Heavy-light form factors on the lattice

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Motivation

B → K*(892)γ decay

- Flavour Changing Neutral Current (FCNC): Loop - suppressed in Standard Model
- Exp. error ≈ 5%:

\[ B(B^0 \rightarrow K^{*0}\gamma) = \begin{cases} (3.92 \pm 0.20 \pm 0.24) \times 10^{-5} & \text{BaBar '04} \\ (4.01 \pm 0.21 \pm 0.17) \times 10^{-5} & \text{Belle '04} \end{cases} \]

⇒ Need precise (nonperturbative) QCD matrix elements to constrain BSM physics.
Problem in lattice calculations: $am_b > 1$
$
\Rightarrow$ Extrapolation in heavy quark mass $m_h \rightarrow m_b$

[Becirevic, Lubicz, Mescia (2007)].

Tensor form factor at $q^2 = 0$

$$\langle K^*(p'; \varepsilon)|i\bar{s}\sigma^{\mu\nu}b|B(p)\rangle = \varepsilon^*_\alpha \varepsilon^{\alpha\mu\nu\beta} (p_\beta + p'_\beta) T^{B \rightarrow K^*}(q^2 = 0)$$

with $q = p - p'$

Their result:

$$T^{B \rightarrow K^*}(q^2 = 0; \mu = m_b) = 0.24 \pm 0.03\,(ext.)^{+0.04}_{-0.01}\,(sys.)$$

$\Rightarrow > 10\%$ error from quark mass extrapolation.
Effective theories for heavy quarks

Alternative solution to \(a m_b > 1\):

\[\downarrow\]

Integrate out high energy modes

\[\downarrow\]

Effective Lagrangian expansion in \(1/m_b\)

\[
\mathcal{L}_{\text{eff}} = Q^\dagger iD_0 Q - Q^\dagger \frac{D^2}{2m_b} Q - C(\mu)g Q^\dagger \frac{\sigma^{\mu\nu} G_{\mu\nu}}{2m_b} Q + \mathcal{O}(1/m_b^2)
\]

\[Q(x) \sim \int d^4p \ Q(p)e^{ipx} \ \text{with} \ p \sim \Lambda_{\text{QCD}}\]

\[\Rightarrow \ \text{expect discretisation errors} \sim a\Lambda_{\text{QCD}} \ll 1 < a m_b\]
NRQCD on the lattice

Lattice action [Lepage et al. (1992)]

\[
S_{\text{eff}}^{(\text{lat})} = \sum_{\mathbf{x}, t} \left[ Q_t^\dagger Q_t - Q_t^\dagger \left( 1 - \frac{a\delta H}{2} \right)_t \left( 1 - \frac{\tilde{H}_0}{2n} \right)_t \right.
\]

\[
\times \left. \left( U_4^\dagger \right)_{t-a} \left( 1 - \frac{a\tilde{H}_0}{2n} \right)_{t-a} \left( 1 - \frac{a\delta H}{2} \right)_{t-a} Q_{t-a} \right]
\]

\[
H_0 = -\frac{\Delta^{(2)}}{2m_b}, \quad \tilde{H}_0 = H_0 - \frac{a}{4n} H_0^2, \quad \delta H = -C(a)g \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{2m_b}
\]

- Stability parameter \( n = 1, 2, \ldots \)
- Lattice derivative \( \Delta^{(2)} \), chromomagnetic field \( \mathbf{B}_k = -\frac{1}{2} \epsilon_{ijk} G_{ij} \)
- Can be systematically improved to include \( \mathcal{O}(1/m_b^2, a^2, \ldots) \)
Current matching I

**Hamiltonian for** $b \to s$ transition:

- **Full theory in the continuum**
  \[
  \mathcal{H}_{b \to s} = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_n C_n(\mu) Q_n
  \]

- **Wilson coefficients** $C_n(\mu)$ known at NLO

- **Effective Lattice theory**
  \[
  \mathcal{H}^{(\text{lat})}_{b \to s} = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_n d_n(a, \mu) C_n(\mu) Q_n^{(\text{lat})}
  \]

**Matching coefficients** $d_n(a, \mu)$ determined by

\[
\langle s\gamma|\mathcal{H}_{b \to s}|b\rangle \equiv \langle s\gamma|\mathcal{H}^{(\text{lat})}_{b \to s}|b\rangle^{\text{lat}}
\]
Current matching II - full theory

$b \rightarrow s$ current in full theory, e.g. [Greub, Hurth, Wyler (1996)]

\[ \mathcal{H}_{b\rightarrow s} = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left( C_2(\mu) Q_2 + C_7(\mu) Q_7 + C_8(\mu) Q_8 + \ldots \right) \]

