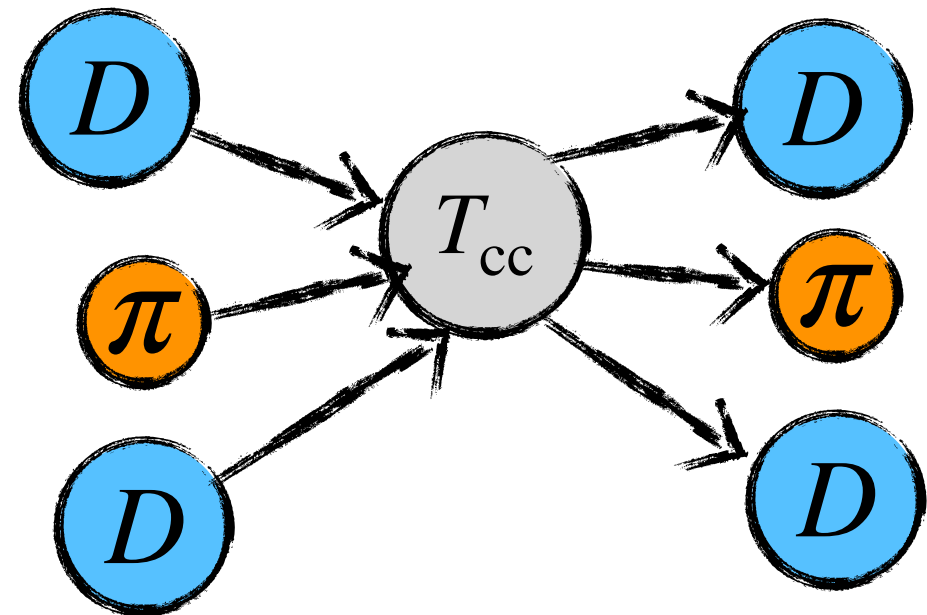
Dawid, Romero-López, SRS, [2409.17059](#)

Overview: present frontier



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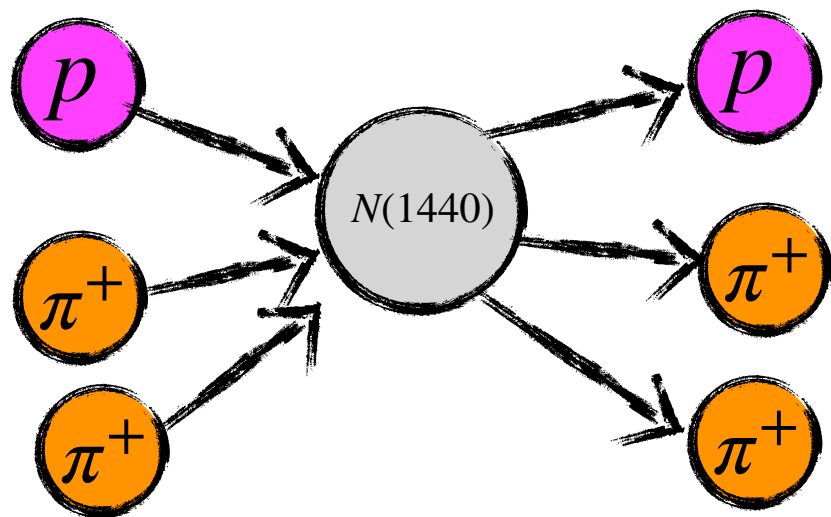


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Padmanath, Prelovsek, [2202.10110](#)

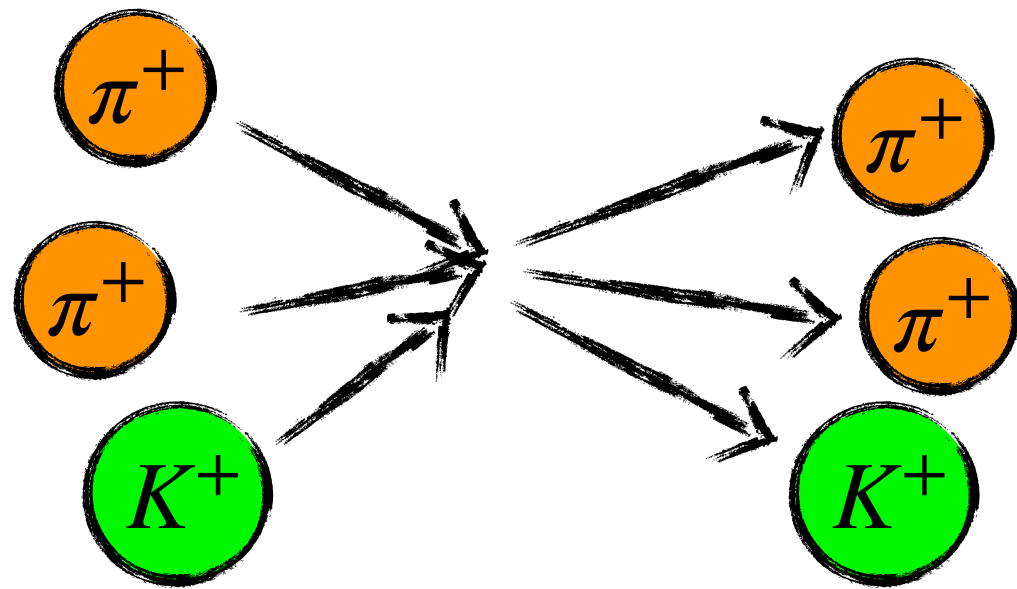
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Dawid, Romero-López, SRS, [2409.17059](#)



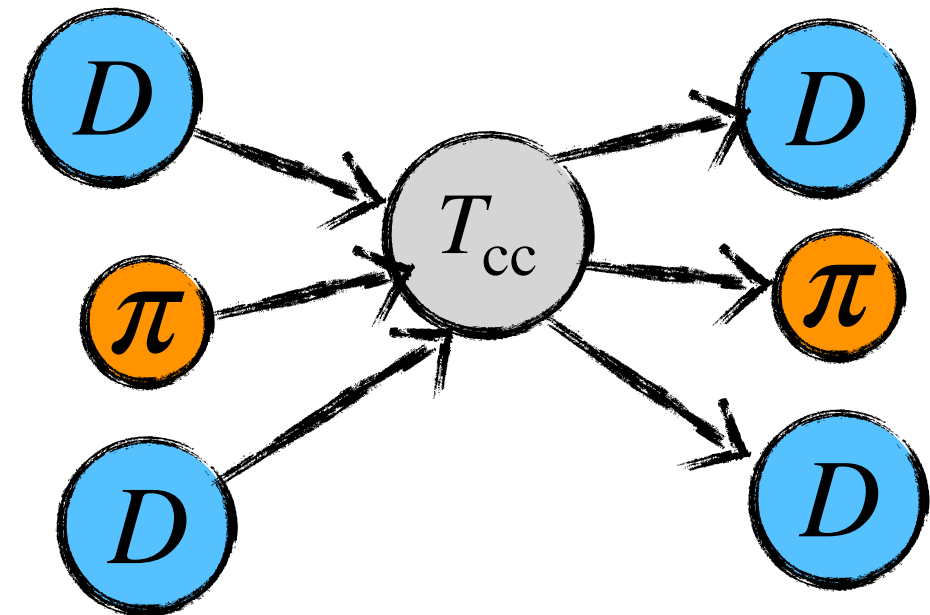
Formalism: Hansen, Romero-López, SRS, [2509.24778](#)

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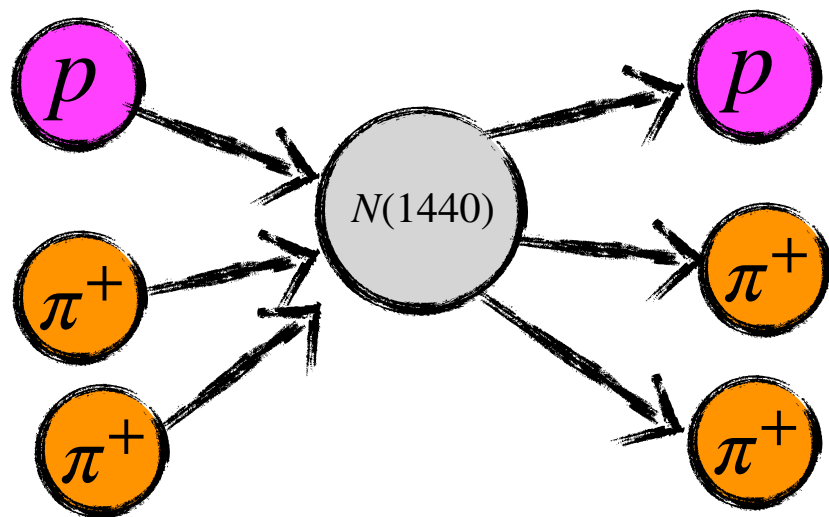


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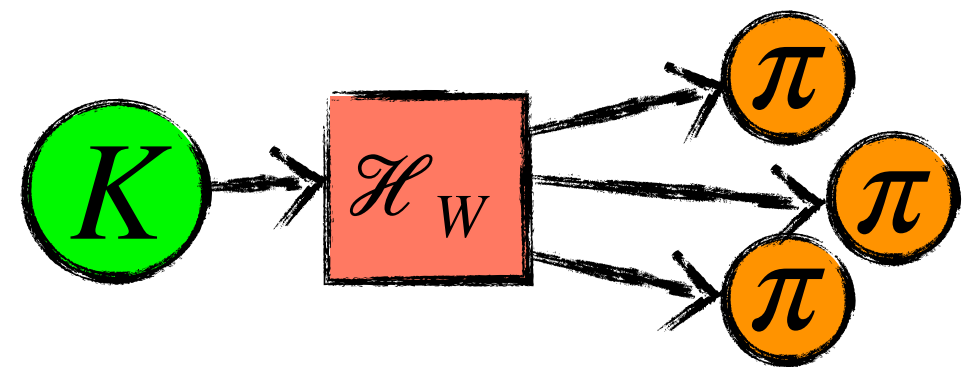
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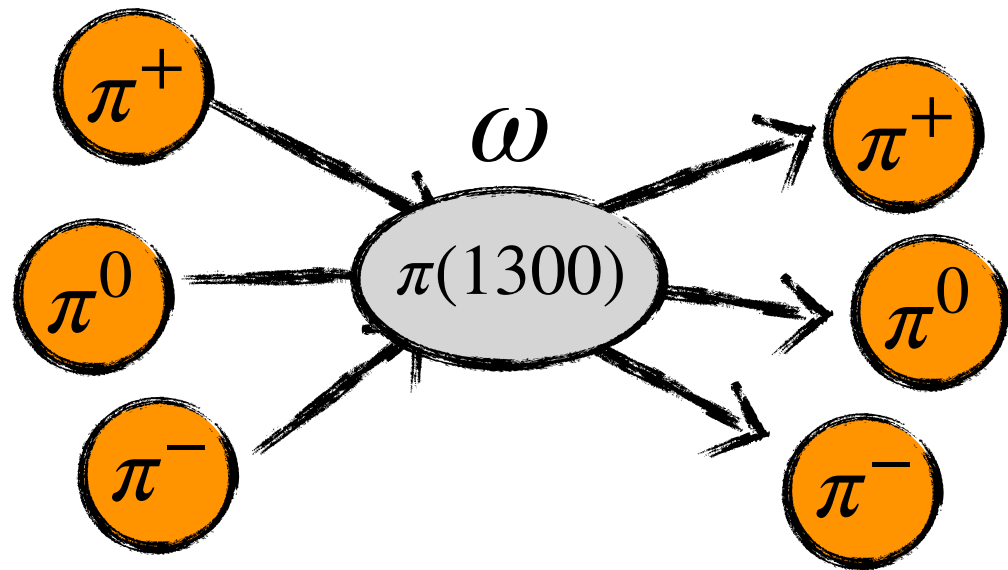
Formalism: Müller, Rusetsky, [2012.13957](#)

Hansen, Romero-López, SRS, [2101.10246](#)

Müller, Pang, Rusetsky, Wu, [2211.10126](#)

Pang, Bubna, Müller, Rusetsky, Wu, [2312.04391](#)

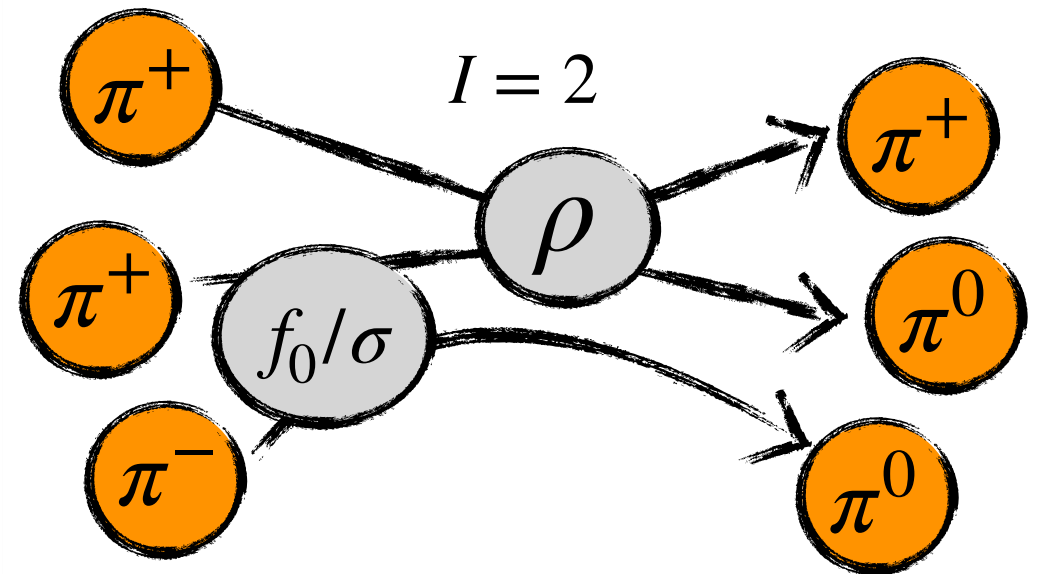
Overview: present frontier



Formalism: Hansen, Romero-López, SRS, 2003.10974

Alternative formalism & first LQCD results:

Yan, Mai, Garofalo, Meißner, Liu, Liu, Urbach, 2407.16659; Yan, Mai, Garofalo, Feng, Döring, Liu, Liu, Meißner, Urbach, 2510.0947

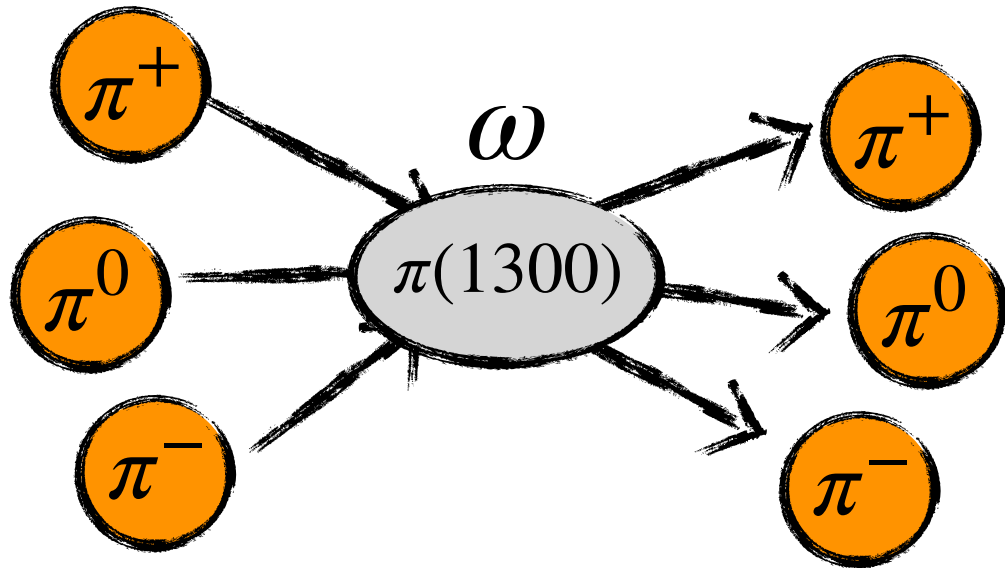


Formalism: Hansen, Romero-López, SRS, 2003.10974; Jackura, 2208.10587

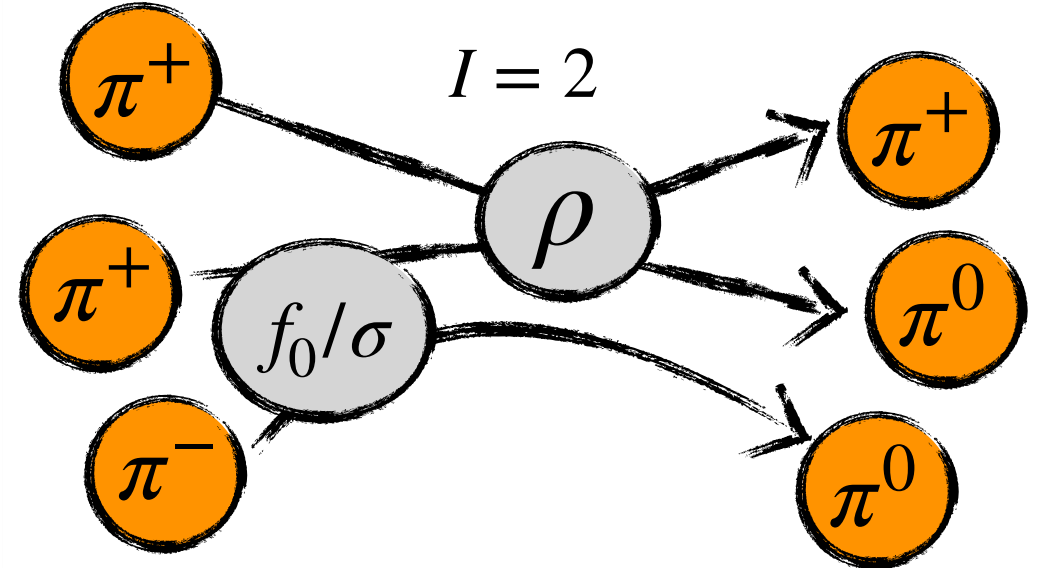
First LQCD results:

Briceño, Hansen, Jackura, Edwards, Thomas, 2510.24894

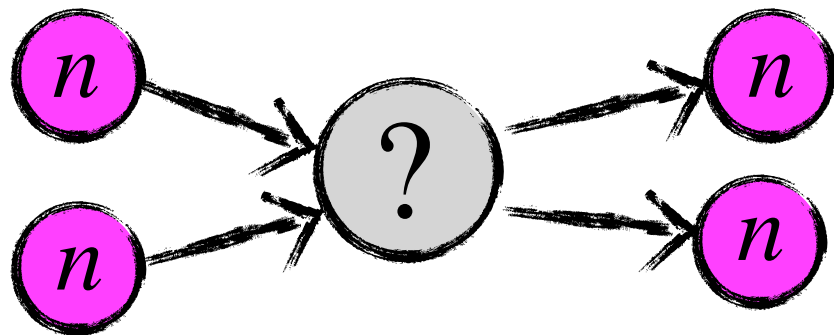
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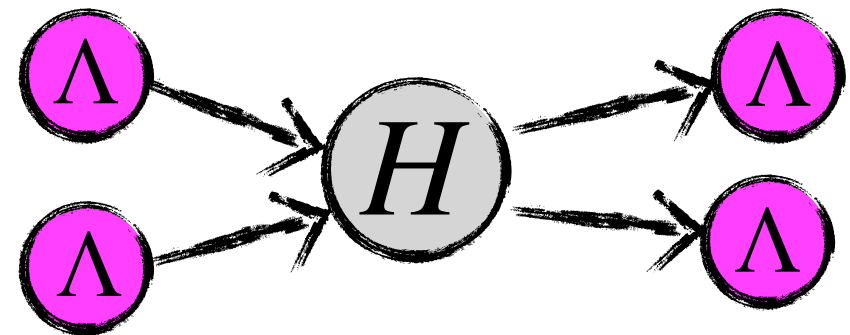
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Di-neutron bound state?

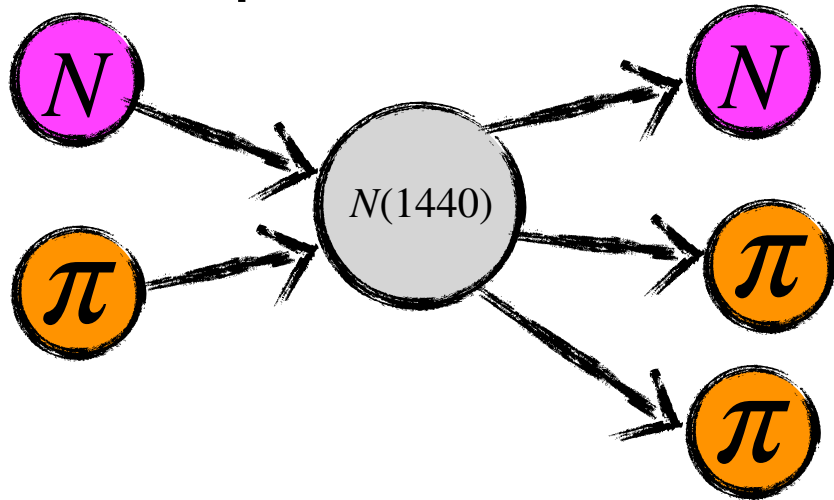


Is H dibaryon bound?

Overview: long-term dreams

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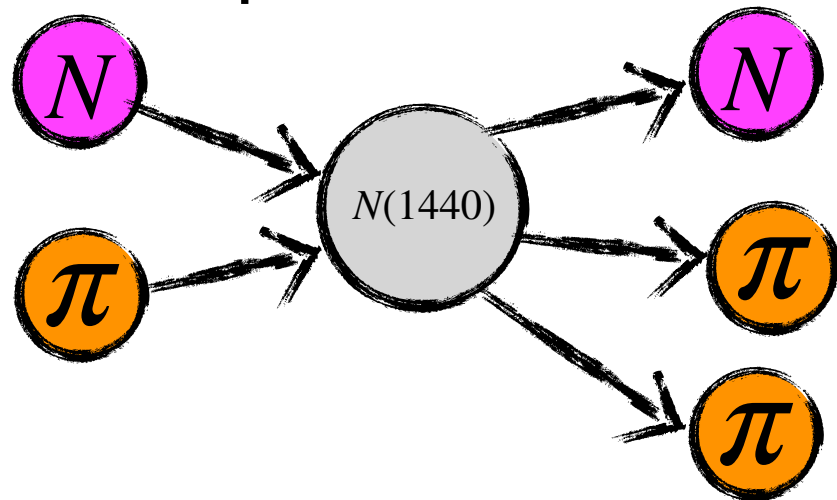
Roper resonance



Formalism: Hansen, Romero-López, SRS, ... in prep.

