Can High Density Effective Theory be used to simulate dense QCD ?

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The Objective

Most of this presentation is based on the series of papers by Hong and Hsu.

- At non-zero quark number chemical potential, μ, the Det$[\not{D}+\gamma_0\mu]$ is in general complex which precludes Lattice QCD studies at finite baryon density
- Conventional wisdom suggests that in the deconfined phase QCD the low energy degrees of freedom should be light fermionic excitations with $|\vec{k}| \simeq p_F \sim \mu$ (e.g. Schaefer Schwenzer (2006)) Is it possible to see it from the bare Lagrangian?
- The idea is to re-write Det $[i\,\gamma_0\,\not{D}+\mu]$ in a physically motivated basis and identify the physically relevant parts of the original determinant.
- Then one has to either check that they are non-negative or argue that neglecting operators responsible for the complexity will not significantly change the theory.
- One does not purport to describe non-analyticities in μ, i.e. to solve the sign problem. This only aims at the LQCD description of the low energy properties of the deconfined phase at high density.

Quark Determinant in a Different Basis

- Let us start by re-writing a Dirac fermion field as

$$\psi(x) = \int_{\hat{v}} e^{i\vec{x}\cdot\hat{v}\mu}(q_v + \chi_v) \qquad \hat{v}\cdot\hat{v} = 1, \tag{1}$$

where $q_v = \mathrm{P}(v)\psi$, $\chi_v = \mathrm{P}(-v)\psi$, where $\mathrm{P}(v) = \frac{1}{2}(1+\gamma_0\vec{\gamma}\cdot\hat{v})$. The $\mathrm{P}(\pm v)$ are projectors for the positive/negative energy components of a Dirac spinor with 3 momentum $\vec{k} : \hat{v} = \vec{k}/|\vec{k}|$. (cf. HQET)

- The quark Lagrangian becomes then

$$\begin{aligned}\mathcal{L}_q &= \psi^\dagger\left(\gamma_0\gamma\cdot(i\partial - A) + \mu\right)\psi = \sum_{\hat{v},\hat{u}} q_v^\dagger\, D_{v,u}\, q_u + \\ &+ \chi_v^\dagger\left(\tilde{D}_{v,u} + 2\mu\right)\chi_u - q_v^\dagger\, A_\perp^{v,u}\chi_u + h.c.,\end{aligned} \tag{2}$$

where

$$\begin{aligned}D_{v,u} &= \delta_{v,u}\left(i\partial_t + i\hat{v}\cdot\vec{\nabla}\right) - \mathrm{P}(v)\gamma_0\,\gamma\cdot A\,\mathrm{P}(u)\,e^{-i\vec{x}\cdot(\hat{v}-\hat{u})\mu} \\ \tilde{D}_{v,u} &= \delta_{v,u}\left(i\partial_t - i\hat{v}\cdot\vec{\nabla}\right) - \mathrm{P}(-v)\gamma_0\,\gamma\cdot A\,\mathrm{P}(-u)\,e^{-i\vec{x}\cdot(\hat{v}-\hat{u})\mu} \\ A_\perp^{v,u} &= \mathrm{P}(v)\gamma_0\,\gamma\cdot A\,\mathrm{P}(-u)\,e^{-i\vec{x}\cdot(\hat{v}-\hat{u})\mu}.\end{aligned} \tag{3}$$

Note that the derivative term in $A_\perp$ takes the form

$$\gamma_\perp\cdot\partial q(\vec{x}\cdot\hat{v}) \equiv 0$$

with $\gamma_\perp^\nu = (0, \gamma^i - v^i\,\vec{\gamma}\cdot\hat{v})$.

- Then the determinant is

$$\begin{aligned}\mathrm{Det}\,[\gamma_0\, i\not{D}+\mu] &= \mathrm{Det}\left(D_{v,u} + A^\dagger_{\perp\,v,w}\left(\tilde{D}_{w,s} + 2\mu\delta_{w,s}\right)^{-1} A_\perp^{s,u}\right) \\ &\times\ \mathrm{Det}\left(\tilde{D}_{v,u} + 2\mu\delta_{v,u}\right).\end{aligned} \tag{4}$$

The two distinct pieces in (4) are the particle-hole and anti-particle determinants. Consider them one at a time.