Operators:

\[ Q_2 = (\bar{c}_L \gamma^\mu b_L)(\bar{s}_L \gamma_\mu c_L) \]

\[ Q_7 = \frac{e}{16\pi^2} m_b (\bar{s}_L \sigma^{\mu\nu} b_R) F_{\mu\nu} \]

\[ Q_8 = \frac{g}{16\pi^2} m_b (\bar{s}_L T^a \sigma^{\mu\nu} b_R) G_{a\mu\nu} \]
Current matching III - effective theory

\[ b \rightarrow s \text{ current in the effective lattice theory} \text{ (at LO in } 1/m_b) \]

\[ \mathcal{H}_{b \rightarrow s}^{(\text{lat})} = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left( d_2 C_2(\mu) Q_{2}^{(\text{lat})} + d_7 C_7(\mu) Q_{7}^{(\text{lat})} + d_8 C_8(\mu) Q_{8}^{(\text{lat})} + O(a, 1/m_b) \right) \]

with lattice operators

\[ Q_{2,7,8}^{(\text{lat})} = Q_{2,7,8} \bigg|_{b \rightarrow Q} \]

\( Q \): heavy quark field on the lattice
Perturbative matching

Expansion in $\alpha_s = \alpha_s(m_b) \ll 1$

$$d_n = d_n^{(0)} + \alpha_s d_n^{(1)} + \ldots$$

- **Tree level**: $d_7^{(0)} = 1$, $d_{2,8}^{(0)} = 0$
- $O(\alpha_s)$: Mixing matrices

$$\langle s\gamma|Q_i|b\rangle = \left( \delta_{ij} + \alpha_s Z_{ij}^{(1)} + \ldots \right) \langle s\gamma|Q_j|b\rangle_{\text{tree}}$$

Matching coefficient

$$C_7(\mu)d_7^{(1)} = C_2(\mu)\delta Z_{27}^{(1)} + C_7(\mu)\delta Z_{77}^{(1)} + C_8(\mu)\delta Z_{87}^{(1)}$$

$$\delta Z_{ij}(\mu, a) = Z_{ij}^{(1)}|_{\text{full,MS,}\mu} - Z_{ij}^{(1)}|_{\text{lat,}\mu}$$
Mixing matrix evaluation

\[ \delta Z_{87}^{(1)}(\mu, a) = b_s \bigg|_{\text{full, } MS, \mu} - b_s \bigg|_{\text{lat, } a} + \ldots \]

- **Full theory:** Calculation of \( Z_{i7}^{(1)}, i = 2, 7, 8 \) completed \[Greub, Hurth, Wyler (1996)]\.
  - UV & IR regularised in dim. Reg.
- **Lattice calculation:**
  - UV regularised by lattice, IR by gluon mass \( \lambda \)

**IR regulator has to be the same**
\[ \Rightarrow \text{Repeat continuum calculation with gluon mass} \]
Feynman rules on the lattice very complicated

⇒ Numerical integration:
Adaptive Monte-Carlo algorithm VEGAS [Lepage (1978)]

Integrand peaks in some regions of phase space
⇒ separate IR - divergence first:

\[
\int \frac{d^4 k}{(2\pi)^4} I^{(\text{lat})} = \int \frac{d^4 k}{(2\pi)^4} I^{(\text{sub})} + \int \frac{d^4 k}{(2\pi)^4} \left( I^{(\text{lat})} - I^{(\text{sub})} \right)
\]

with suitable subtraction function

\[
I^{(\text{sub})}(k) \approx I^{(\text{lat})}(k) \quad \text{for } k \to 0
\]
To do I

Next steps

1. Repeat calculation in full theory with finite gluon mass $\lambda$
   (Includes 2-loop diagrams at $O(\alpha_s)$)

2. Subtract lattice calculation
   - Improved gluonic action
   - ASQTAD (HISQ?) light quarks

Done:

- matching calculation for the (partially conserved!) axial-vector current $\overline{b}\gamma_\mu\gamma_5 u$ in the static limit
- compared to results in [Dalgic, Shigemitsu, Wingate (2004)]
Further ahead

1. Include $1/m_b$ corrections, improve to $O(a)$
2. Momentum of $K^*$ as large as $\approx m_b/2$, discretisation errors for light quarks in final state significant

⇒ Work in moving frame → mNRQCD

[Dougall, Foley, Davies, Lepage (2005,2006)], see poster by Stefan Meinel at this school.

3. Finally: Compute matrix elements using $\mathcal{H}_{b\to s}^{(\text{lat})}$
Conclusion

- **Heavy quarks on the lattice:**
  Effective theory to avoid high energy modes at $m_b$ which cannot be represented on the lattice.