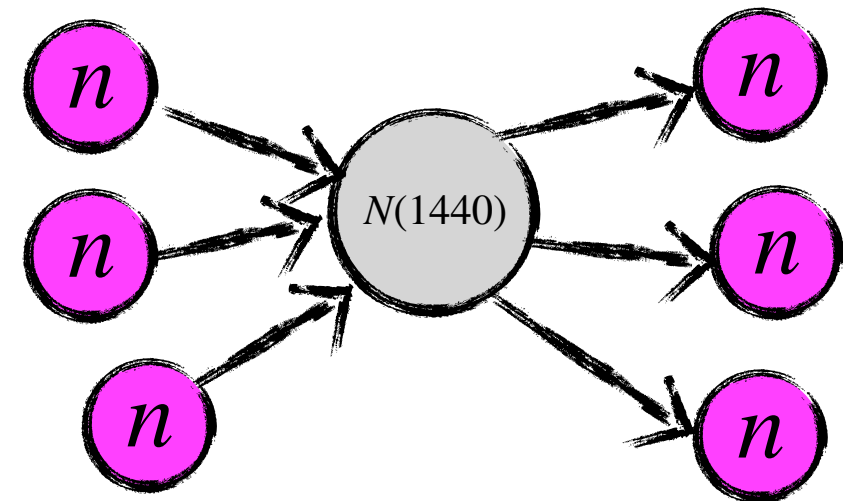
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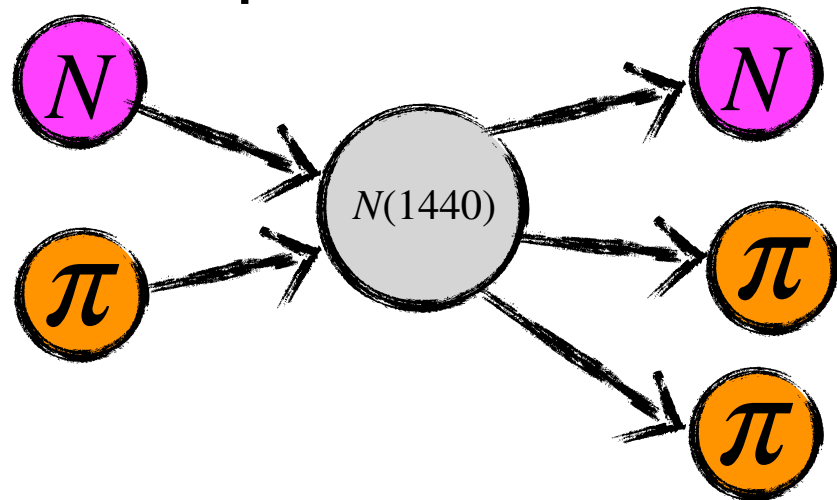
3 neutron interactions



Formalism: Draper, Hansen, Romero-López, SRS, 2303.10219
Toy application: Schaaf, SRS, 2512.24508

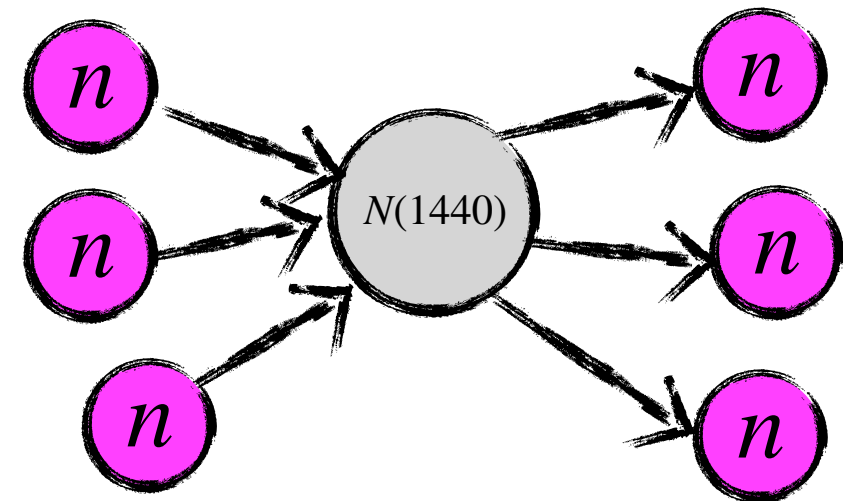
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Roper resonance



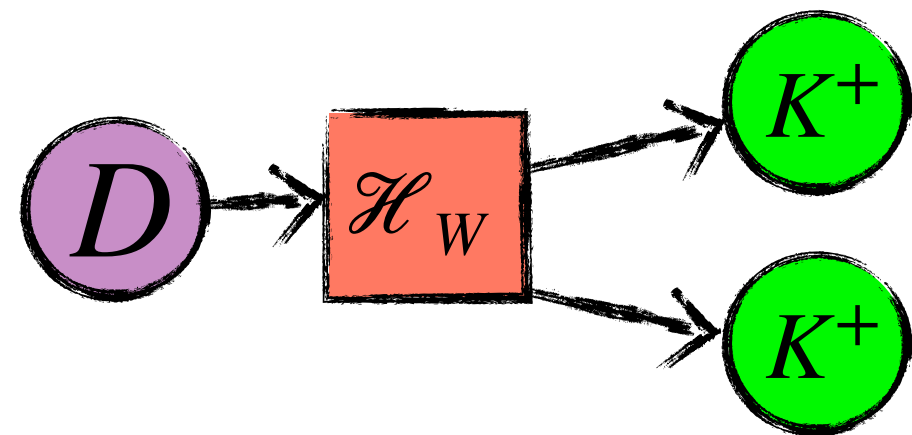
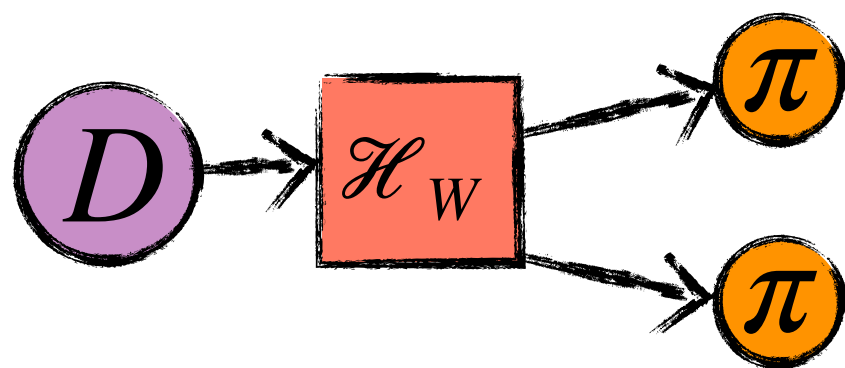
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3 neutron interactions



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Toy application: Schaaf, SRS, 2512.24508

CP violation in D decays



Outline

- Motivations
- The fundamental issue & its resolution
- Recent applications of three-particle formalism
 - Three-particle amplitudes involving pions & kaons at physical quark masses
 - $DD\pi$ scattering, relevant for the T_{cc}^+
 - Dreaming of three neutrons
- Summary & Outlook

Motivations

Underlying motivations

- Determine properties of strong interaction resonances from QCD
 - E.g. exotics such as $T_{cc}(3875)^+ \rightarrow DD^* \rightarrow DD\pi$
 - Long-standing puzzles, e.g. Roper $N(1440) \rightarrow N\pi + N\pi\pi$

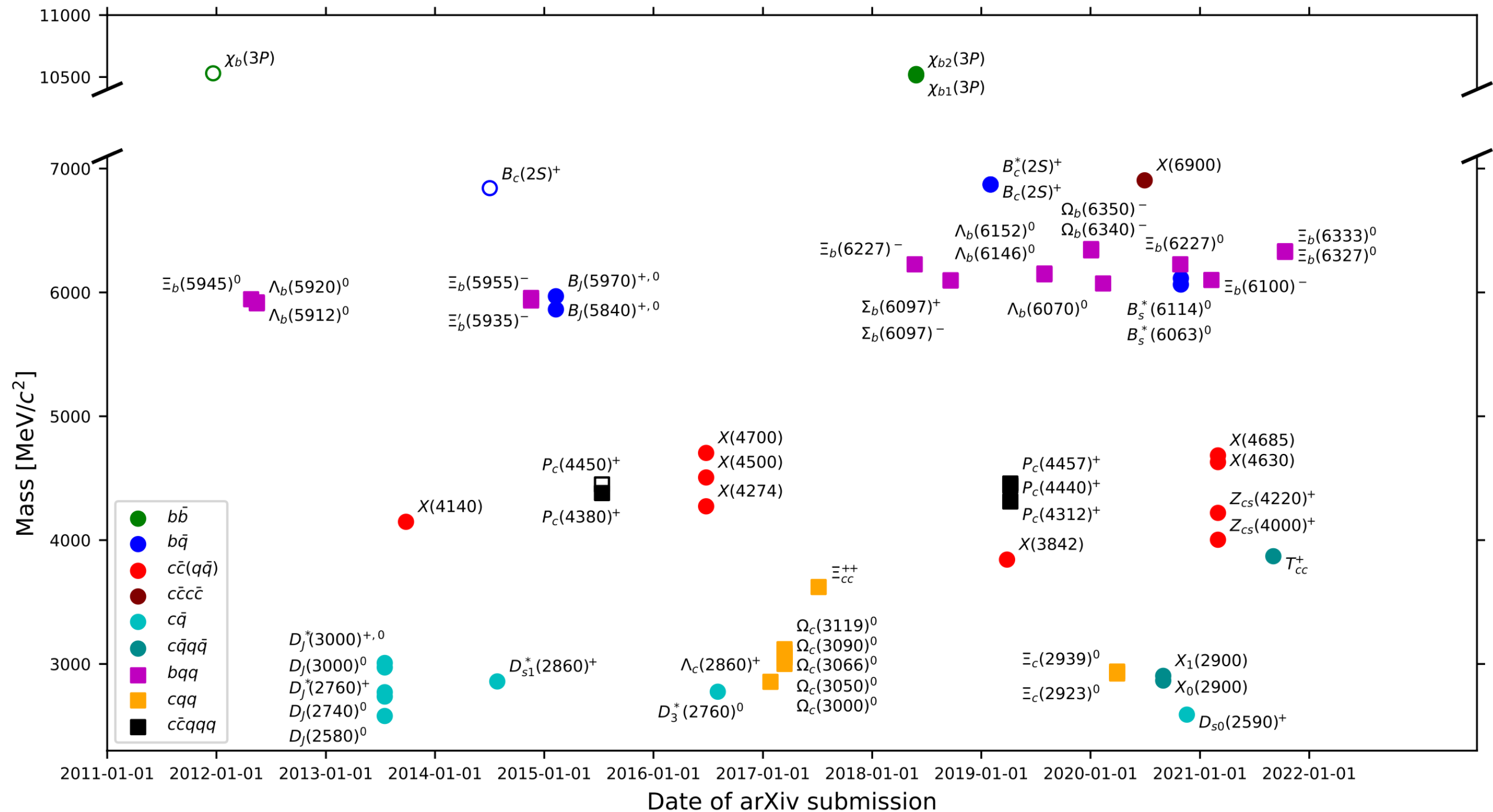
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 - Needed to understand neutron star EoS, properties of large nuclei, ...

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- Determine three particle “forces” for $3n$, 3π , $3K$, ...
 - Needed to understand neutron star EoS, properties of large nuclei, ...
- Calculate weak decay amplitudes within the Standard Model, in order to search for new physics
 - E.g. $K \rightarrow 2\pi$ (essentially done), $K \rightarrow 3\pi$ (method known), & $D \rightarrow \pi^+\pi^-$, K^+K^- (open question)

Cornucopia of exotics



[I. Danilkin, talk at INT workshop, March 23]

+ data from Babar, Belle, COMPASS, BESIII, ...

The fundamental issue & its resolution

On the one hand...

- LQCD determines energies and properties of finite-volume eigenstates
 - Obtained by fits to (numerically-evaluated) Euclidean correlation functions:

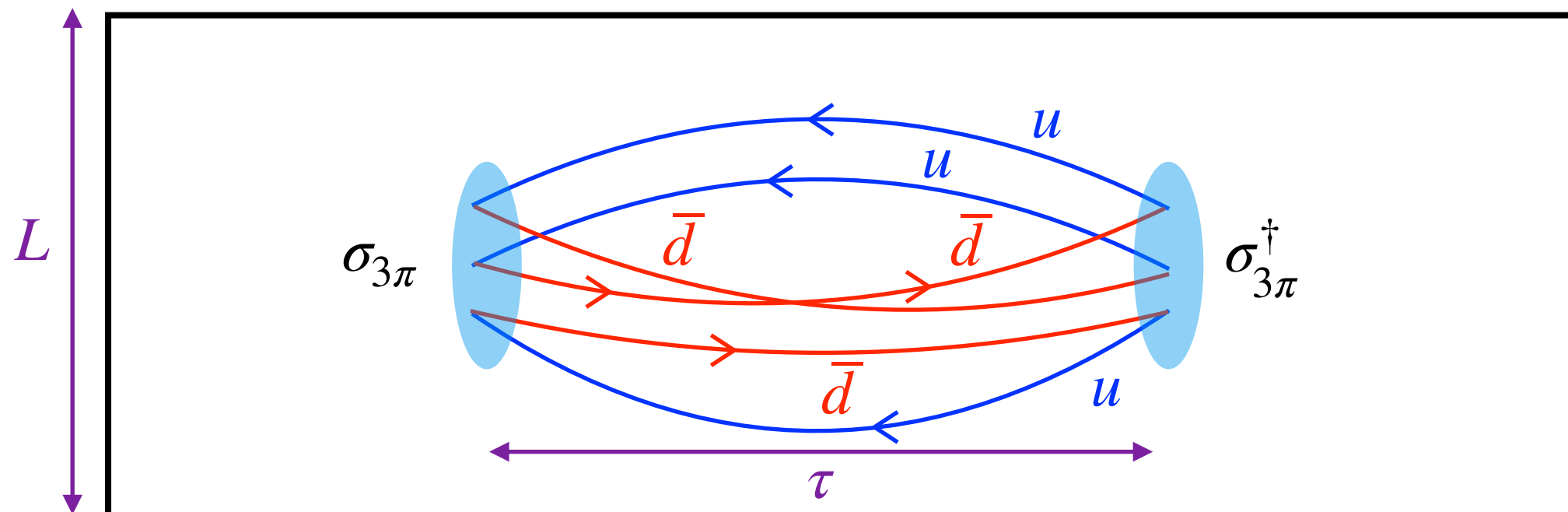
$$\int_L d^3x e^{-i\vec{P}\cdot\vec{x}} {}_L\langle\Omega|\sigma_{3\pi}(\tau,\vec{x})\sigma_{3\pi}^\dagger(0)|\Omega\rangle_L \propto \sum_n \left| {}_L\langle 0|\sigma_{3\pi}^\dagger(0)|3\pi,\vec{P},n\rangle_L \right|^2 e^{-E_n\tau}; \quad (\tau > 0)$$

$\vec{P} = 2\pi\vec{n}/L$
 Assuming L^3 box with PBC

$\sigma_{3\pi} \sim 3\pi^+$
 Lives on timeslice

Tower of finite-volume states
 with quantum numbers of $3\pi^+$,
 with momentum $\vec{P}$, and living
 in irreps of cubic group

Energies of said states



On the one hand...

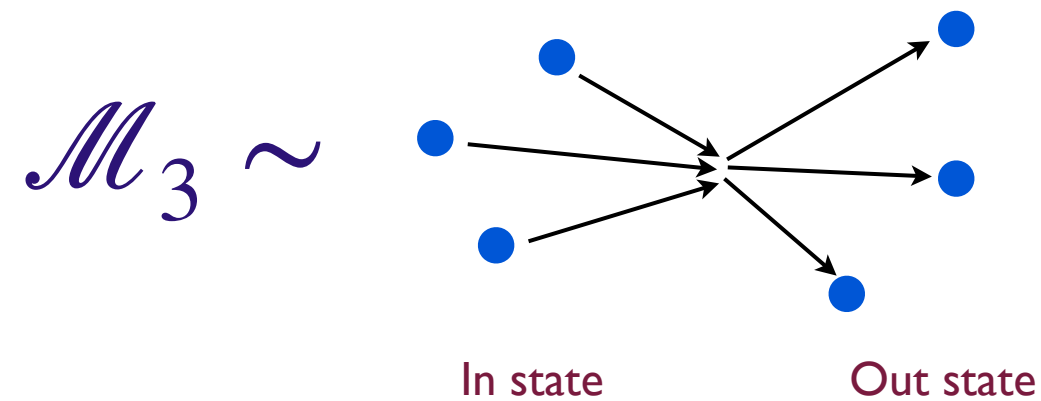
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- E_n are physical quantities!
 - Can determine 5-10 levels for each choice of quantum numbers ($\vec{P}$, irrep, ...)
 - Can now begin to calculate with physical quark masses
 - Results come with statistical & systematic errors (e.g. need $a \rightarrow 0$)
 - Today, I assume that the physical E_n are provided by LQCD simulations

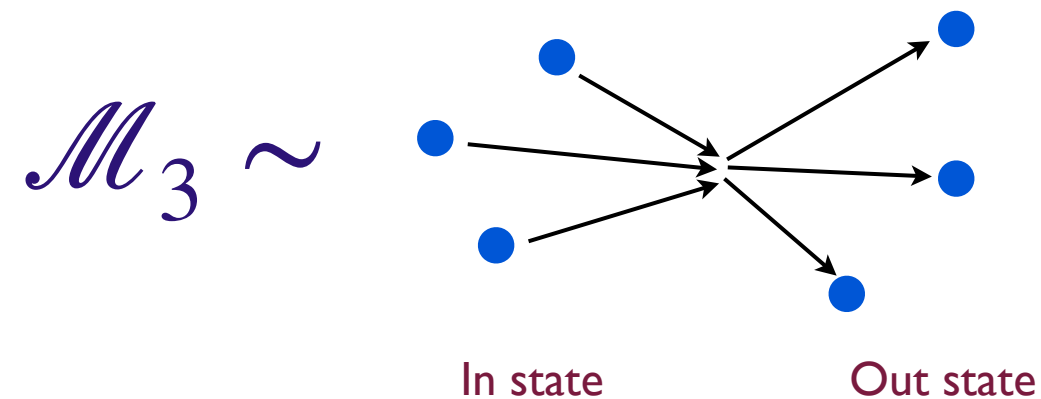
...while on the other

- We want infinite-volume scattering amplitudes, e.g.

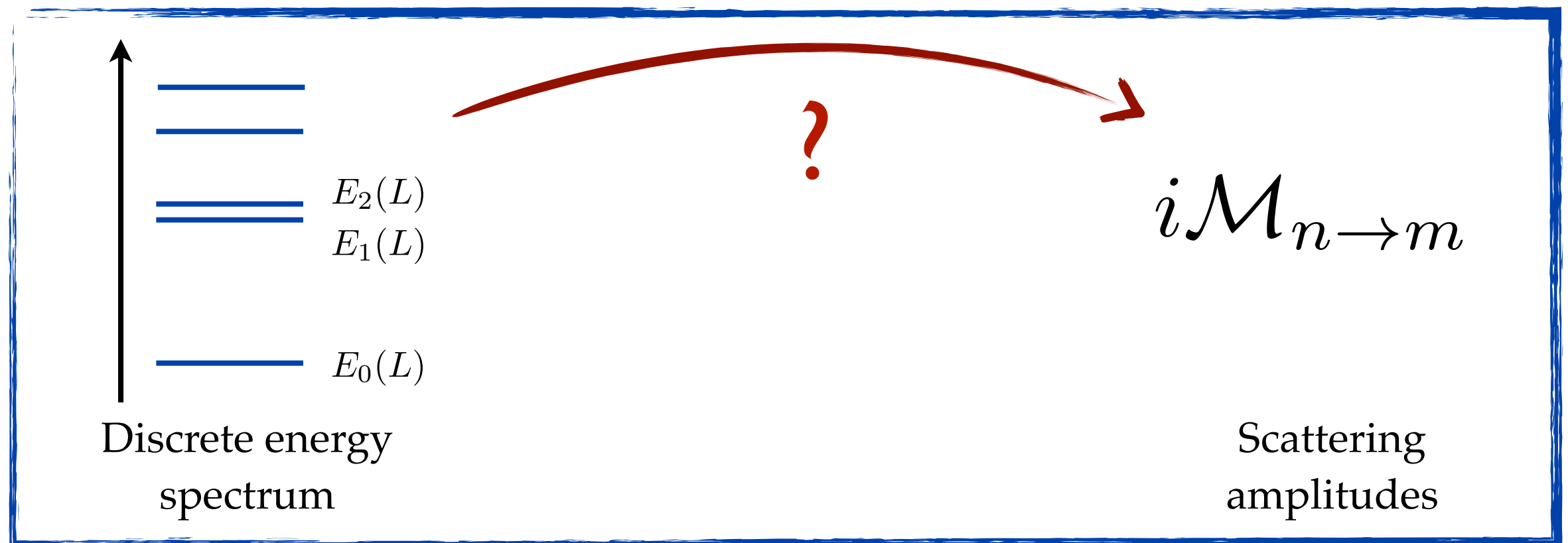


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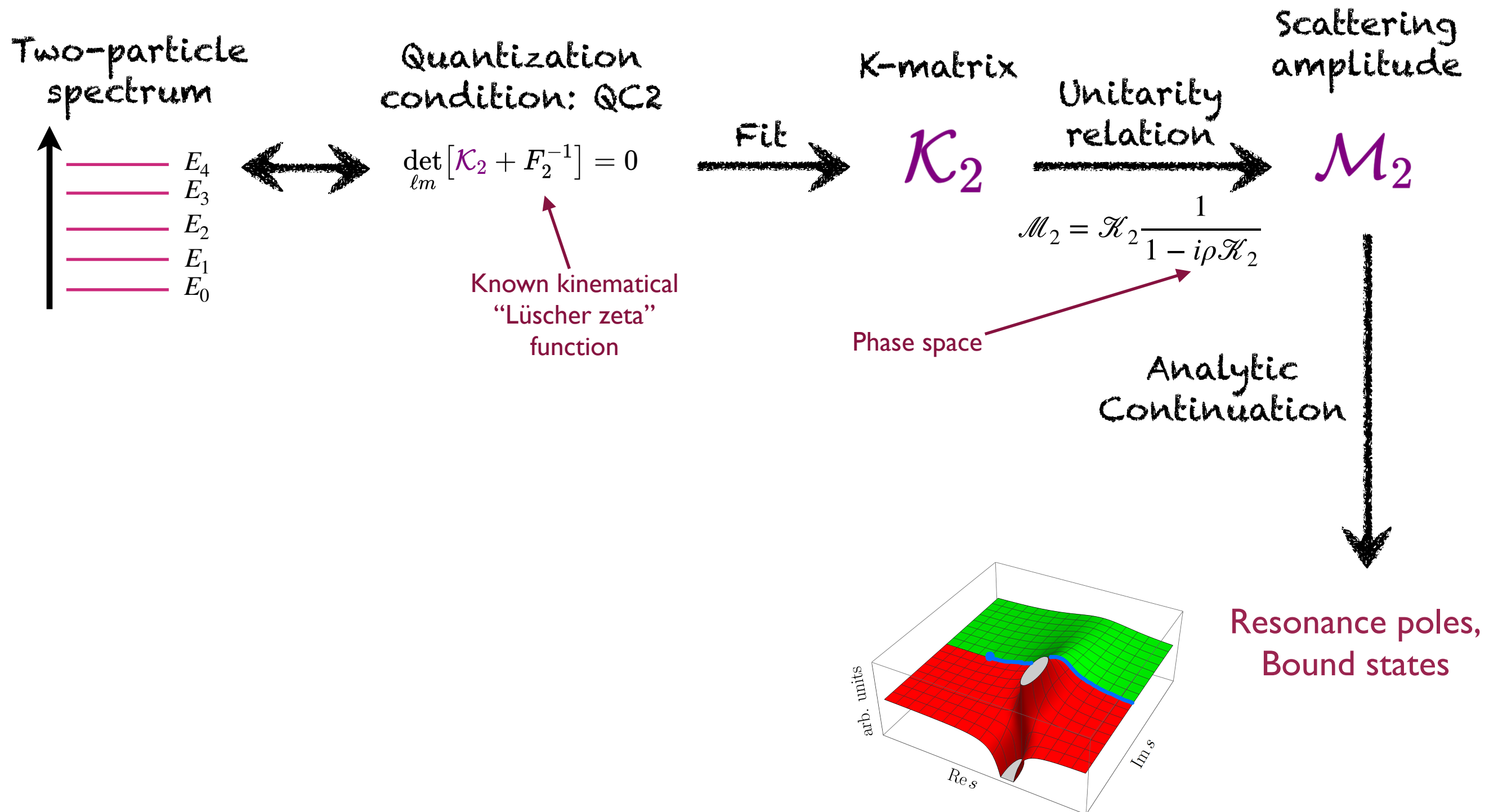


- How do we relate these? A finite-volume QFT problem.



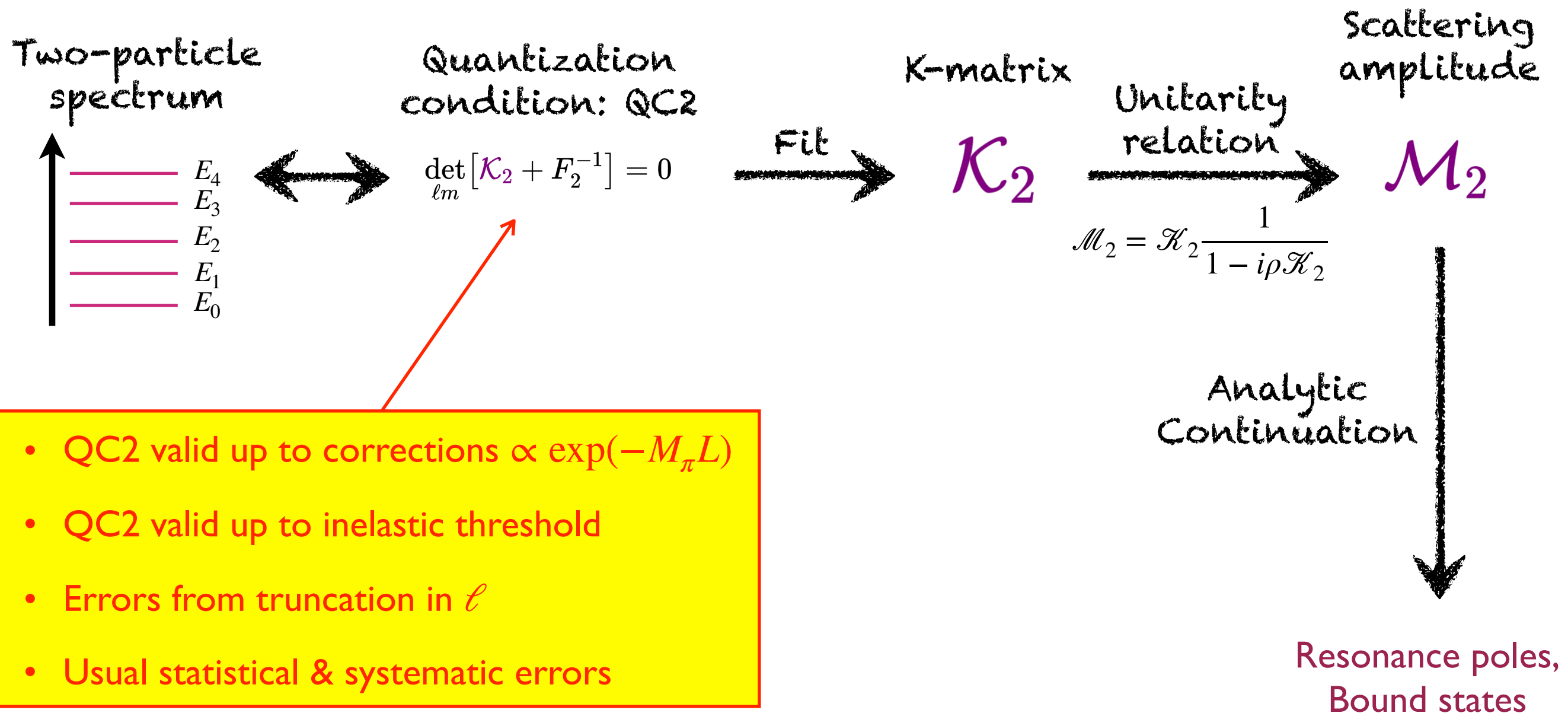
2-particle formalism

[Lüscher, 1986-91 + many subsequent works]



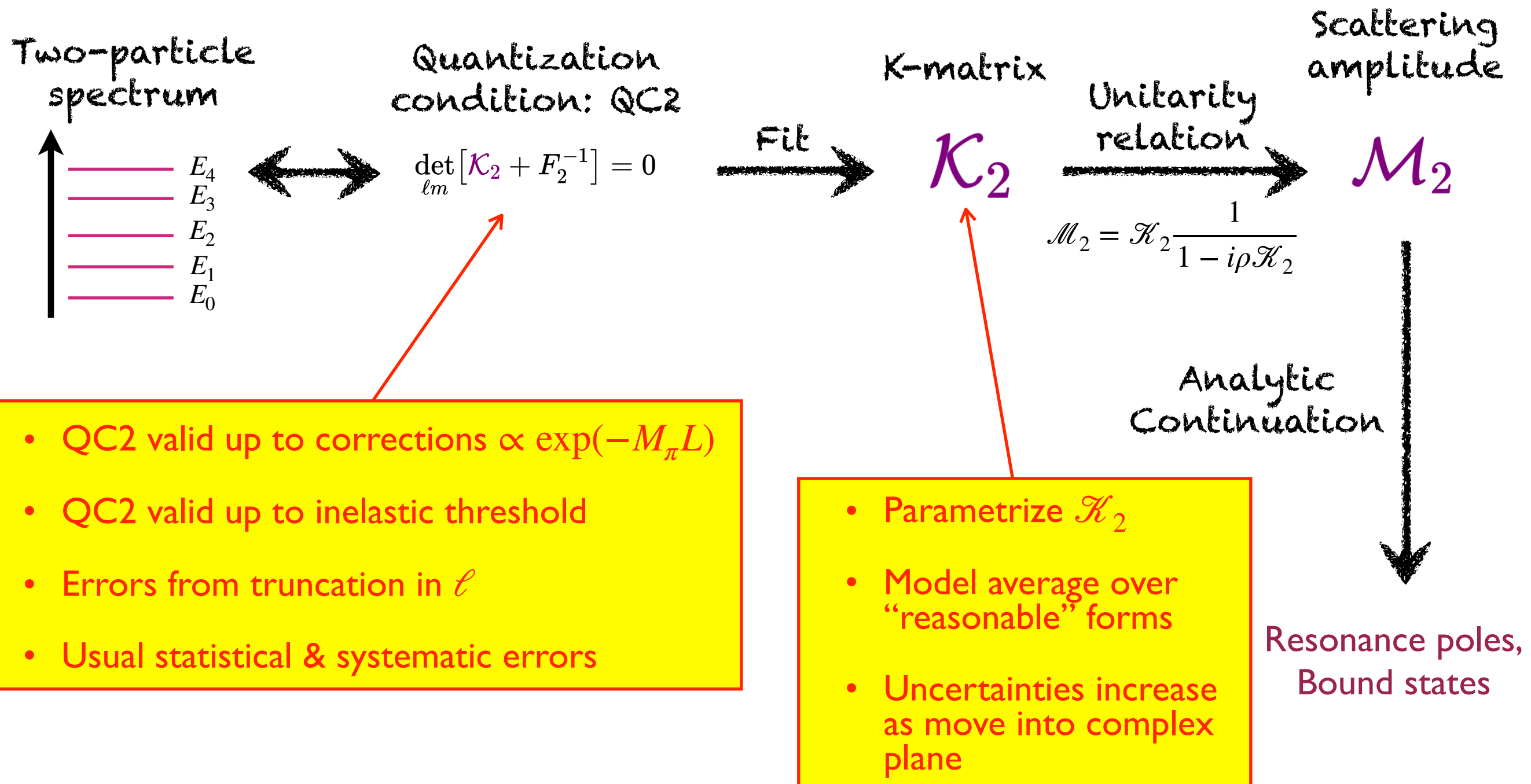
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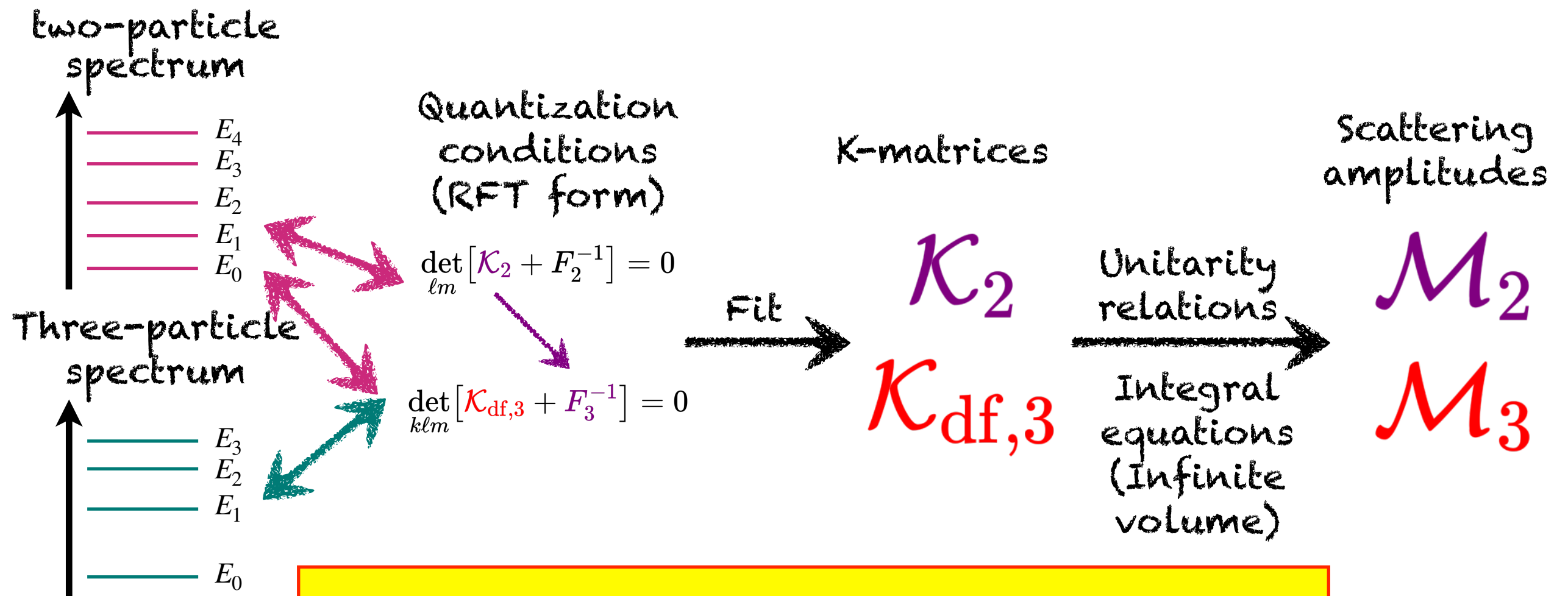
2-particle formalism

[Lüscher, 1986-91 + many subsequent works]



3-particle formalism

[Hansen & SRS, 2014 & 2015 + many subsequent works]



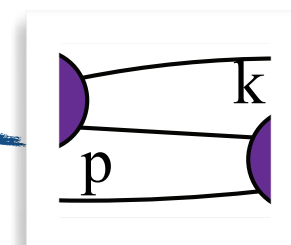
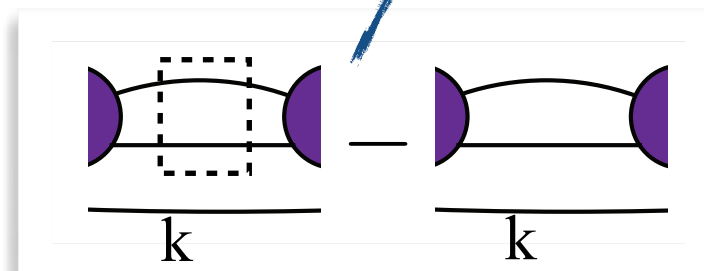
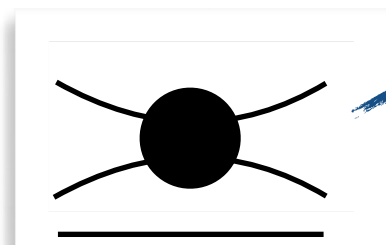
- Using more data than QC2, but fitting much more challenging
- More quantities to parametrize, with limited phenomenological knowledge for $\mathcal{K}_{\text{df},3}$
- Solving integral equations involves new challenges

QC3

$$\det_{k\ell m} \left[F_3^{-1} + \mathcal{K}_{\text{df},3} \right] = 0$$

- QC3 valid up to corrections $\propto \exp(-M_\pi L)$
- QC3 valid up to first inelastic threshold, e.g. $E^* = 5M_\pi$ for 3 pion system
- Matrix indices are spectator finite-volume momentum k , and pair CM-frame ℓ, m
- RFT formalism includes smooth cutoff in k ; must truncate “by hand” in ℓ

$$F_3 = \frac{1}{2\omega L^3} \left[\frac{F}{3} - F \frac{1}{1/\mathcal{K}_{2,L} + F + G} F \right]$$



Relating $\mathcal{K}_{\text{df},3}$ to $\mathcal{M}_3$

$$\mathcal{M}_3 = \lim_{L \rightarrow \infty} \mathcal{S} \left\{ \mathcal{D}_L^{(u,u)} + \mathcal{M}_{\text{df},3,\text{L}}^{(u,u)} \right\} = \mathcal{D} + \mathcal{M}_{\text{df},3}$$

- $\mathcal{D}$ contains all divergent contributions to $\mathcal{M}_3$, but depends on cutoff function $H(\vec{k})$

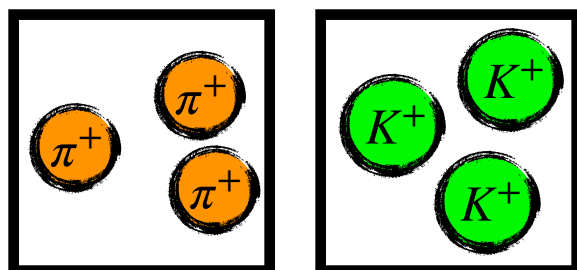
$$\mathcal{D} = \mathcal{S} \left\{ \begin{array}{c} \text{Diagram 1: } \mathcal{M}_2 \text{ on top line, } G^\infty \text{ on middle line, } \mathcal{M}_2 \text{ on bottom line} \\ \text{Diagram 2: } \mathcal{M}_2 \text{ on top line, } G^\infty \text{ on middle line, } \mathcal{M}_2 \text{ on bottom line, } G^\infty \text{ on top line, } \mathcal{M}_2 \text{ on middle line} \\ \vdots \end{array} \right\}$$

- $\mathcal{M}_{\text{df},3}$ is divergence-free, equals $\mathcal{K}_{\text{df},3}$ at leading order, and is also cutoff-dependent

$$\mathcal{M}_{\text{df},3} = \mathcal{K}_{\text{df},3} + \mathcal{S} \left\{ \begin{array}{c} \text{Diagram 1: } \mathcal{M}_2 \text{ (oval) connected to } \mathcal{K}_{\text{df},3} \text{ (oval) via } \rho \text{ (purple line)} \\ \text{Diagram 2: } \mathcal{K}_{\text{df},3} \text{ (oval) connected to } \mathcal{M}_2 \text{ (oval) via } \rho \text{ (purple line)} \\ \text{Diagram 3: } \dots \end{array} \right\}$$

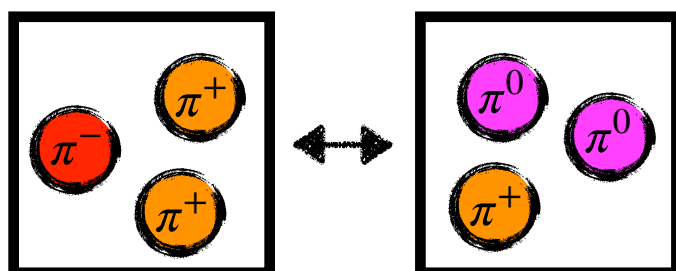
- “Decorations” ensure that $\mathcal{M}_3$ is unitary

Status: formalism



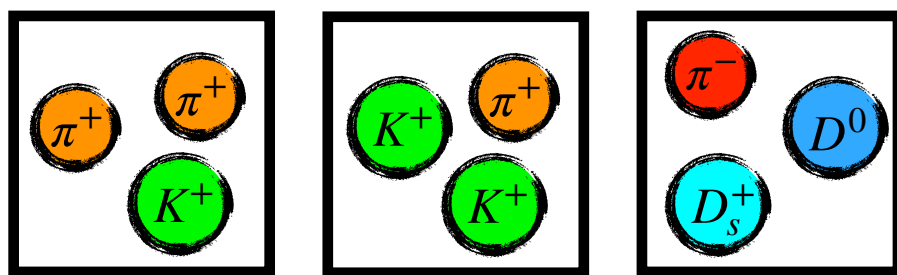
[Hansen & SRS 14,15 (RFT); Hammer, Pang, Rusetsky 17 (NREFT); Mai, Döring 17 (FVU)]

[Many subsequent refinements & tests]



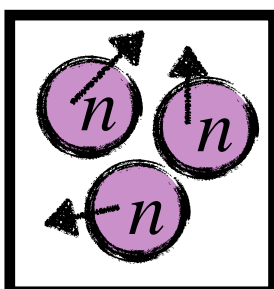
[Hansen, Romero-López, SRS (RFT) 20;

Mai, Alexandru, Brett, Culver, Döring, Lee, Sadasivan (FVU) 21; Feng, Gil, Döring, Molina, Mai, Shastry, Szczepaniak (FVU) 24; Yan, Mai, Garofalo, Meißner, Liu, Liu, Urbach (FVU) 24; Yan, Mai, Garofalo, Feng, Döring, Liu, Liu, Meißner, Urbach (FVU) 25]

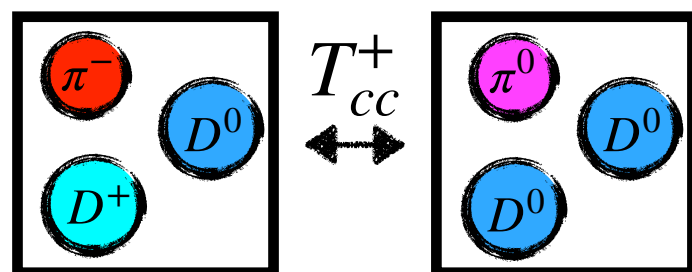


[Blanton, Sharpe (RFT) 20, 21;

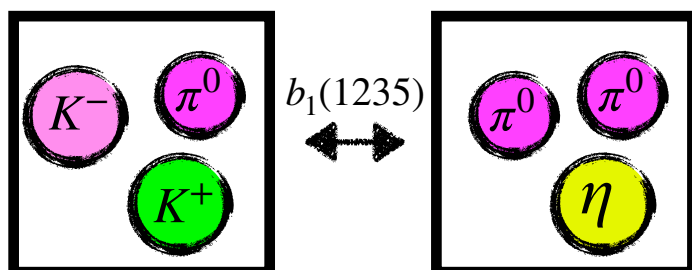
Feng et al. (FVU) 24]



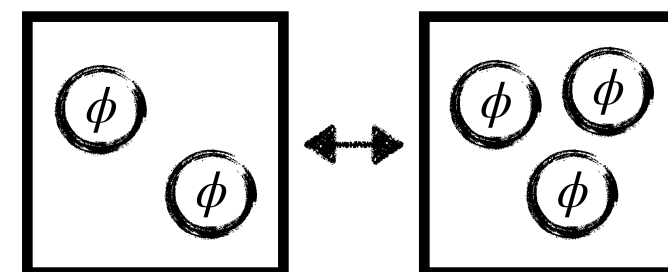
[Draper, Hansen, Romero-López, SRS 23]



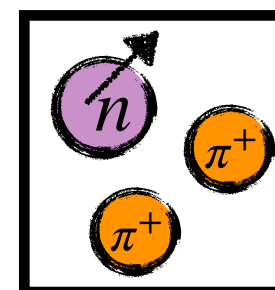
[Hansen, Romero-López, SRS 24]



[Draper, SRS 24]

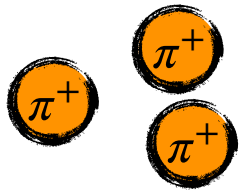


[Briceño, Hansen, SRS 17]

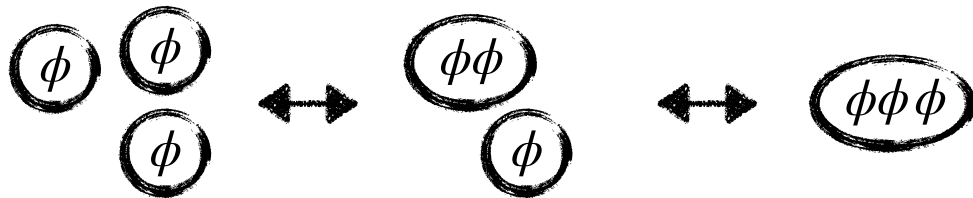


[Hansen, Romero-López, SRS 25]

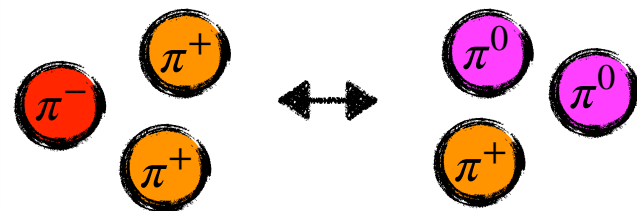
Status: solving integral equations



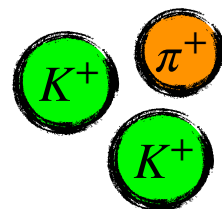
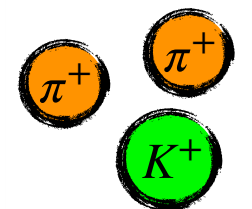
[Hansen, Briceño, Edwards, Thomas, Wilson (RFT) 20]



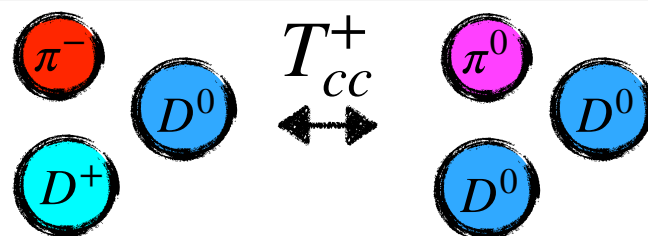
[Jackura, Briceño, Dawid, Islam, McCarty, 20;
Dawid, Islam, Briceño (RFT) 23;
Dawid, Islam, Briceño, Jackura (RFT) 23]



[Mai, Alexandru, Brett, Culver, Döring, Lee, Sadasivan (IVU) 21;
Jackura, Briceño, 23; Briceño, Costa, Jackura (RFT) 24; Jackura, Chambers,
Briceño (RFT) 25; Briceño, Hansen, Jackura, Edwards, Thomas (RFT) 25;
Yan, Mai, Garofalo, Feng, Meißner, Liu, Liu, Urbach (IVU) 24;
Yan, Mai, Garofalo, Feng, Döring, Liu, Liu, Meißner, Urbach (IVU) 25]



[Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, SRS, Skinner, 25]



[Dawid, Romero-López, SRS 24, 25]

Recent applications of three-particle formalism

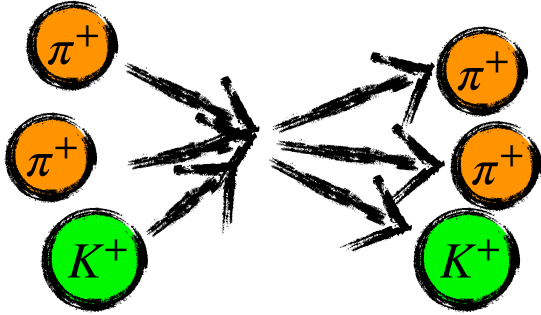
QCD predictions for physical multimeson scattering amplitudes

Sebastian M. Dawid,¹ Zachary T. Draper,¹ Andrew D. Hanlon,² Ben Hörz,³ Colin Morningstar,⁴ Fernando Romero-López,^{5,6} Stephen R. Sharpe,¹ and Sarah Skinner⁴

[2502.14348 (PRL) & 2502.17976 (PRD)]



Three PGBs at maximal isospin



$$3\pi^+, \quad 3K^+, \quad \pi^+\pi^+K^+, \quad K^+K^+\pi^+$$

- Benchmark system, with simple repulsive dynamics
 - First calculation at physical quark masses
 - Study $J^P = 0^-, 1^+, 2^-$, including subchannels with $\ell = 0, 1, 2$
 - Compare to expectations from ChPT
- Use CLS ensembles ($\mathcal{O}(a)$ improved Wilson) + GEVP