- **Matching coefficients for heavy-light currents:**
  Perturbative matching.
  - Outlined calculation for $b \rightarrow s$ current.
  - Numerical evaluation of lattice diagrams using VEGAS
  - IR divergences have to be understood

- **Systematic improvement** possible
  $\mathcal{O}(1/m_b)$ corrections, mNRQCD
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- **Heavy quarks on the lattice:**
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  - Perturbative matching.
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Travel grant from The University of Edinburgh Campaign
Enlightenment in the 21st century
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Eike Mueller

Heavy-light form factors on the lattice
Matching of axial $b \rightarrow u$ current

**Continuum calculation:**

Heavy quark momentum: $p = m_b v + p_{\text{res}}$

$$J^{(1)}_{\mu} = \bar{u} \gamma_5 \gamma_\mu b$$
$$J^{(2)}_{\mu} = \bar{u} \gamma_5 v_\mu b$$

Matrix elements

$$\langle u(p') | J^{(1)}_{\mu} | b(p) \rangle = \left[ 1 + \frac{\alpha_s}{3\pi} \left( \frac{3}{2} \log \frac{m_b^2}{\lambda^2} - \frac{11}{4} \right) \right] \langle u | J^{(1)}_{\mu} | b \rangle_{\text{tree}}$$
$$+ \frac{2\alpha_s}{3\pi} \langle u | J^{(2)}_{\mu} | b \rangle_{\text{tree}}$$

- **NO scale dependence** as current is partially conserved.
- **Logarithmic IR divergence**, gluon mass $\lambda$
Lattice calculation

Matrix elements on the lattice

\[ \langle u | J_0^{(1,2)} | b \rangle_{\text{lat}} = \left[ 1 + \alpha_s \left( \xi^{(\text{lat})} + \frac{1}{2} \left( R_q^{(\text{lat})} + R_h^{(\text{lat})} \right) \right) \right] \langle u | J_0^{(1,2)} | b \rangle_{\text{tree}} \]

(Split off UV/IR divergent terms)

\[ = \left[ 1 + \alpha_s \left( \xi^{(\text{reg})} + \frac{1}{2} \left( R_q^{(\text{reg})} + R_h^{(\text{reg})} \right) \right) \right] \langle u | J_0^{(1,2)} | b \rangle_{\text{tree}} \]

\[ \left( \frac{1}{2\pi} \left( \log a^2 + \gamma_E - \log 2 \right) - \frac{1}{2\pi} \log \lambda^2 \right) \]

- \( \xi^{(\text{reg})} \), \( R_q^{(\text{reg})} \) and \( R_h^{(\text{reg})} \): IR and UV finite
- **IR divergence**: same as in full theory → cancel!
- **UV divergence**: → \( \log a^2 m_b^2 \) terms in matching coefficients
Matching coefficient

Subtract to get one loop matching coefficient of $\bar{u}\gamma_5\gamma_{\mu}b$:

$$d^{(1)} = \frac{1}{2\pi} \log a^2 m_b^2 + \frac{1}{2\pi} \left(-\frac{11}{6} + \gamma_E - \log 2\right)$$

$$- \left(\xi^{(\text{reg})}_{1\text{PI}} + \frac{1}{2}(R^{(\text{reg})}_q + R^{(\text{reg})}_h)\right)$$

Numerical results

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<tr>
<th></th>
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<th>[Dalgic et al. ('04)] $^1$</th>
<th>[Morningstar ('93)] $^{1,2}$</th>
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<tr>
<td>$R^{(\text{reg})}_h$</td>
<td>0.3933(5)</td>
<td>0.31(2)</td>
<td>0.404(2)</td>
</tr>
<tr>
<td>$R^{(\text{reg})}_q$</td>
<td>—</td>
<td>$-0.924(3)$</td>
<td>—</td>
</tr>
<tr>
<td>$\xi^{(\text{reg})}_{1\text{PI}}$</td>
<td>0.4797(2)</td>
<td>0.49(1)</td>
<td>—</td>
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$^1$ Extrapolated to $m_b \rightarrow \infty$. $^2$ Gluon action different at $O(a^4)$
Zero point energy

\[ aE_0^{(\text{lat})} \] for finite \( m_b \) and \( m_b \to \infty \)

\[ \begin{array}{|c|c|}
\hline
1/(am_b) & aE_0^{(\text{lat})} \\
\hline
0 & 0.4 \\
0.2 & 0.6 \\
0.4 & 0.8 \\
0.6 & 1.0 \\
0.8 & 1.2 \\
1 & 1.4 \\
\hline
\end{array} \]

[Dalgic et al. ('04)]

my data