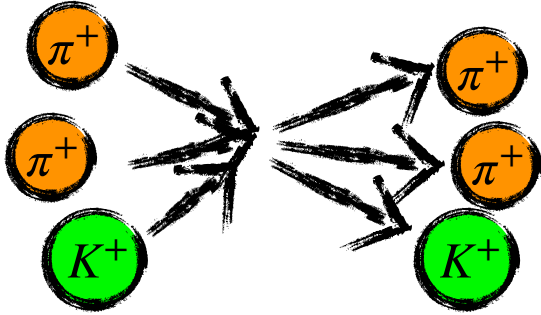
	$(L/a)^3 \times (T/a)$	$M_\pi[\text{MeV}]$	$M_K[\text{MeV}]$	N_{cfg}	$M_\pi L$
N203	$48^3 \times 128$	340	440	771	5.41
N200	$48^3 \times 128$	280	460	1712	4.42
D200	$64^3 \times 128$	200	480	2000	4.20
E250	$96^3 \times 192$	130	500	505	4.05

[Blanton, SRS, et al.,
PRL 2020 & JHEP 2021]
[Draper, SRS, et al.,
JHEP 2023]

New work

$$a \simeq 0.063 \text{ fm}$$

Three PGBs at maximal isospin



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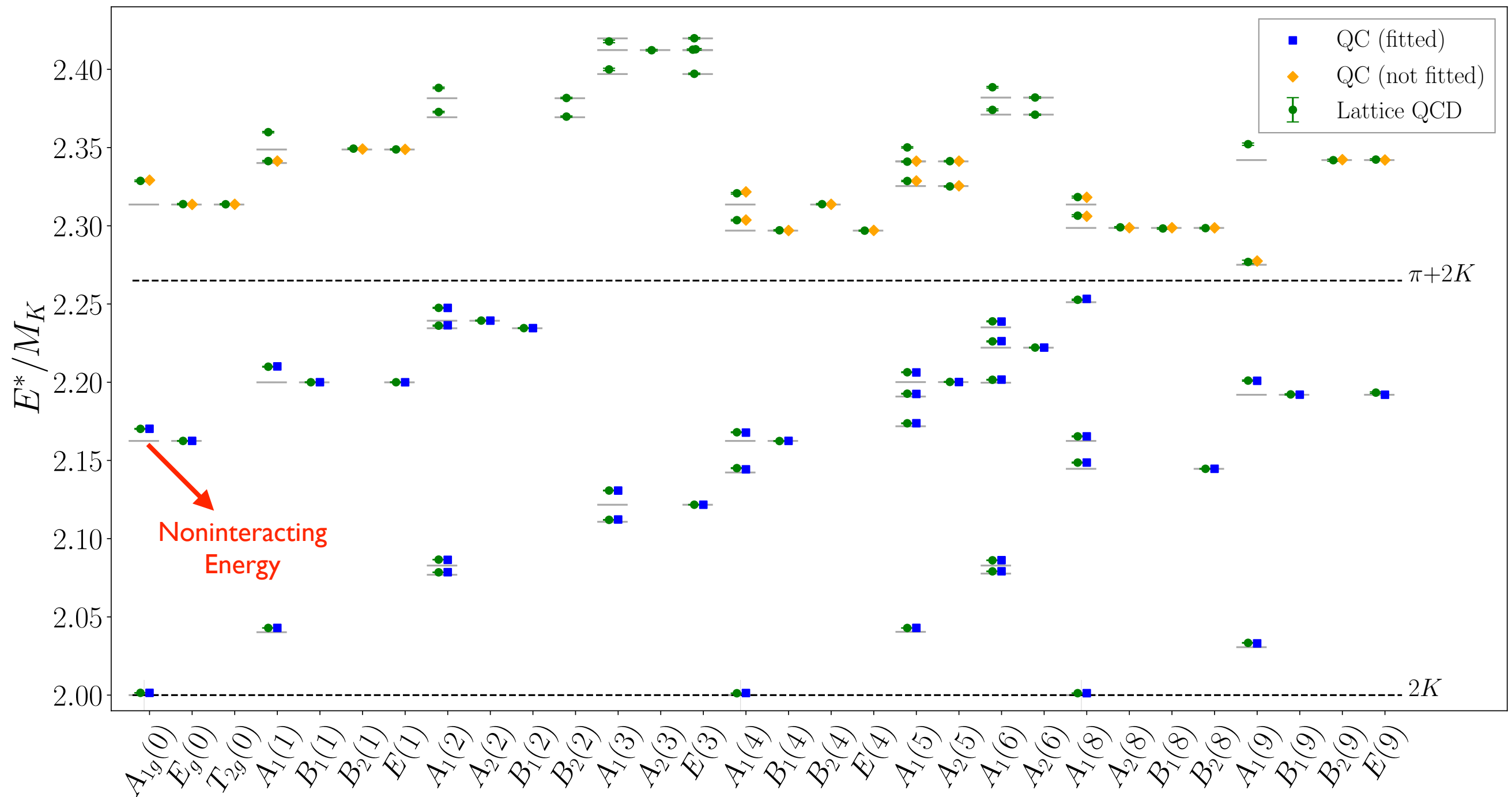
[Blanton, SRS, et al.,
PRL 2020 & JHEP 2021]
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JHEP 2023]

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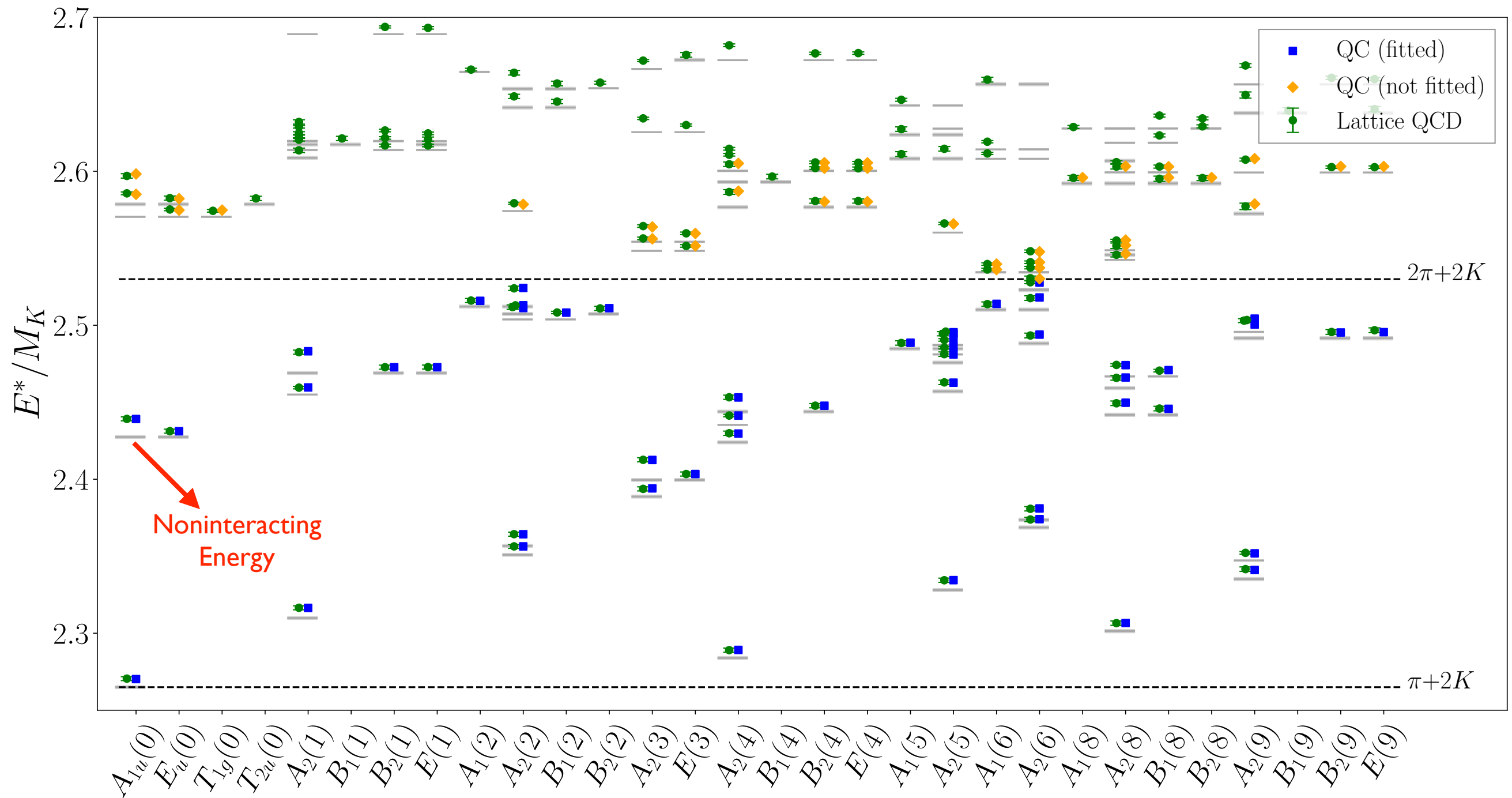
$$\text{tr } m_q = 2m_{ud} + m_s \simeq \text{const}$$

Example of spectrum



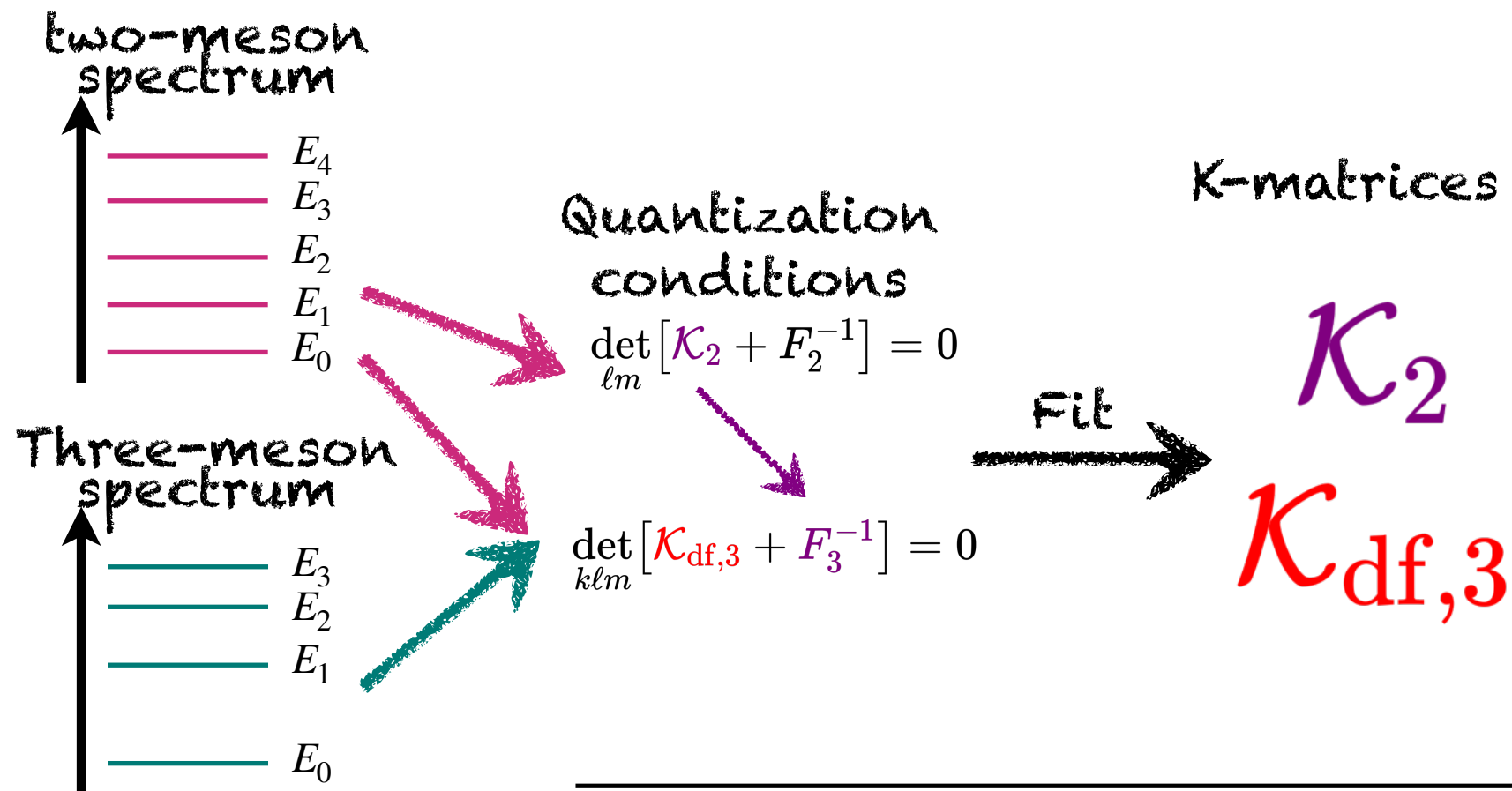
E250: K^+K^+

Example of spectrum



E250: $K^+K^+\pi^+$

Fits to spectra

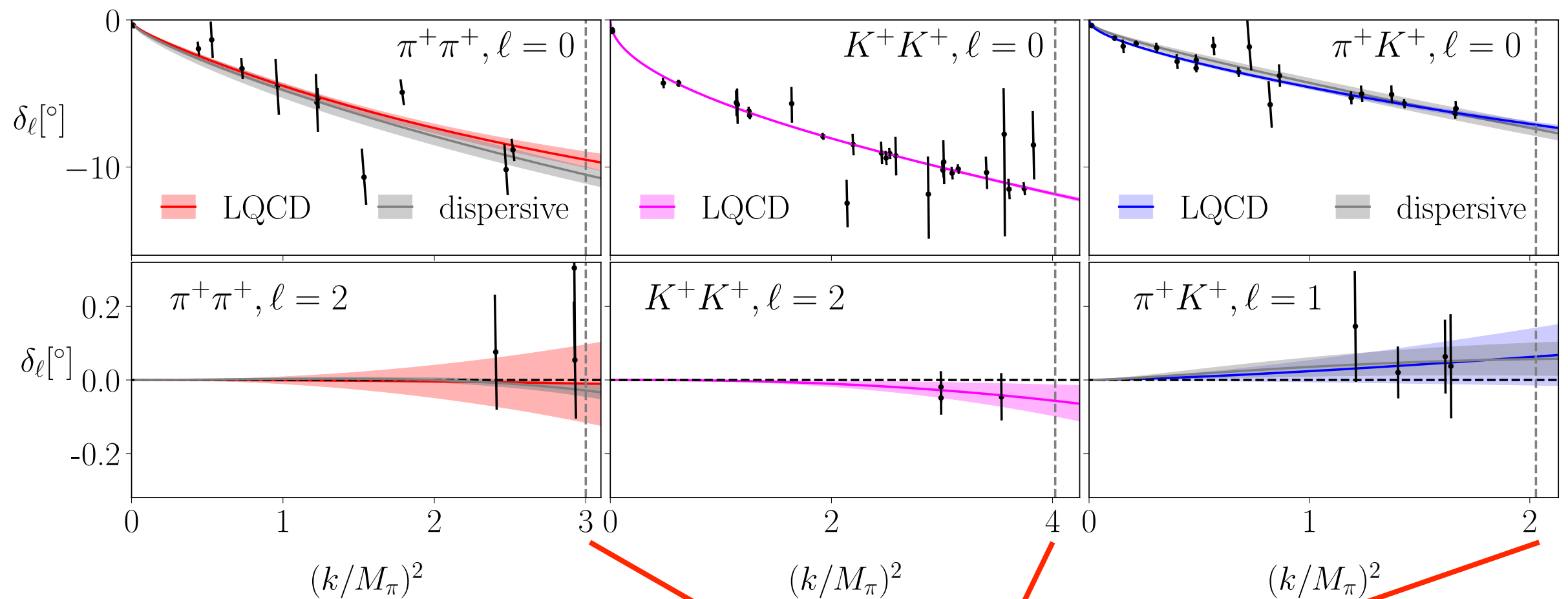


channels	# levels	# parameters	χ^2/DOF
$2\pi/3\pi$	$34 + 32$	6	1.14
$2\pi/\pi K/2\pi K$	$24 + 25 + 23$	9	1.23
$2K/\pi K/2K\pi$	$25 + 40 + 50$	10	1.95
$2K/3K$	$40 + 53$	6	1.49

Parameters in $\mathcal{K}_2$ and $\mathcal{K}_3$: use threshold expansions

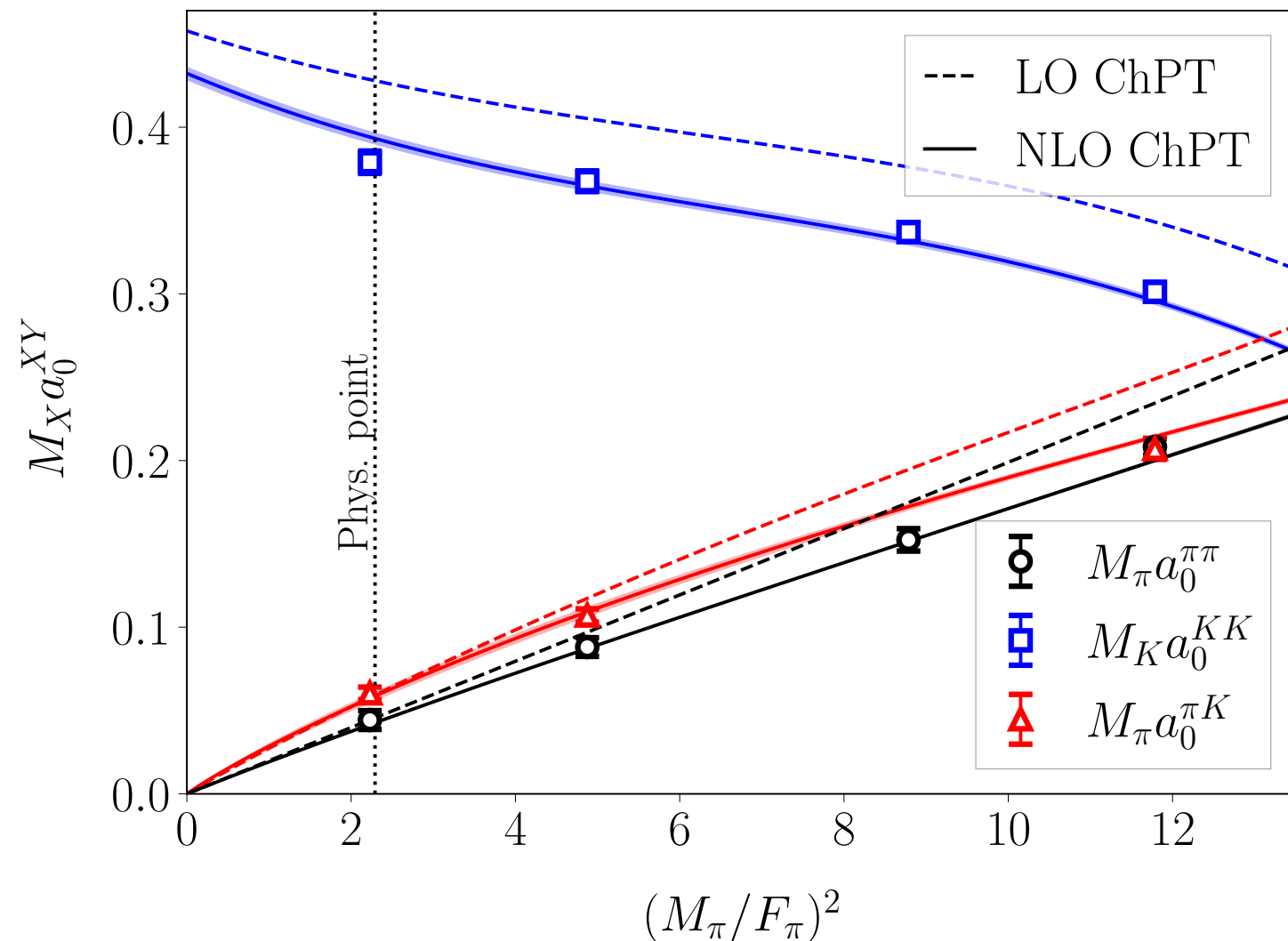
2-particle interactions

E250: $\sim$ physical quark masses



Inelastic thresholds

Scattering lengths

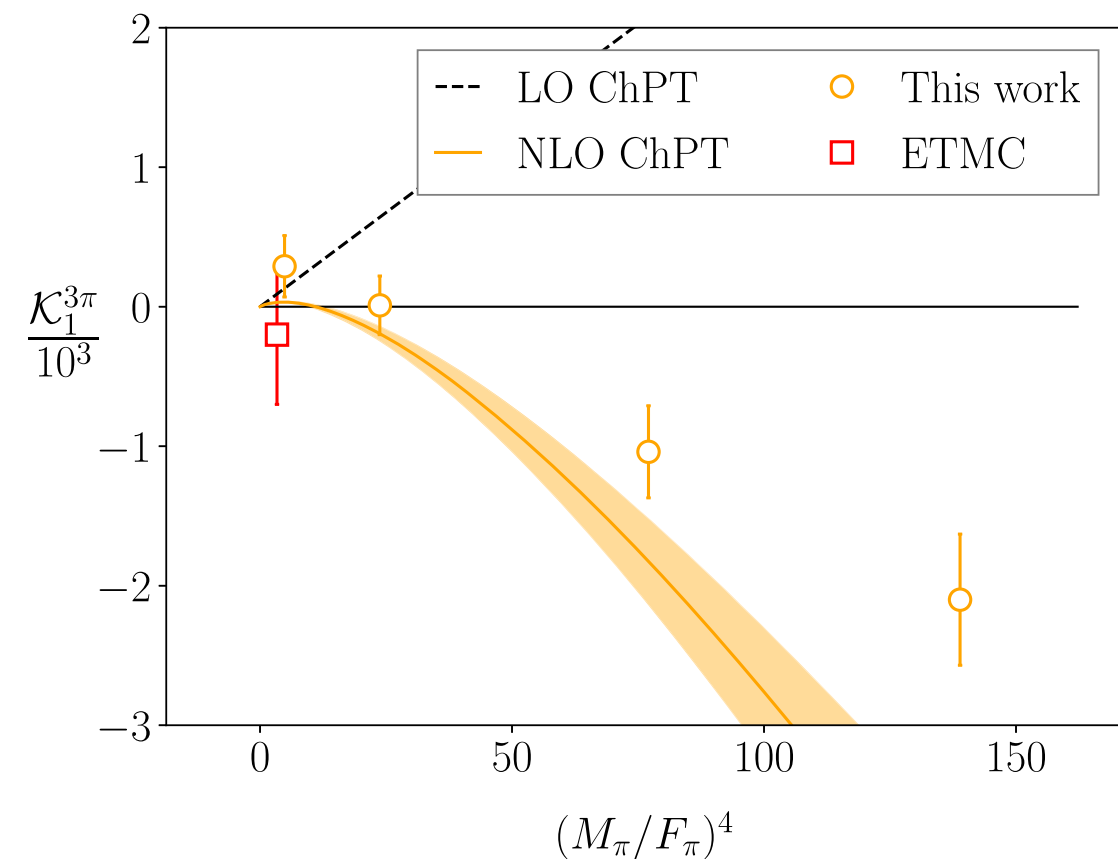
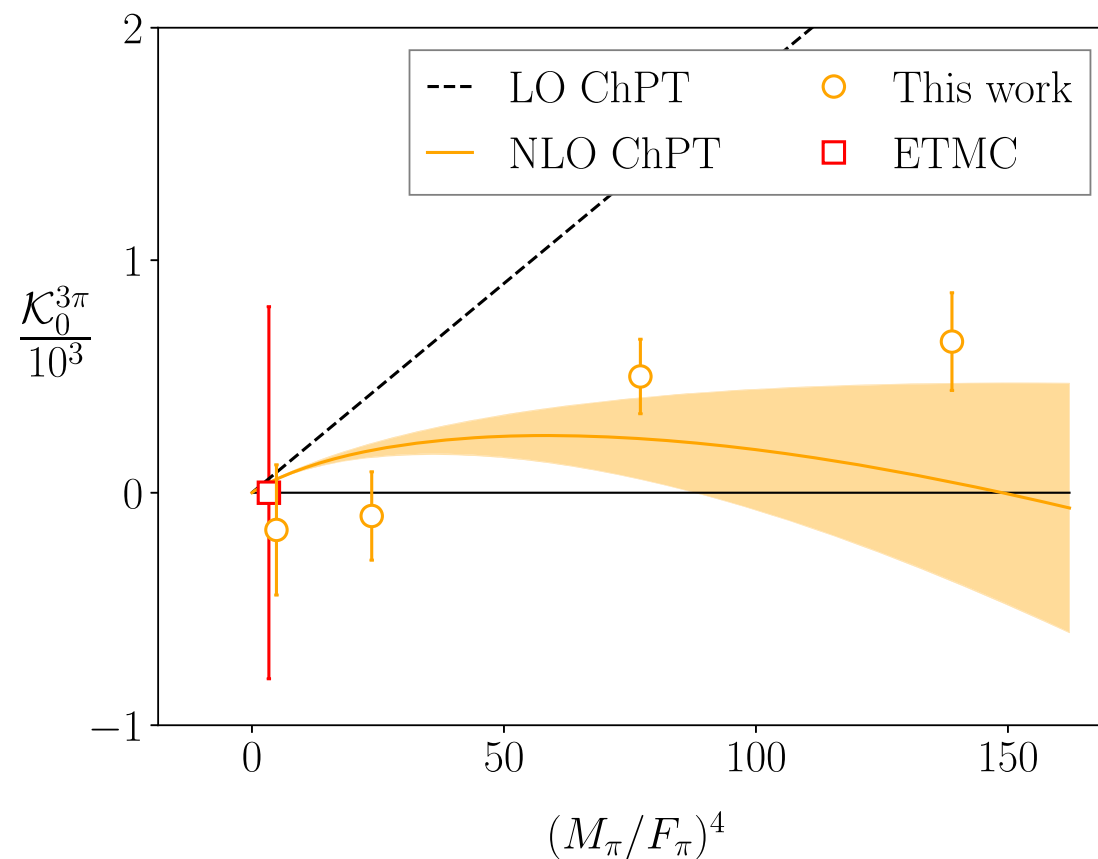


$$\chi^2/\text{dof} = 12.4/(11 - 2)$$

- Simultaneous chiral fit to $\pi\pi$, πK , KK scattering lengths
 - Fit involves two LECs, one of which is determined with 2% stat. errors

Comparing $\mathcal{K}_{\text{df},3}$ to ChPT

- Parametrize $\mathcal{K}_{\text{df},3}$ in a threshold expansion, keeping first two terms
- From fits to spectra, we find non vanishing results for these terms at heavier quark masses
- Compare to NLO Chiral Perturbation theory calculation of $\mathcal{K}_{\text{df},3}$ for $3\pi \rightarrow 3\pi$
[Baeza-Ballesteros, Bijmans, Husek, Romero-López, SRS, Sjö, 2303.13206 (JHEP) & 2401.14293 (JHEP)]

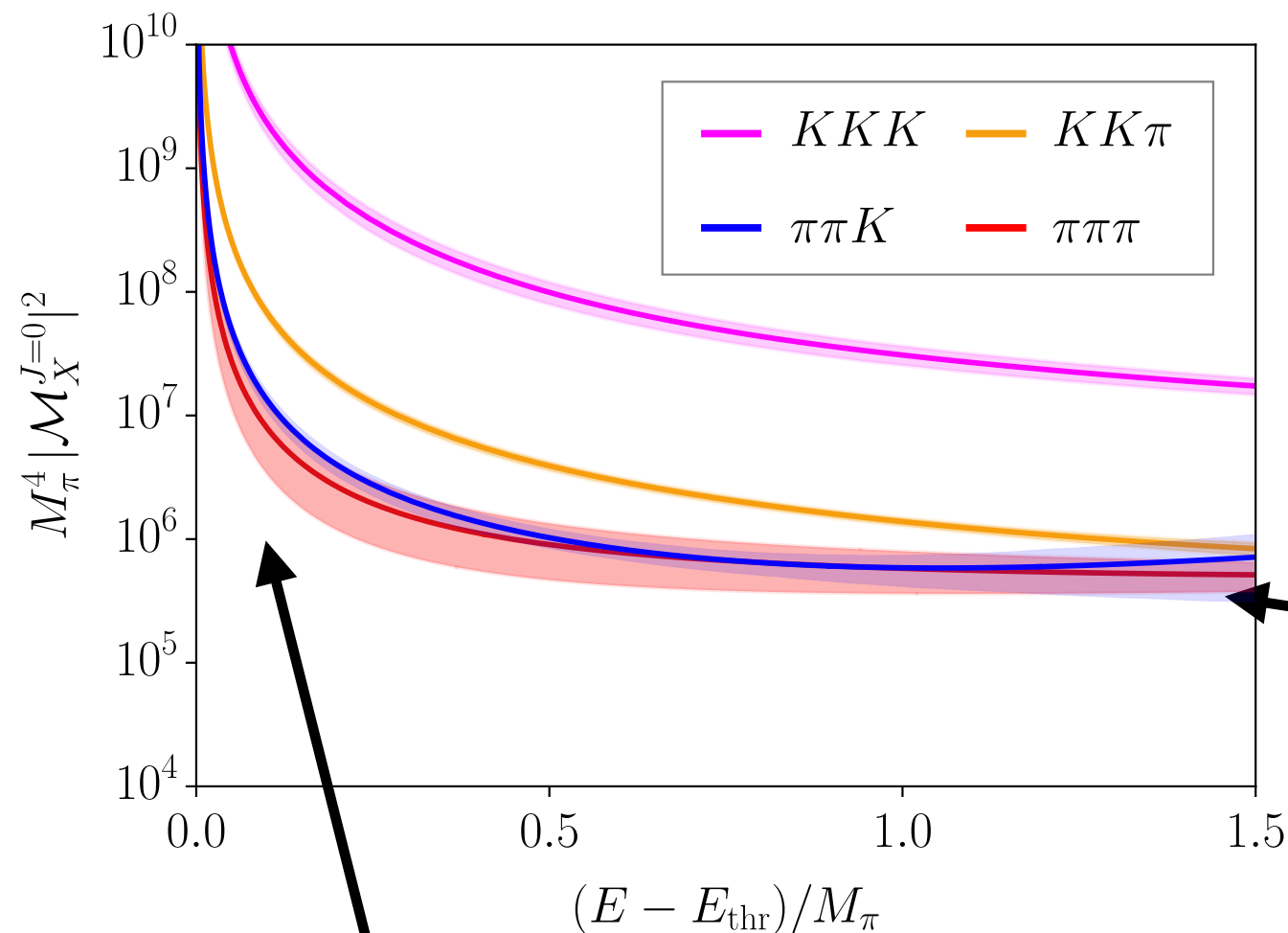


Large NLO corrections in ChPT resolve LO inconsistency

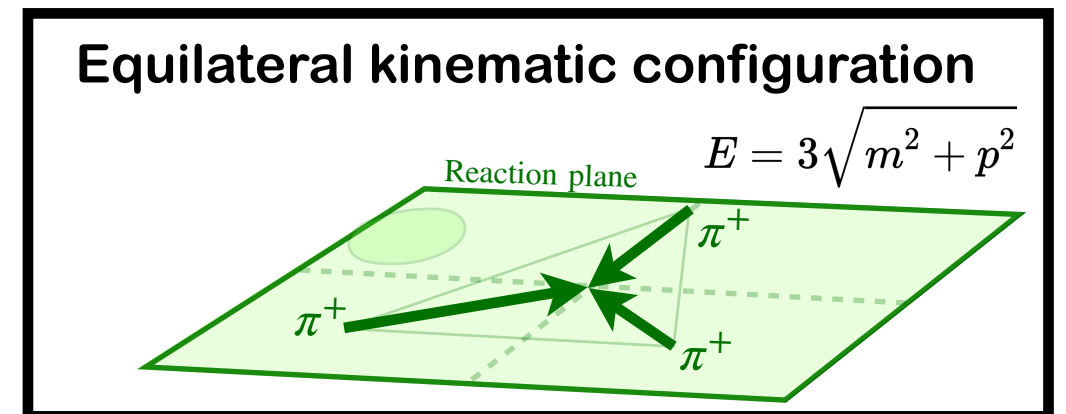
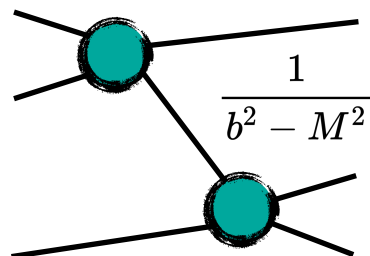
Physical 3-particle amplitudes

$$M_\pi^4 |\text{Amplitude}|^2$$

$$J^P = 0^-$$



**Divergent
at threshold**



**Pion interactions
are chirally suppressed**

$$\mathcal{M}_3 = \mathcal{D} + \mathcal{M}_{\text{df},3}$$

$$\mathcal{D} = \mathcal{S} \left\{ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \dots \right\}$$

Diagram 1: Two vertices labeled $\mathcal{M}_2$ connected by a line labeled G .

Diagram 2: Three vertices labeled $\mathcal{M}_2$ connected by two lines labeled G .

$$\mathcal{M}_{\text{df},3} = \mathcal{K}_{\text{df},3} + \mathcal{S} \left\{ \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \dots \right\}$$

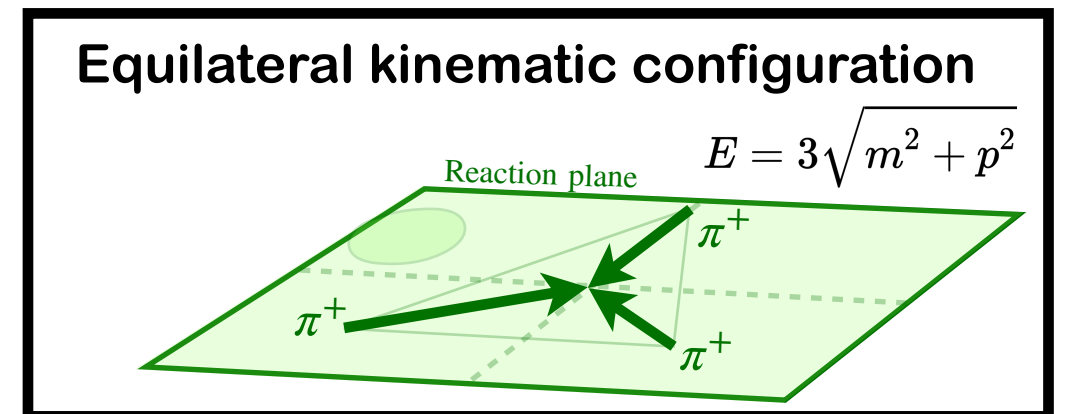
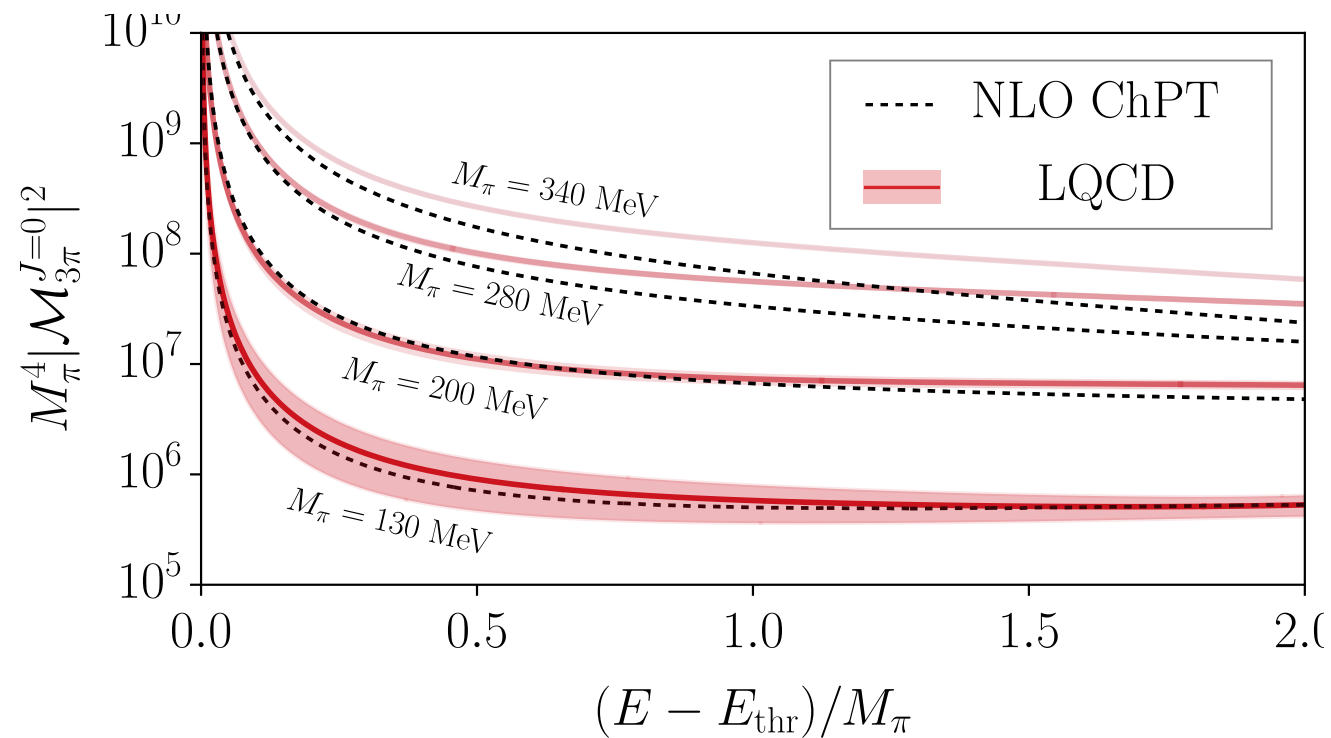
Diagram 3: Two vertices labeled $\mathcal{M}_2$ and $\mathcal{K}_{\text{df},3}$ connected by a line labeled ρ .

Diagram 4: Two vertices labeled $\mathcal{K}_{\text{df},3}$ and $\mathcal{M}_2$ connected by a line labeled ρ .

Comparison with ChPT

$$M_\pi^4 |\text{Amplitude}|^2$$

$$J^P = 0^-$$



Observe expected breakdown of ChPT as M_π or p_π increase

Using three-particle methods to study the $T_{cc}^+(3875)$



Incorporating $DD\pi$ effects and left-hand cuts in lattice QCD studies of the $T_{cc}(3875)^+$

Maxwell T. Hansen,^a Fernando Romero-López ^b and Stephen R. Sharpe ^c

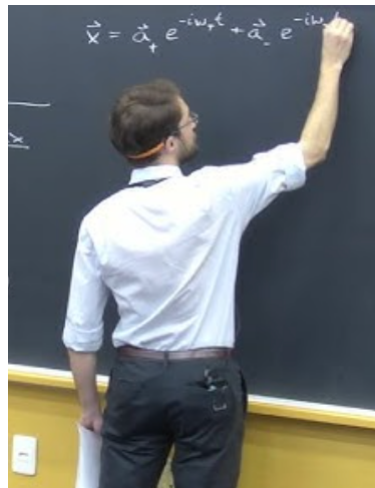
[2401.06609, JHEP]



Finite- and infinite-volume study of $DD\pi$ scattering

Sebastian M. Dawid ^a, Fernando Romero-López ^{b,c} and Stephen R. Sharpe ^a

[2409.17059, JHEP]



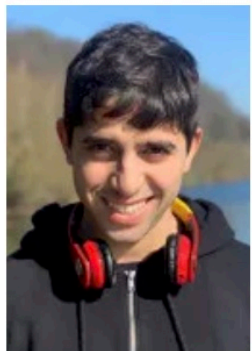
Comparison of integral equations used to study T_{cc}^+ for a stable D^*

Sebastian M. Dawid ^a, Fernando Romero-López ^b and Stephen R. Sharpe ^a

[2505.05466, JHEP]

Using three-particle methods to study the $T_{cc}^+(3875)$

+ work in progress



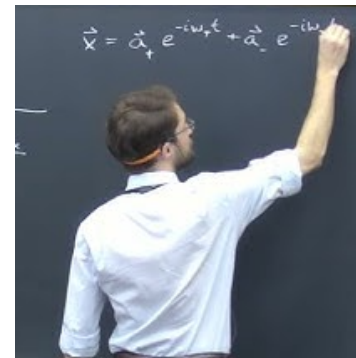
Herzallah Alharazin (Bochum)



André Baião-Raposo (Bochum)



John Bulava (Bochum)



Sebastian Dawid (Indiana)



Jeremy Green (DESY)



Colin Morningstar (CMU)



Davide Laudicina (Bochum)



Fernando Romero-López (U. Bern)



Miguel Salg (U. Bern)



Andres Stump (Humboldt U.)

Doubly-charmed tetraquark

NATURE PHYSICS | VOL 18 | JULY 2022 | 751-754 | www.nature.com/naturephysics

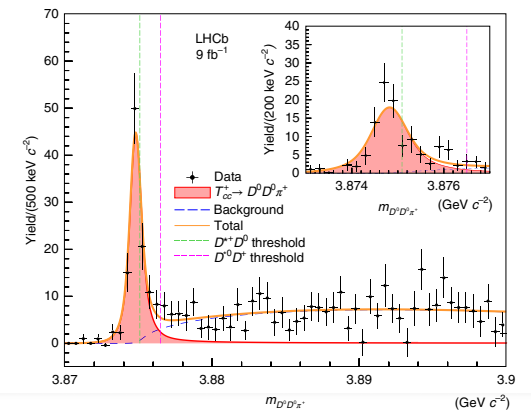
Observation of an exotic narrow doubly charmed tetraquark

LHCb Collaboration*

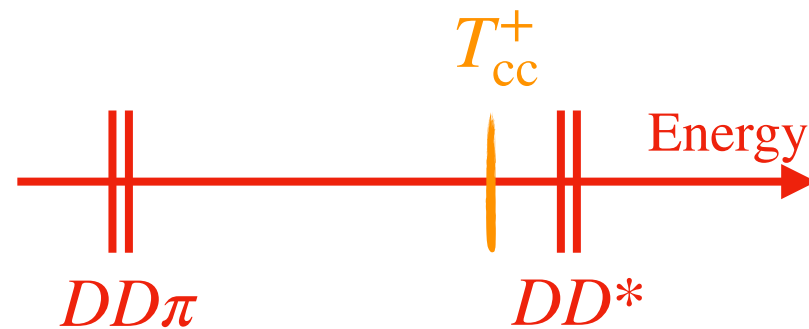
$$T_{cc}^+(3875)$$

$$I = 0, J^P = 1^+$$

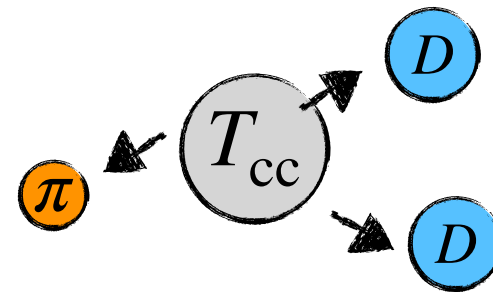
$$cc\bar{u}\bar{d}$$



Experiment



► For physical quark masses is a three-body resonance



need three-body formalism!

Adapted from: Fernando Romero-López

Doubly-charmed tetraquark

NATURE PHYSICS | VOL 18 | JULY 2022 | 751-754 | www.nature.com/naturephysics

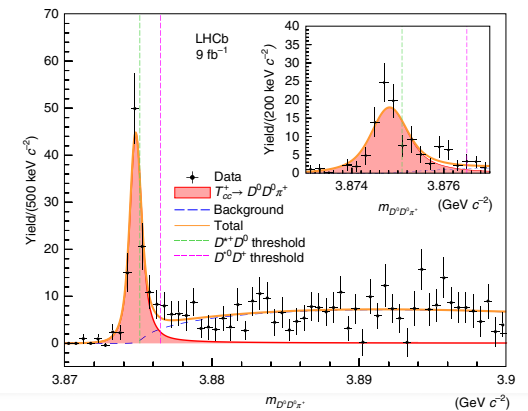
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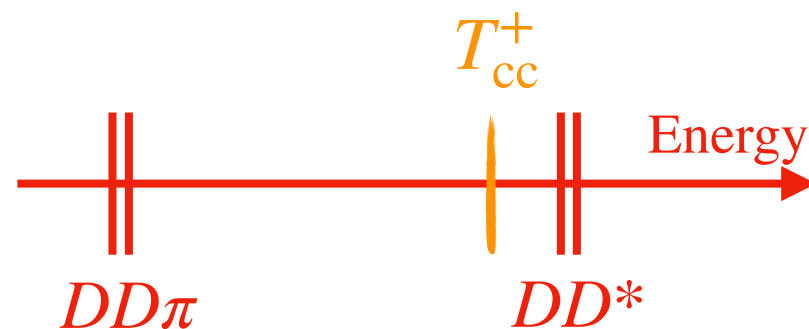
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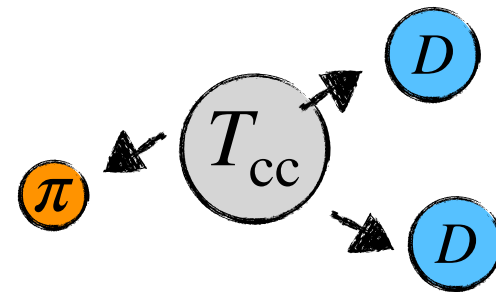
$$cc\bar{u}\bar{d}$$



Experiment

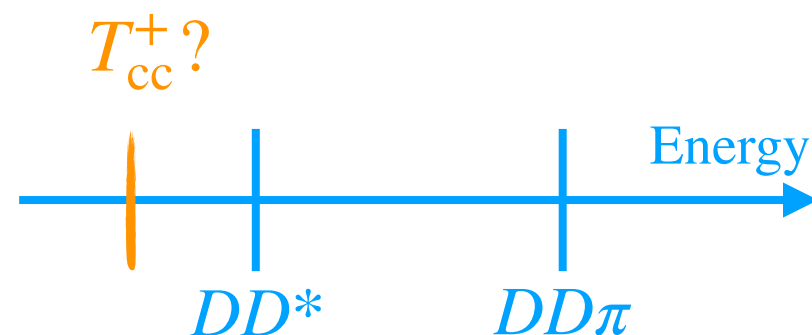


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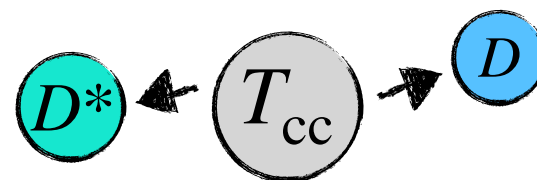


need three-body formalism!

LQCD calculations with heavier-than-physical quarks



► D* is stable for slightly heavier-than-physical quarks



suitable for the two-body finite-volume formalism?

Adapted from: Fernando Romero-López

DD^* scattering

- Several LQCD studies using QC2 in T_{cc} channel

[Chen et al., 2206.06185]

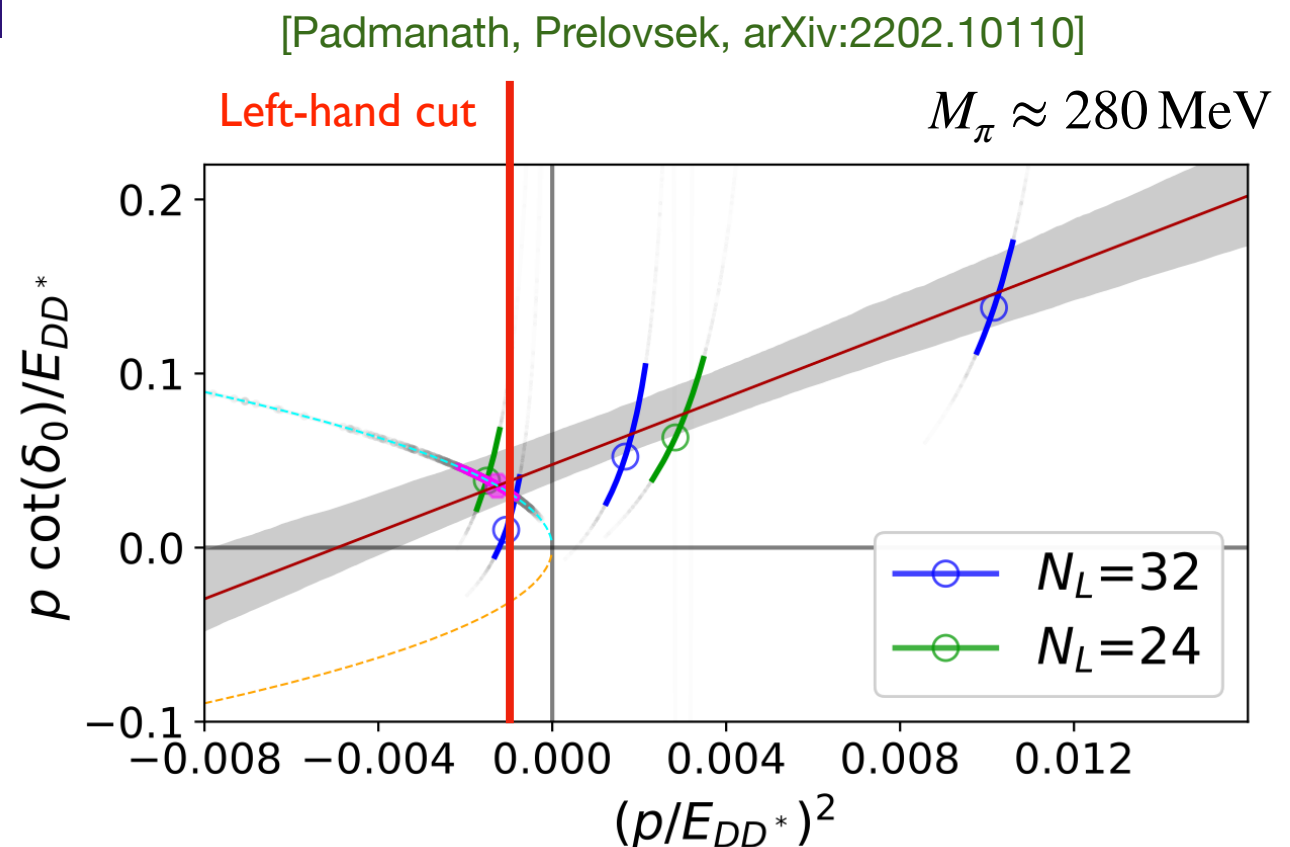
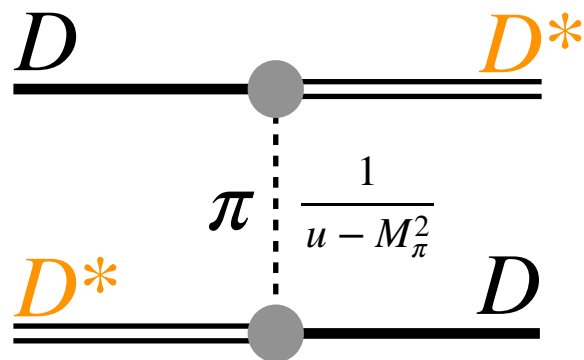
[Lyu et al. (HALQCD), 2302.04505]

[Padmanath & Prelovsek, 2202.10110]

[Whyte, Thomas, Wilson, 2405.15741]

[Several talks at Lattice 2025]

- Use heavier-than-physical quarks
- Find signature of virtual bound state
- But QC2 fails at left-hand cut, below which phase shift becomes complex



- Several solutions to failure have been proposed, by generalizing the QC2

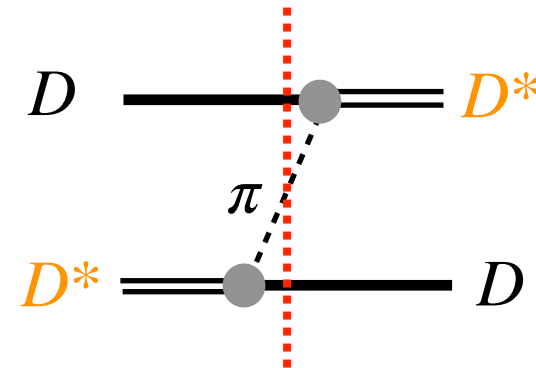
[Du et al (2408.09375); Abolnikov et al. (2407.04649), Bubna et al. (2402.12985);

Meng et al. (2312.01930); Raposo, Hansen (2311.18793); Raposo, Hansen, Briceño, Jackura (2502.19375)]

3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

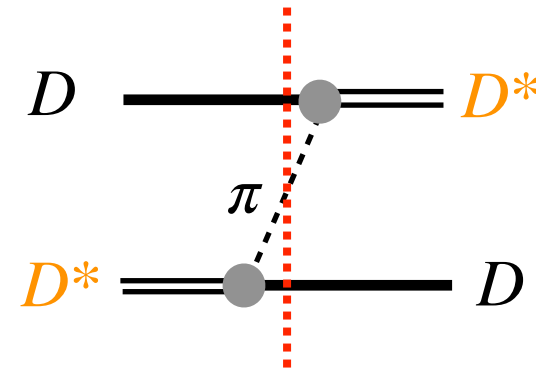
- Left-hand cut is a three-body effect



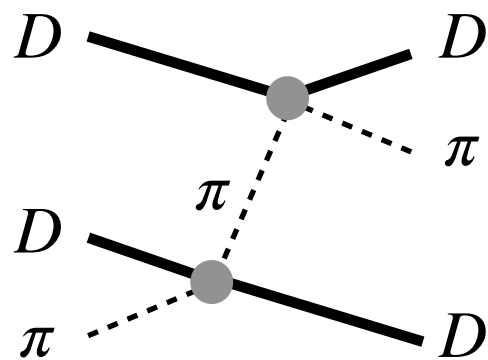
3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

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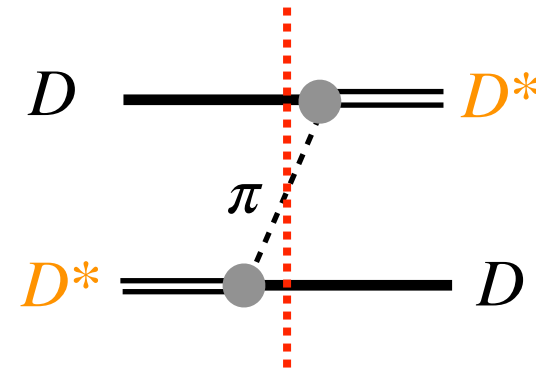
- Use $DD\pi$ QC3, but include D^* as a bound state in p -wave $D\pi$ channel



3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

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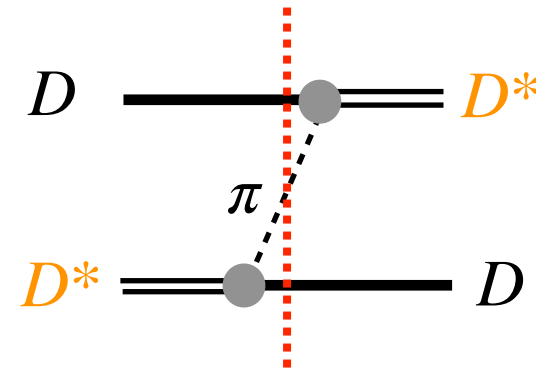


- Finite-volume effects from u -channel pion exchange naturally incorporated

3-body solution

[Hansen, Romero-López, SRS, 2401.06609, JHEP]

- Left-hand cut is a three-body effect



- Use $DD\pi$ QC3, but include D^* as a bound state in p -wave $D\pi$ channel



- Finite-volume effects from u -channel pion exchange naturally incorporated
- Another advantage of QC3 approach:
 - Works for all choices of quark masses, including physical case of unbound D^*
- Successfully implemented in [Dawid, Romero-López, SRS, 2409.17059, JHEP & 2505.05466, JHEP; plus work in progress]

Simulation details

- Use two CLS ensembles with same pion mass but different volume

	L/a	T/a	$m_\pi L$	m_π [MeV]	a [fm]	M_D [MeV]	$(M_{D^*} - M_D)$ [MeV]	Configs
X252	36	128	3.4	280	0.064	1880	130	175
X253	40	128	3.7	280	0.064	1880	130	301

Physical values:

135

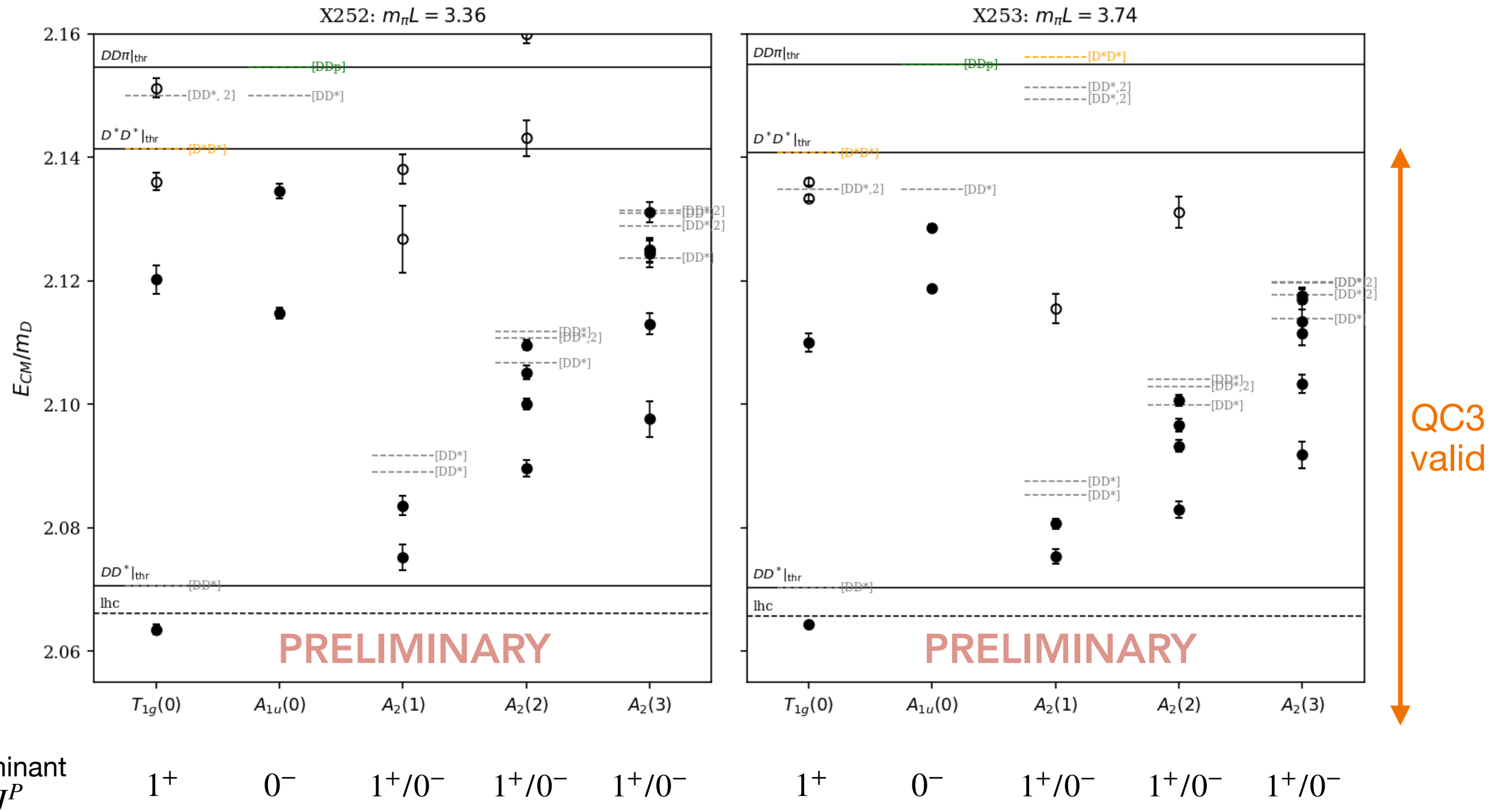
1870

140-145

- Use frames (0,0,0), (0,0,1), (1,1,0), (1,1,1) and irreps coupling dominantly to $J^P = 1^+, 0^-$
- Use distillation, with multiple operators, and GEVP
 - All results are PRELIMINARY

$DD\pi$ spectra

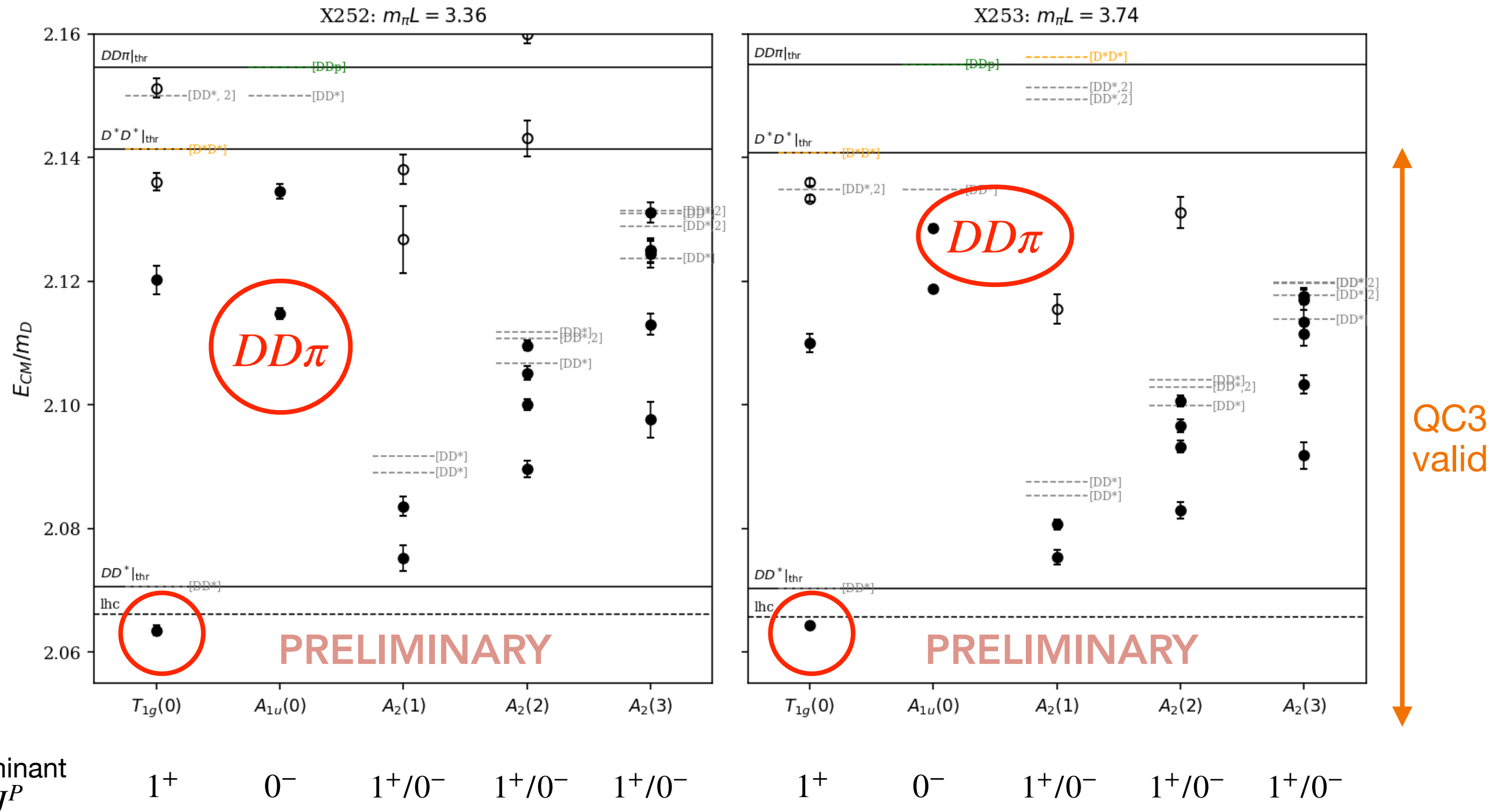
Keep 16+16 $DD\pi$ levels in fits, together with 16+16 $D\pi$ levels



$DD\pi$ spectra

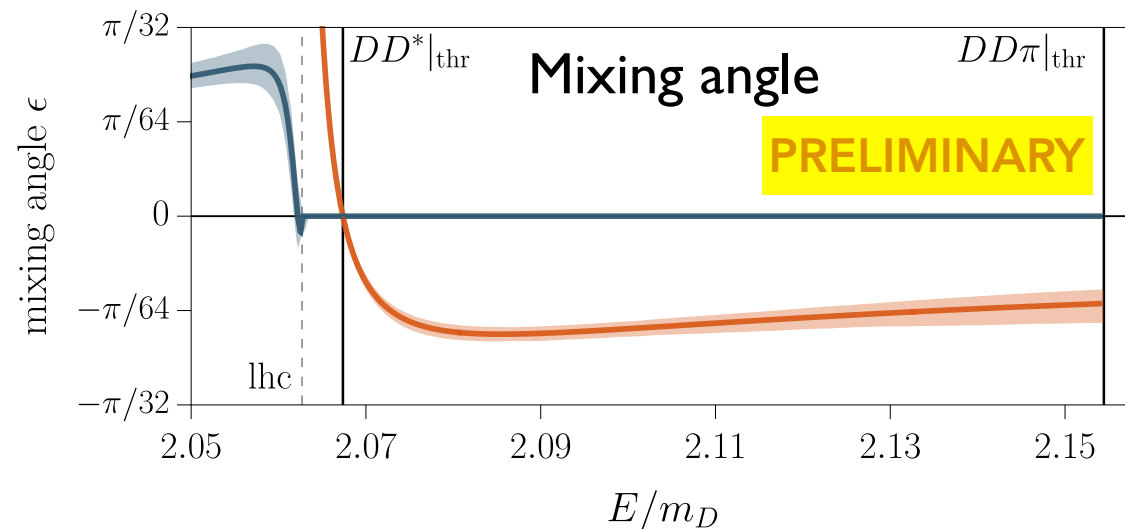
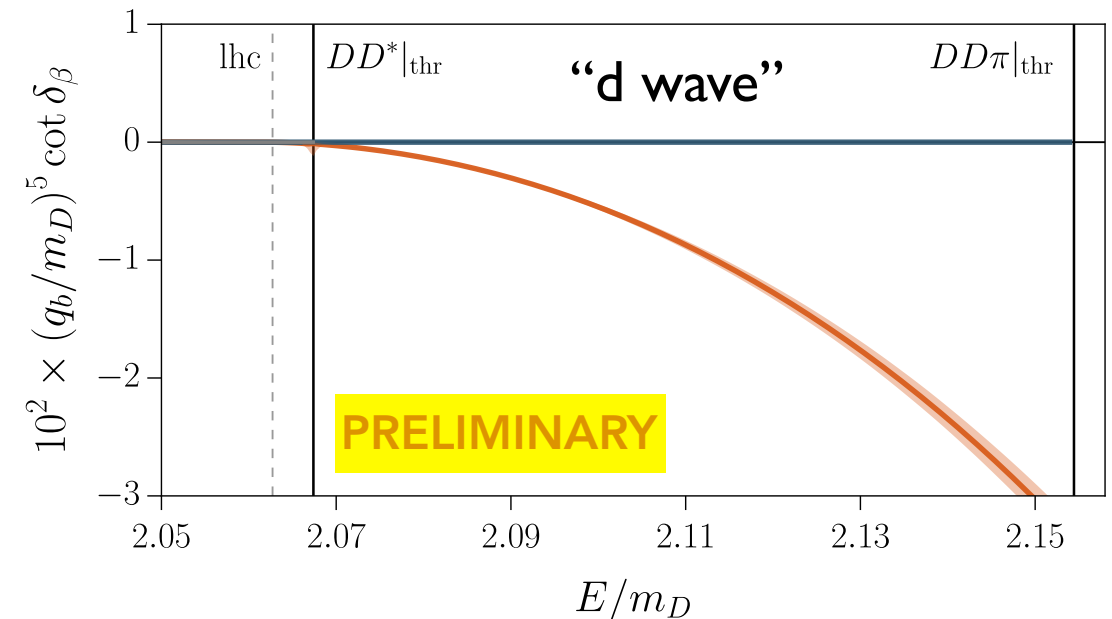
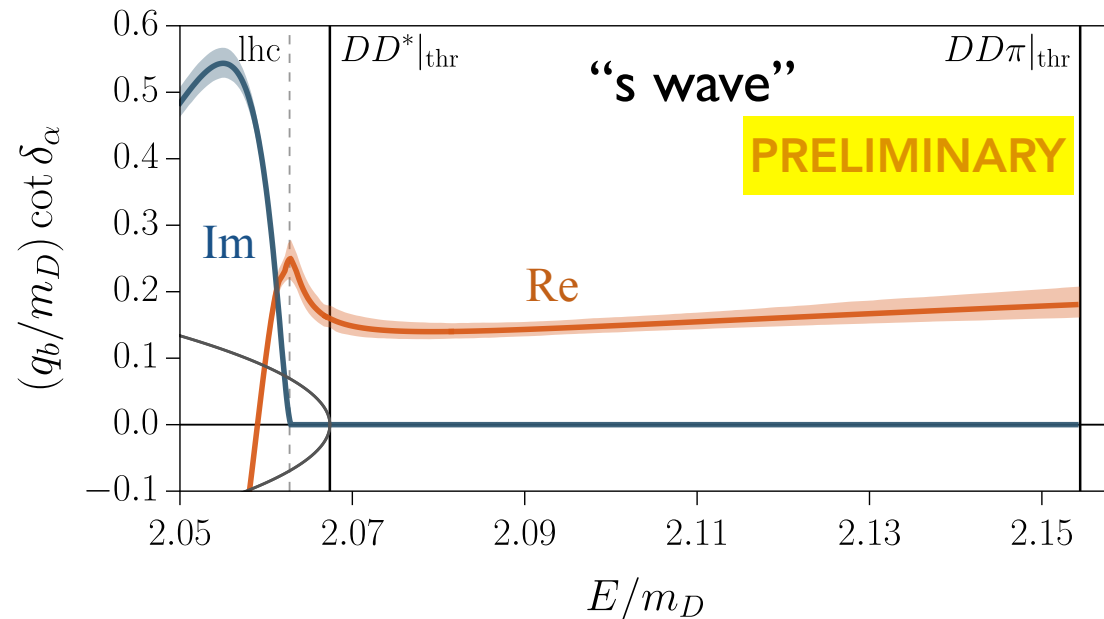
Keep 16+16 $DD\pi$ levels in fits, together with 16+16 $D\pi$ levels

3-particle
effects crucial



DD^* amplitude

- Given $\mathcal{K}_2$ and $\mathcal{K}_{\text{df},3}$, can solve integral equations to obtain DD^* amplitude using LSZ
- Using results from (imperfect) global fit as an example

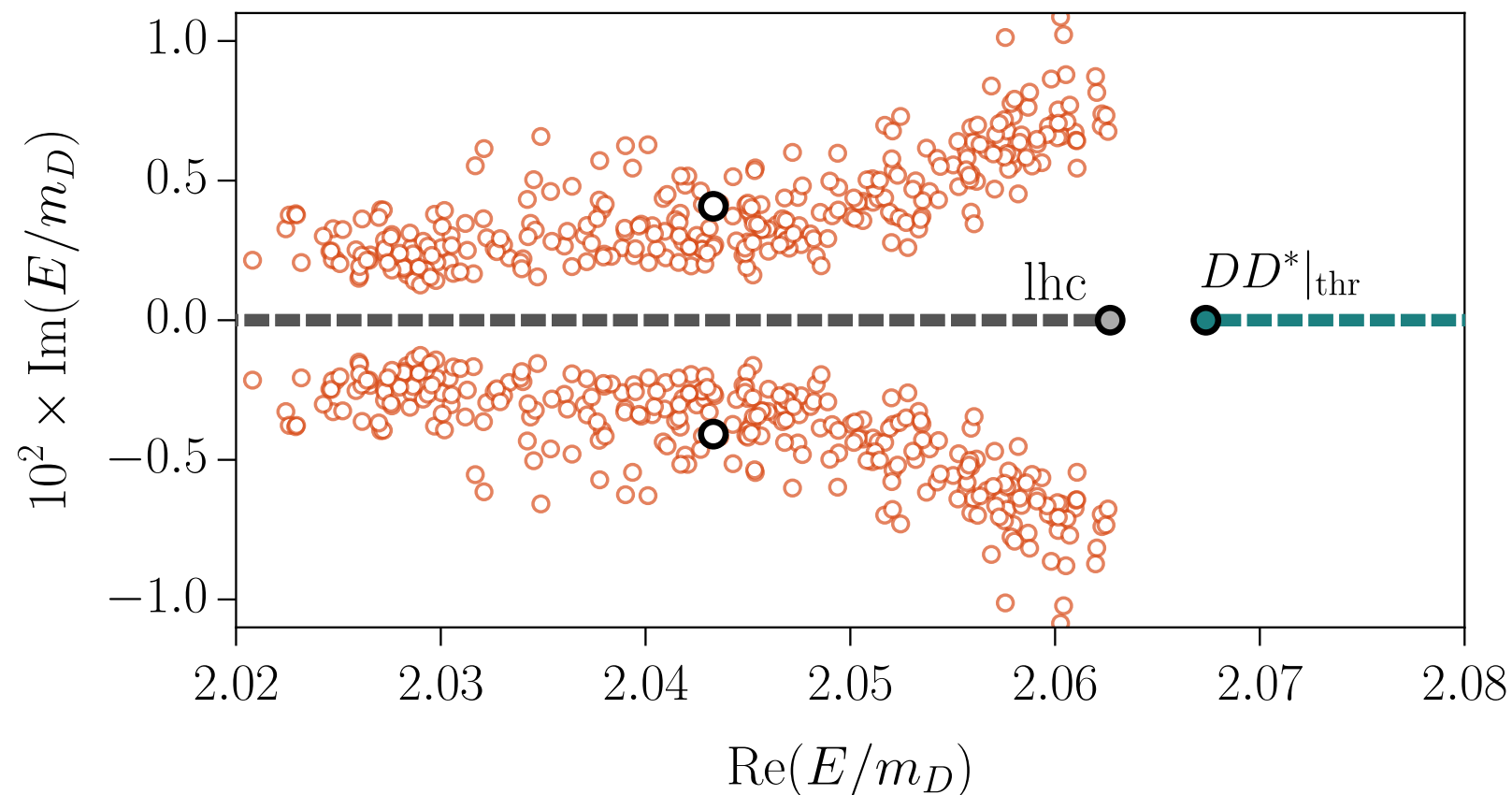


- Both s & d waves combine with spin of D^* to give $J^P = 1^+$
- Use Blatt-Biedenbarn parametrization of K matrix

$$(8\pi E) [\mathcal{K}_{DD^*}^{-1}] = \begin{pmatrix} \cos(\epsilon) & -\sin(\epsilon) \\ \sin(\epsilon) & \cos(\epsilon) \end{pmatrix} \begin{pmatrix} q_b \cot(\delta_\alpha) & 0 \\ 0 & q_b \cot(\delta_\beta) \end{pmatrix} \begin{pmatrix} \cos(\epsilon) & \sin(\epsilon) \\ -\sin(\epsilon) & \cos(\epsilon) \end{pmatrix}$$

DD^* amplitude

- T_{cc}^+ appears as subthreshold resonance pole
 - Use 300 samples drawn from correlation matrix for K-matrix parameters from global fit



Dreaming of 3 neutrons

Three relativistic neutrons in a finite volume

Zachary T. Draper,^a Maxwell T. Hansen,^b Fernando Romero-López^c
and Stephen R. Sharpe^a

[2303.10219, JHEP]



Implementation of the three-neutron quantization condition

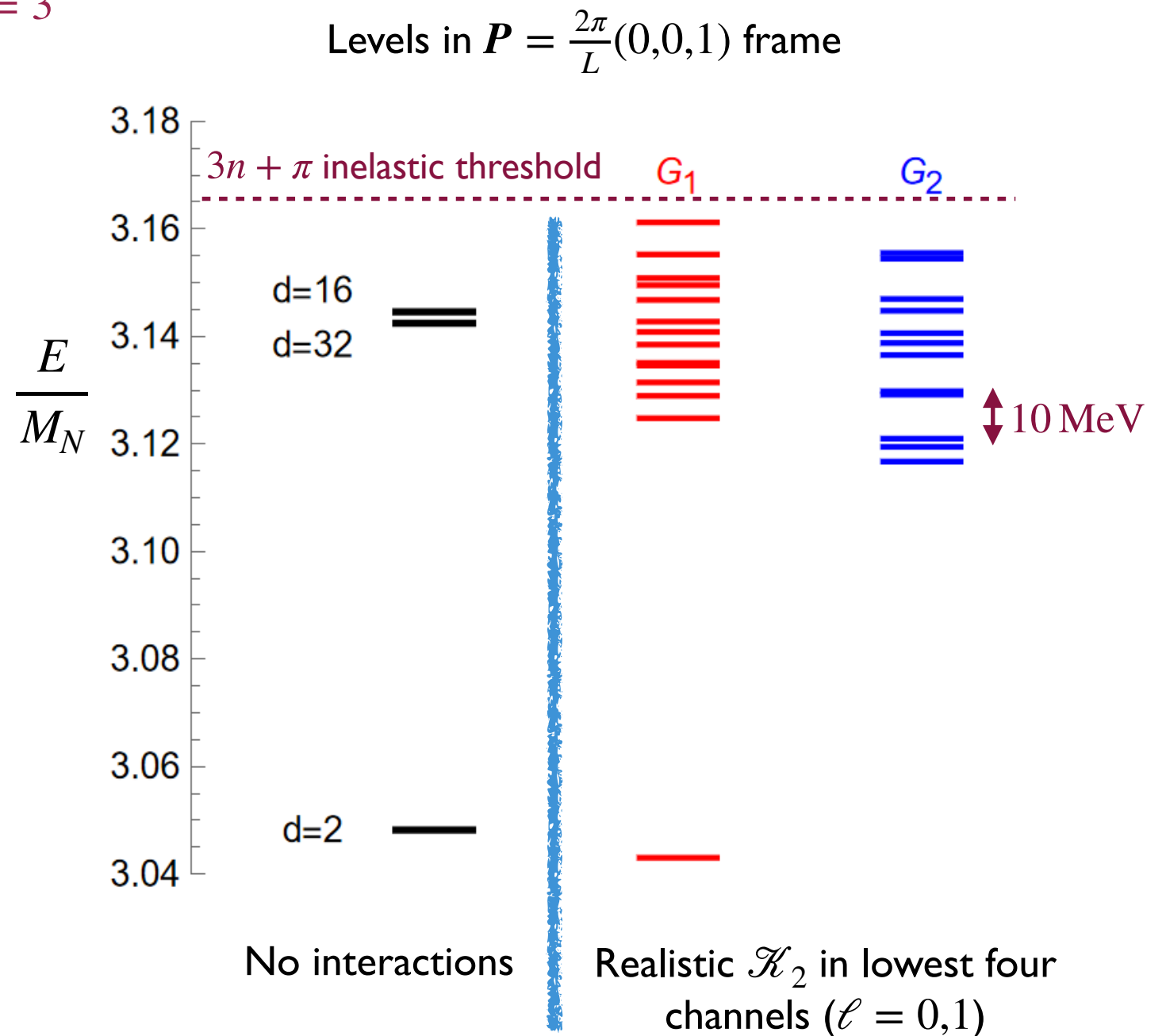
Wilder Schaaf^a and Stephen R. Sharpe^a

[2512.24508]

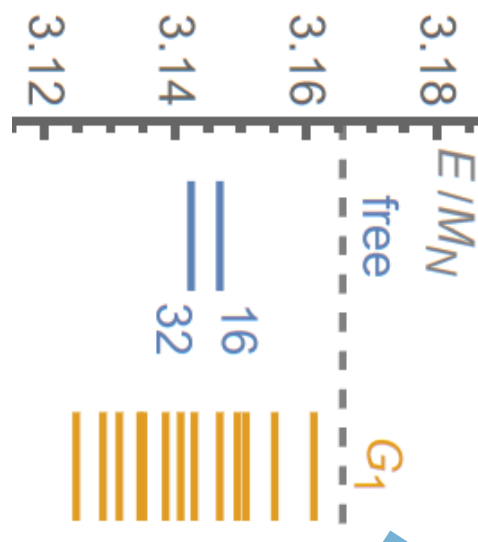


Spectrum in realistic setup

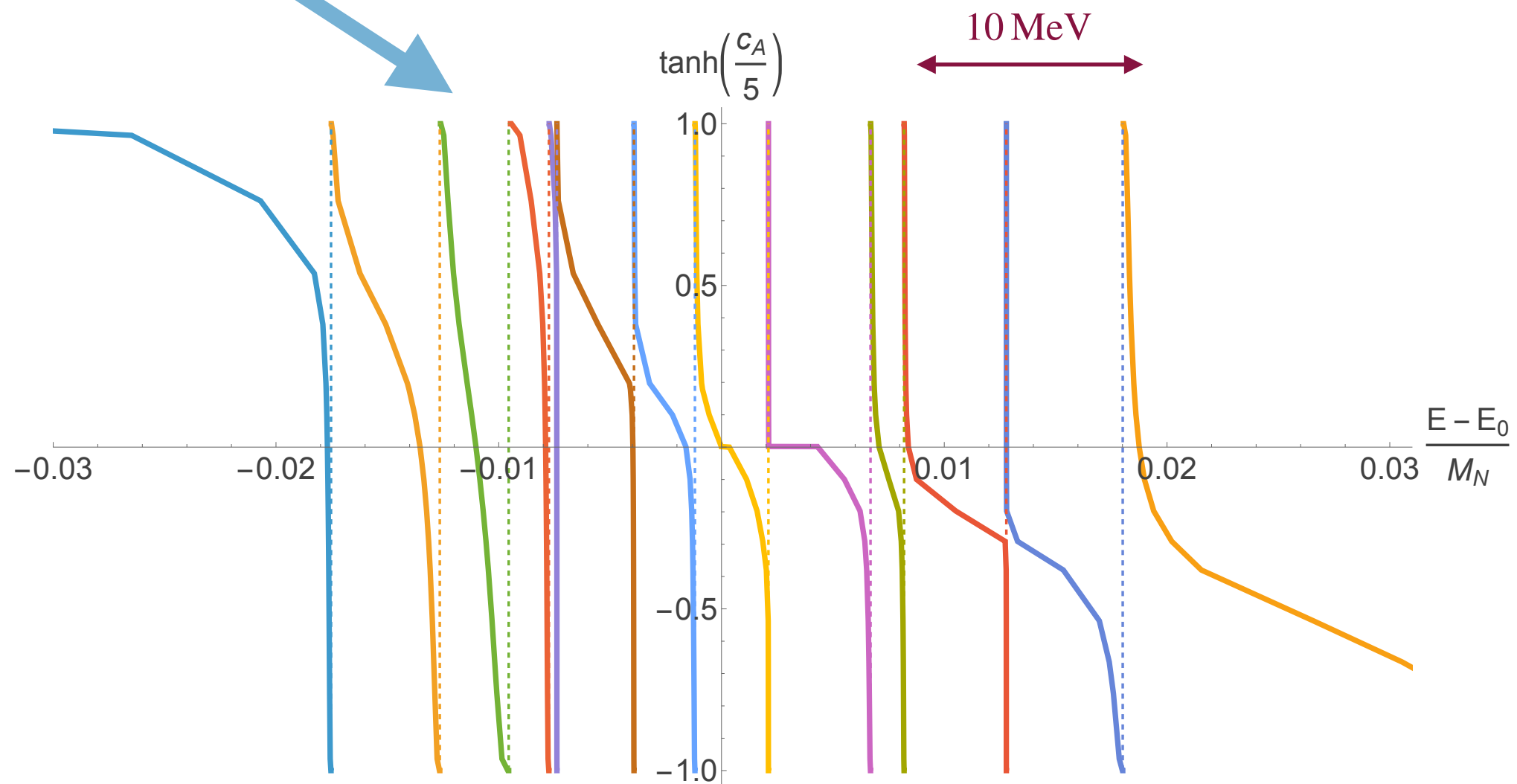
- Physical masses, $L \approx 4.3$ fm:
 - $M_N L = 20$, $M_\pi = 0.15 M_N \Rightarrow M_\pi L = 3$



Spectrum in realistic setup



Add leading-order $\mathcal{K}_{\text{df},3}$ in threshold expansion
Keep one term (out of two) with coefficient c_A



Summary & Outlook

Summary & Outlook

- Major steps have been taken in the three-particle sector
 - Formalism well established & cross checked, and almost complete
 - Several pioneering end-to-end applications to three-particle spectra from LQCD
 - Path to a calculation of $K \rightarrow 3\pi$ decay amplitudes is now open
- Near-term steps in implementation
 - $T_{cc}^+ \rightarrow D^*D \rightarrow DD\pi$ for lighter quark masses
 - Extended calculations for $3\pi(I = 0,1,2)$
 - $N\pi\pi \leftrightarrow \Delta\pi(I = 5/2)$
- Final steps in three-particle formalism
 - $3\pi(I = 0) \leftrightarrow K\bar{K}$ (WZW term)
 - $N\pi\pi + N\pi$ [for Roper], $NN\pi + NN$, and $NNN(I = \frac{1}{2})$
- Very recent review—talk by SRS at Lattice 2025 [2601.04147]

Challenges & Future

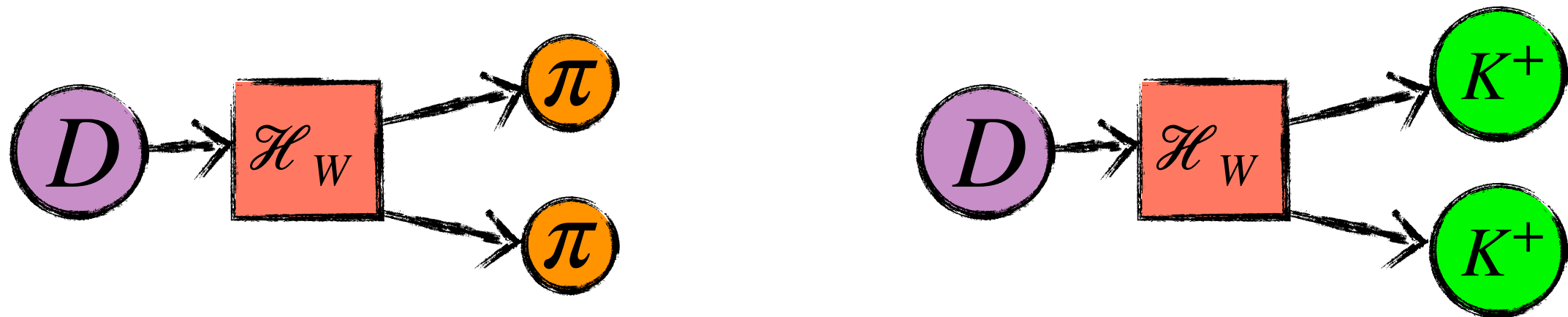
- Reducing model-dependence in fitting & analytic continuation
 - [Salg, Romero-López, Jay, [2506.16161](#)] propose using Bayesian reconstruction + Nevanlinna interpolation
- Controlling all systematic errors (when will results be FLAG-ready?)
 - For which processes is precision important?
 - For three-nucleon interaction even a semiquantitative result would be useful
- Comparing formalisms in detail (e.g. RFT vs FVU)
 - Equivalent up to technical details (e.g. smooth vs hard cutoff functions)
 - Share data for cross-checking analysis? Bootcamp?
- Moving to more particles (needed, e.g., for $D \rightarrow \pi\pi, K\bar{K}$ & physical $\pi(1300)$)
 - QC4 (which is not yet known except in threshold expansion) may be practical
 - QCN seems impractical—combine QC2-4 with inclusive methods in some yet-to-developed manner?
 - “Scattering amplitudes from finite-volume spectral functions” [Bulava & Hansen, [1903.11735](#)]
 - Squeeze as much information as we can out of the lattice simulations!
- Investigate alternative approaches
 - “Scattering amplitudes from Euclidean correlators: Haag-Ruelle theory and approximation formulae”, [Patella & Tantalo, [2407.02069](#)]
 - HALQCD approach? [Doi et al., [1106.2276](#): 3 nucleon potentials in NR regime]
- Combine LQCD results with EFT and amplitude-analysis methods (Exohad collab.)

Thank you!
Questions?

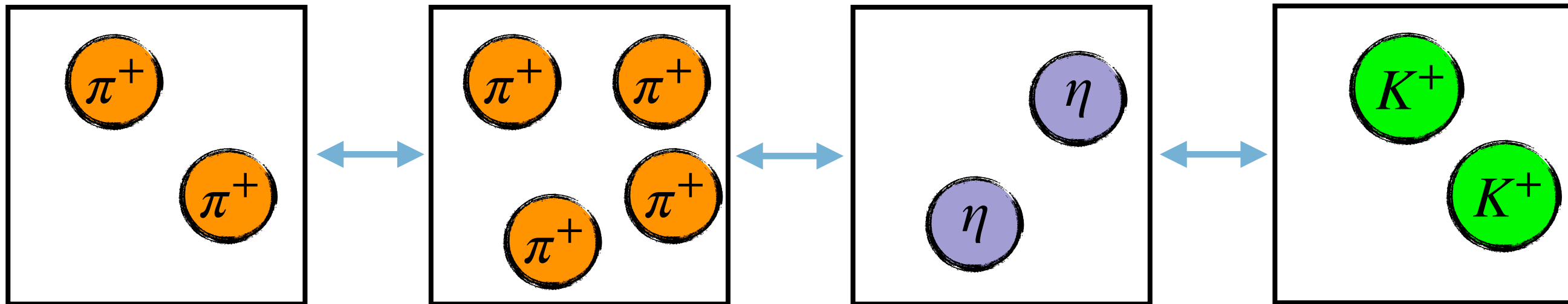
Backup slides

Long-term dream

CP violation in D decays

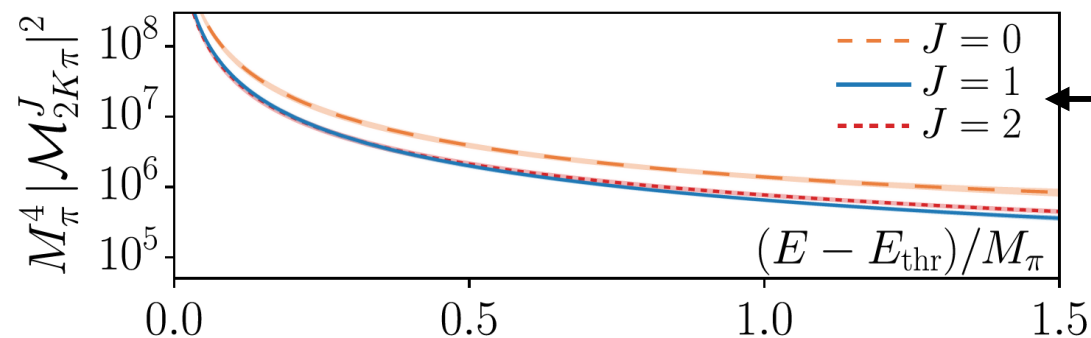


Challenge: finite-volume mixing with 4π , 6π , $\eta\eta$, ...



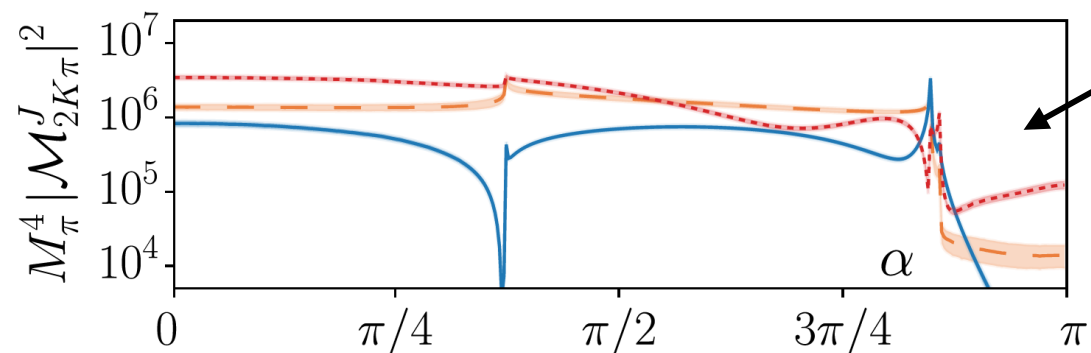
4+-particle formalism not yet developed

Some angular dependence

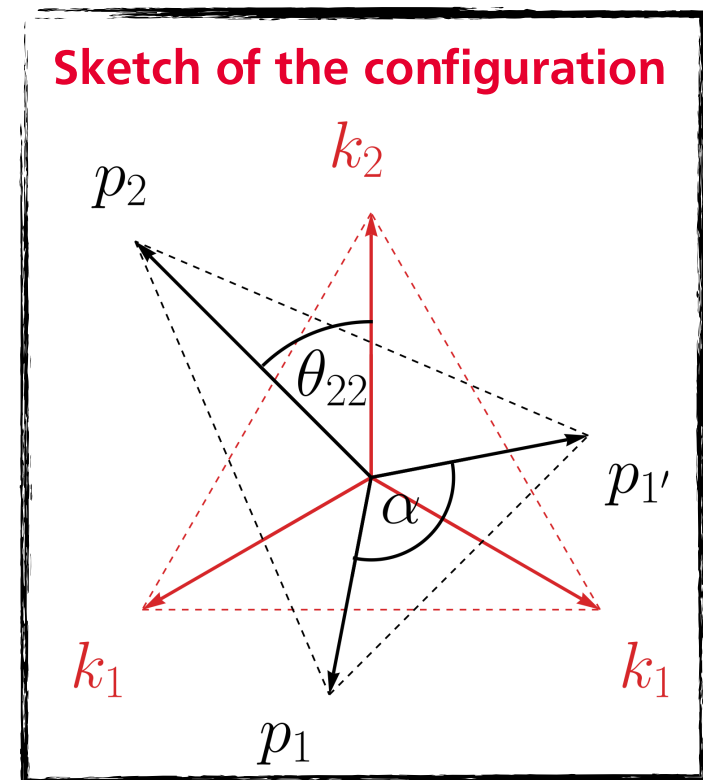
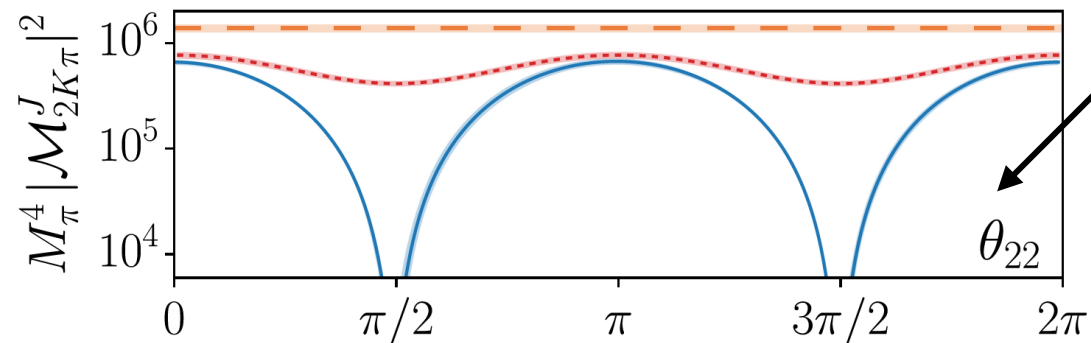


Different partial waves (largest is $J = 0$)

Angle in the triangle:
equilateral ($\alpha = \pi/3$) vs isosceles



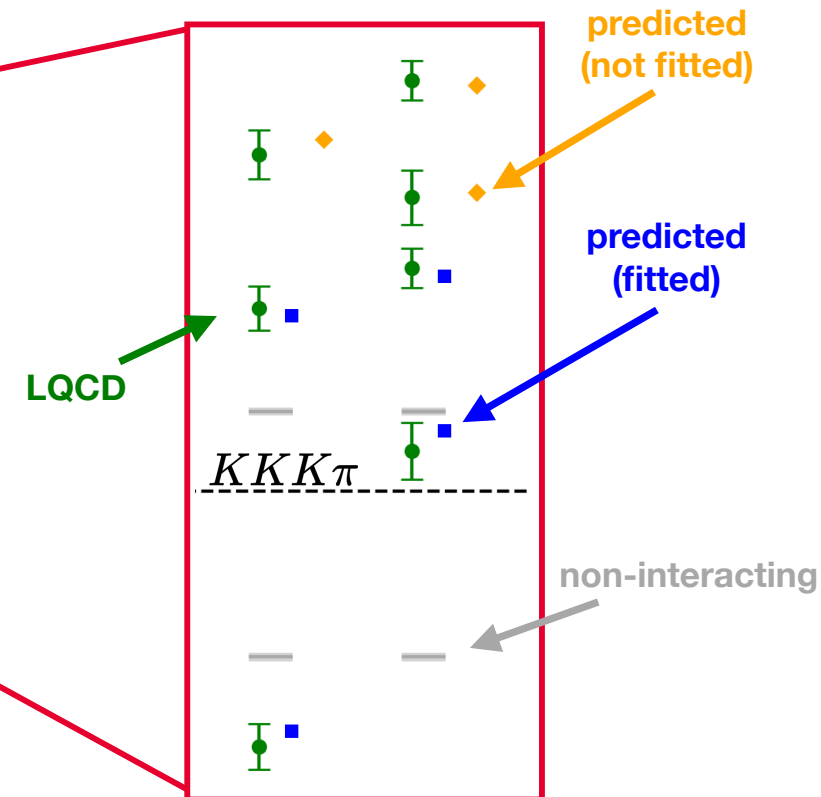
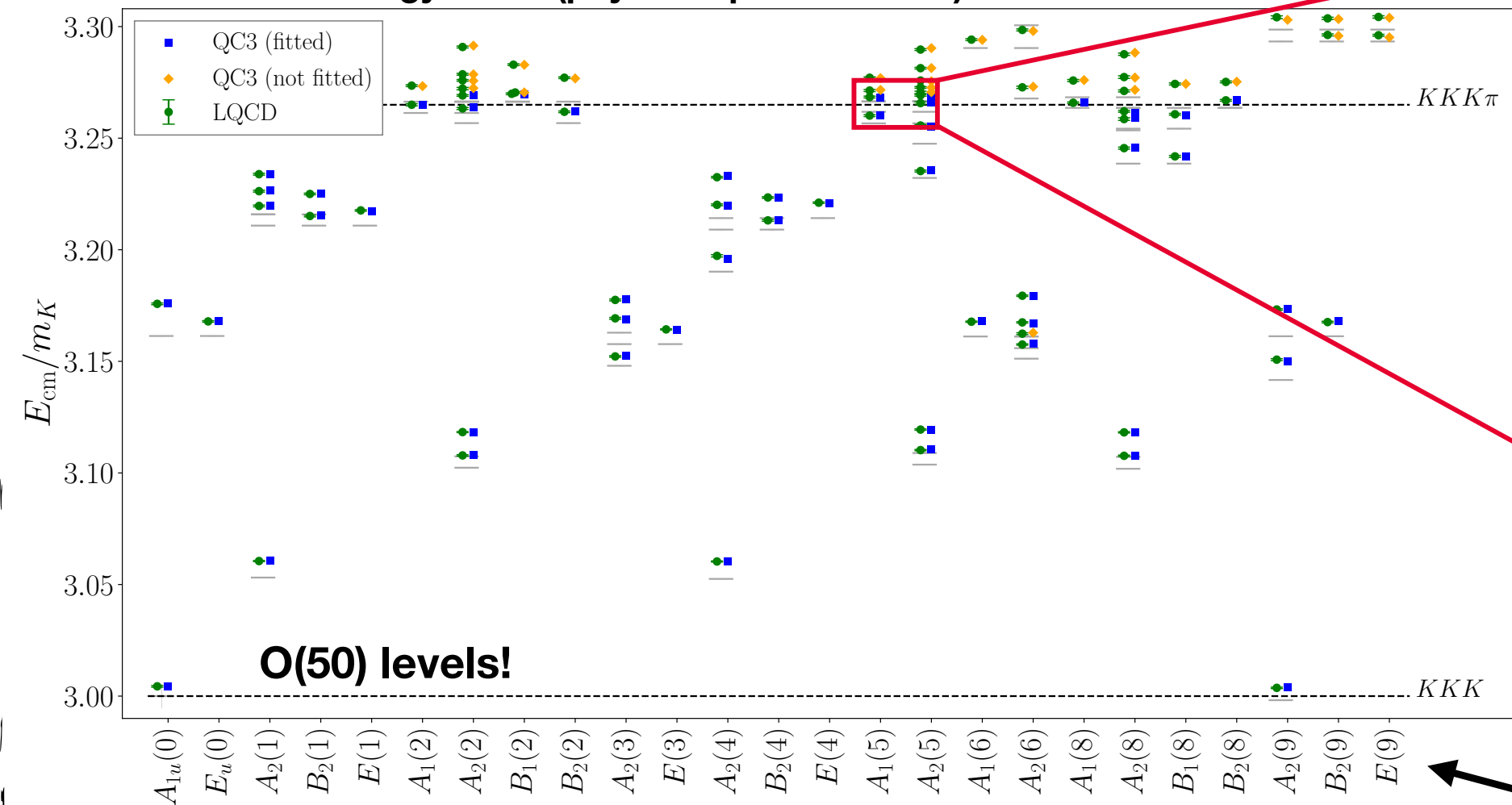
Angle between
in/out momenta



15/17

Three-meson spectrum

Three-kaon energy levels (physical quark masses)



Displayed fit:
 $\chi^2 = 129$, dof = 87

Finite-volume irreps & frame

8 / 17

3-particle resonances

Emergence of the $\pi(1300)$ Resonance from Lattice QCD

Haobo Yan (燕浩波),^{1,2,*} Maxim Mai,^{3,4,†} Marco Garofalo,^{2,‡} Yuchuan Feng,^{4,§} Michael Döring,^{4,¶}
Chuan Liu (刘川),^{1,5,6,**} Liuming Liu (刘柳明),^{7,8,††} Ulf-G. Meißner,^{2,9,10,‡‡} and Carsten Urbach^{2,§§}

[2510.09476, talk by Maxim Mai]

ω Meson from lattice QCD

Haobo Yan (燕浩波),^{1,2,*} Maxim Mai,^{2,3,4,†} Marco Garofalo,^{2,‡}
Ulf-G. Meißner,^{2,5,6,§} Chuan Liu,^{1,7,8,¶} Liuming Liu,^{9,10,**} and Carsten Urbach^{2,††}

[2407.16659, PRL]

Natural candidate for QC3?

$\pi(1300)$	$I^G(J^{PC}) = 1^-(0^{-+})$	[PDGlive]		
$\pi(1300)$ MASS		[1]	1300 ± 100 MeV	▼
$\pi(1300)$ WIDTH		[1]	200 to 600 MeV	▼
$\pi(1300)$ DECAY MODES				
Mode		Fraction (Γ_i / Γ)	Scale Factor/ Conf. Level	P(MeV/c)
Γ_1	$\rho\pi$	seen		404 ▼
Γ_2	$\pi(\pi\pi)_{S\text{-wave}}$	seen		▼
Γ_3	$\gamma\gamma$			650 ▼

[1] Our estimate. See the Particle Listings for details.

Pro: s -wave decay to 3π implies strong 3-particle coupling

Con: Lies well above the inelastic threshold for QC3

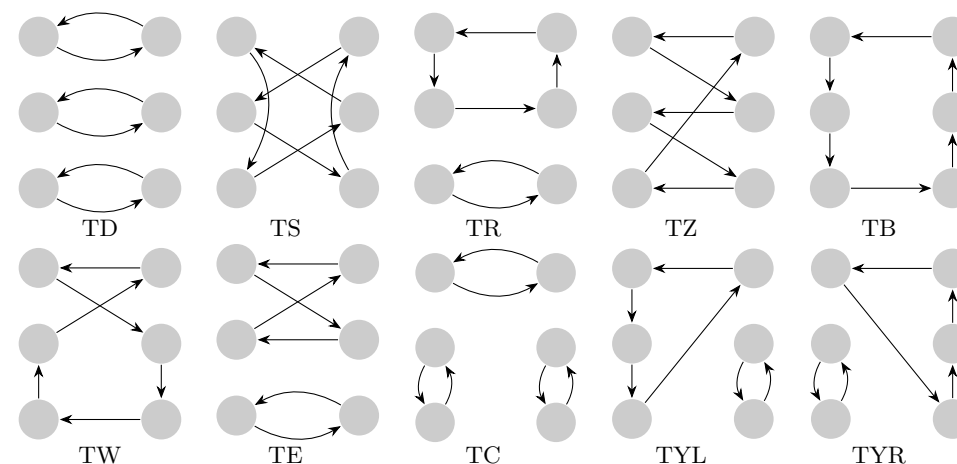
- $M(\pi(1300)) \approx 10M_\pi \gg 5M_\pi$
- QC3 will apply for heavier than physical quark masses

Simulation details

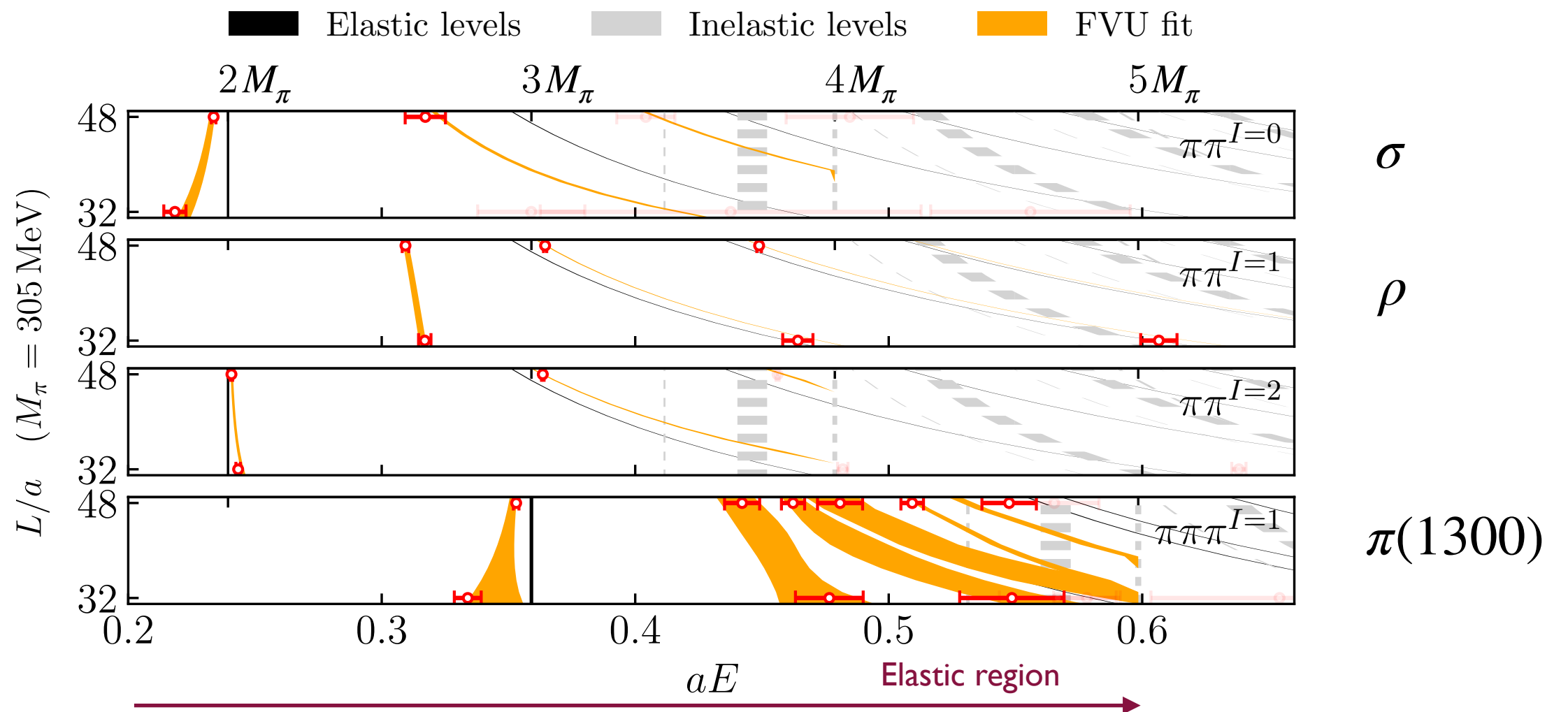
- Use CLQCD ensembles (tadpole improved Clover), fixed $m_s, a = 0.07746(18)\text{fm}$
 - Rest-frame only and single irrep

ens.	size	$M_\pi(\text{MeV})$	$M_K(\text{MeV})$	$M_\pi L$	# 2π levels	# 3π levels
F32P30	$32^3 \times 96$	303	525	3.8	5	3
F48P30	$48^3 \times 96$	303	524	5.7	7	6
F32P21	$32^3 \times 64$	211	492	2.7	5	0
F48P21	$48^3 \times 96$	207	493	3.9	7	3

- Large basis of operators: $\pi, \rho\pi, \sigma\pi, 3\pi$ for three-particle case
 - Large set of contractions, e.g.



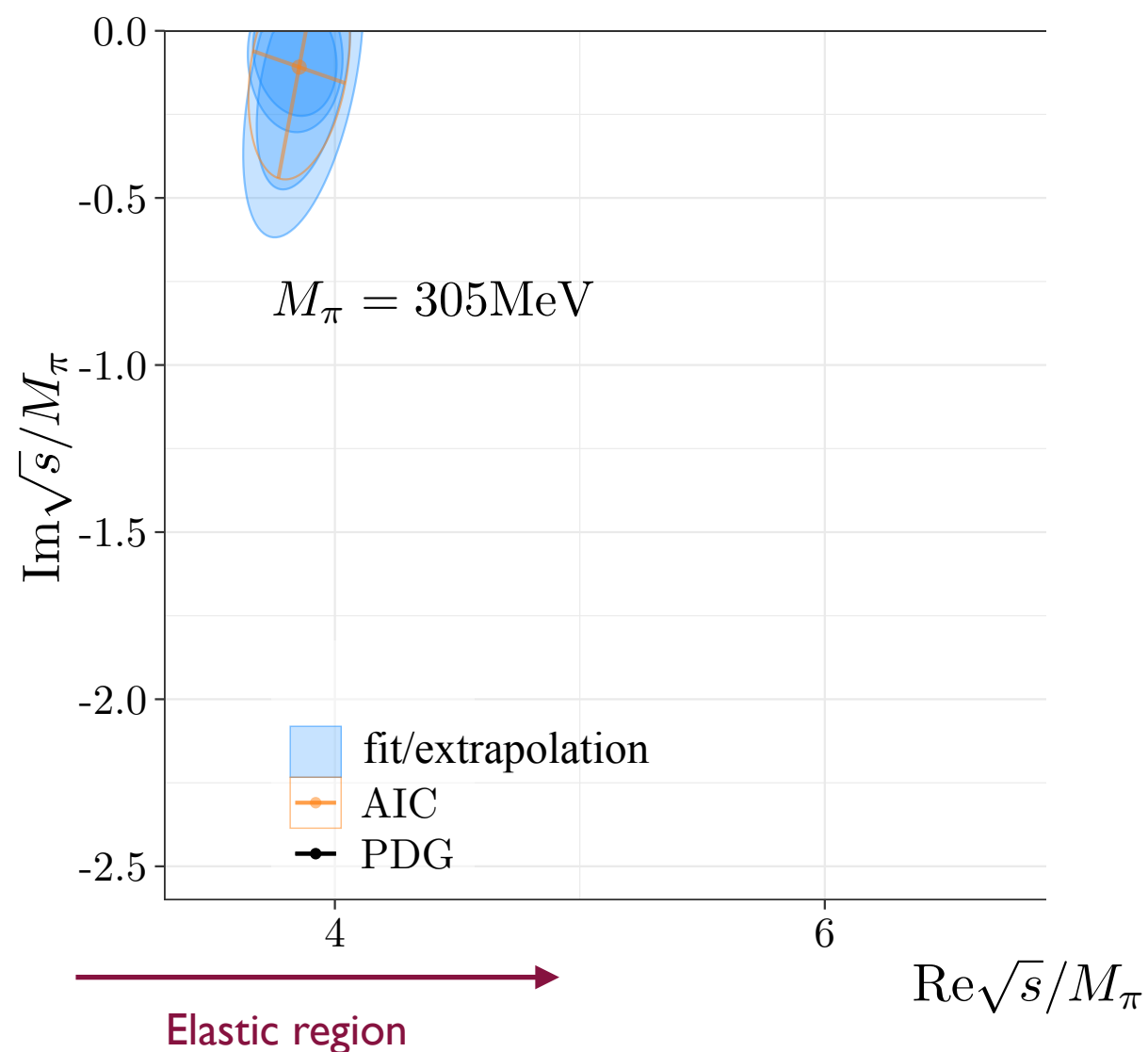
Fits



- Best global fit to both masses and volumes
 - 9 params (3 for 2π , and 6 for 3π) to 36 levels, with $\chi^2/\text{dof} = 1.08$
 - Final results from model averaging with Akaike information criterion

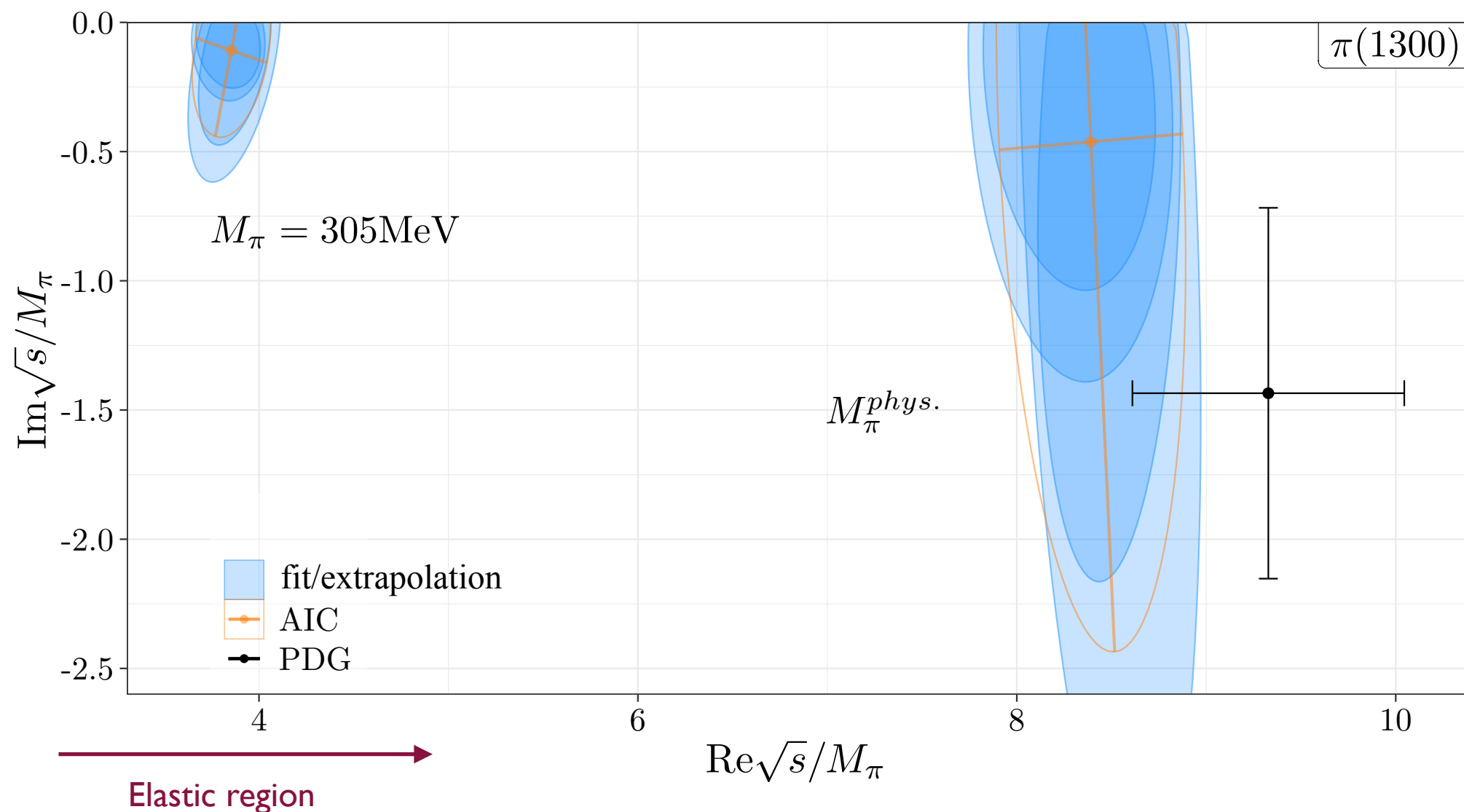
Pole positions

- Solve IVU (=infinite-volume unitarity) form of integral equations
 - Find resonance pole for $M_\pi = 305 \text{ MeV}$!

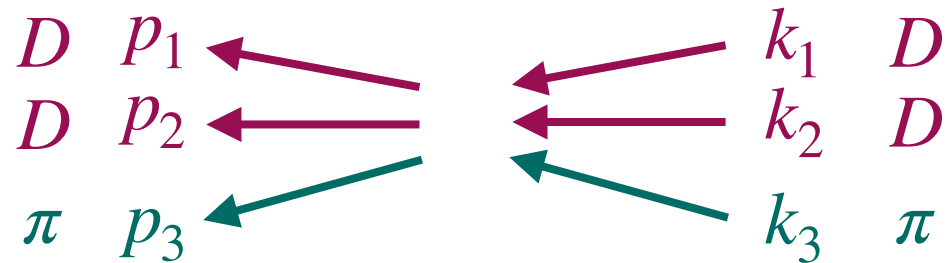


Pole positions

- Solve IVU (=infinite-volume unitarity) form of integral equations
 - Find resonance pole for $M_\pi = 305 \text{ MeV}$!
- Extrapolate to physical masses using EFT-inspired form



$\mathcal{K}_{\text{df},3}$ threshold expansion



$$\Delta = \frac{s - s_{\text{th}}}{s_{\text{th}}}, \quad s_{\text{th}} = (2M_D + M_\pi)^2,$$

$$t_{ij} = \frac{(p_i - k_j)^2 - (M_i - M_j)^2}{s_{\text{th}}}$$

$$\Delta_i = \frac{(k_j + k_k)^2 - (M_j + M_k)^2}{s_{\text{th}}}, \quad \Delta'_i = \frac{(p_j + p_k)^2 - (M_j + M_k)^2}{s_{\text{th}}},$$

For our data: $(-0.01) \lesssim \Delta \lesssim 0.15$

$$\begin{aligned} \mathcal{K}_{\text{df},3} = & c_{\text{iso},0} + \underbrace{c_{\text{iso},1}\Delta + c_B(\Delta_3 + \Delta'_3) + c_E t_{33}}_{\mathcal{O}(\Delta^1): \text{includes s \& p waves}} + \\ & \underbrace{c_{\text{iso},2}\Delta^2 + c'_B\Delta(\Delta_3 + \Delta'_3) + c'_E\Delta t_{33} + c_{E^2}(t_{33})^2 + \sum_{i=5}^{11} c_i \mathcal{O}_i^{(2)} + \dots}_{\mathcal{O}(\Delta^2): \text{includes s, p \& d waves}} \end{aligned}$$

$\mathcal{O}(\Delta^0)$: includes dimer s waves

$\mathcal{O}(\Delta^1)$: includes s & p waves

$\mathcal{O}(\Delta^2)$: includes s, p & d waves

- At $\mathcal{O}(\Delta^1)$, only c_E term couples to DD^* in T_{cc}^+ channel ($T_{1g}(0)$ irrep)
- We fit with all $\mathcal{O}(\Delta^0)$ and $\mathcal{O}(\Delta^1)$ terms + c'_E term