The Particle-Hole Determinant

- Expand $\mathrm{Det}\left(D_{v,u} + A^\dagger_{\perp\,v,w}\left(\tilde{D}_{w,s} + 2\mu\delta_{w,s}\right)^{-1} A_\perp^{s,u}\right)$ in $\mathrm{J} \equiv \left(A_\perp^\dagger\left(\tilde{D} + 2\mu\right)^{-1} A_\perp\right)^{v,u}$

$$\begin{aligned}&\mathrm{Det}\left(\mathrm{D}_{v,u} + A^\dagger_{\perp\,v,w}\left(\tilde{\mathrm{D}}_{w,s} + 2\mu\delta_{w,s}\right)^{-1} A_\perp^{s,u}\right) = \\ &\mathrm{Det\,D}\ \exp[\mathrm{Tr\,D^{-1}J} - \frac{1}{2}\mathrm{Tr\,D^{-1}JD^{-1}J} + \\ &+ \frac{1}{3}\mathrm{Tr\,D^{-1}JD^{-1}JD^{-1}J} + ...]\end{aligned} \tag{5}$$

The Particle-Hole Determinant (cont.)

- One may, in turn, represent non-local operator J by a series

$$\mathrm{J} = \left(\frac{1}{2\mu}A_\perp^\dagger\left(1 - \frac{\tilde{D}}{2\mu} + \left(\frac{\tilde{D}}{2\mu}\right)^2 - \left(\frac{\tilde{D}}{2\mu}\right)^3 + ...\right)A_\perp\right)^{v,u} \tag{6}$$

Of course, if $1/\mu$ expansions are used, any information about non-analyticity in μ will be lost.

- Det D contains most of the information about low-energy dynamics of fermionic quasiparticle excitations near Fermi surface. In particular, it contains contributions from exchanges of hard gluons with 3 momenta $\sim \mu$ which is crucial for the formation of, e.g., a superconducting ground state.
- It may be shown that in Euclidean space Det D ≥ 0. Using chiral representation of gamma matrices

$$\gamma^\nu = \left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}\right) \equiv \begin{pmatrix}0 & \sigma^\nu\\ \bar{\sigma}^\nu & 0\end{pmatrix}$$

we have $\mathrm{P}_R(v) = \frac{1}{2}(1+\vec{\sigma}\cdot\hat{v})$, $\mathrm{P}_L(v) = \frac{1}{2}(1-\vec{\sigma}\cdot\hat{v})$ and in Euclidean space

$$\begin{aligned}\mathrm{D}^R_{u,v} &= \left(\partial_\tau + i\hat{v}\cdot\vec{\nabla}\right)\delta_{u,v} - \cos^2\frac{\Theta_{u,v}}{2}\left(i\,A_0 + \hat{v}\cdot\vec{A}\right) \\ &\quad \exp[-i\mu(\hat{v}-\hat{u})\cdot\vec{x}] \equiv \mathcal{D}_{u,v} \\ \mathrm{D}^L_{u,v} &= \mathcal{D}_{-u,-v}.\end{aligned} \tag{7}$$

then

$$\mathrm{Det\,D} = \mathrm{Det}\begin{pmatrix}\mathcal{D}_{u,v} & \\ & \mathcal{D}_{-u,-v}\end{pmatrix} \geq 0. \tag{8}$$

- Consider higher order terms. Weak coupling analysis suggests that the "tadpole" $\mathrm{Tr\,D^{-1}J}$ is cutoff sensitive and, so, not suppressed by $1/\mu$ (e.g. gluon Meissner mass in CFL). The "susceptibility" $\mathrm{Tr\,D^{-1}JD^{-1}J}$ and the higher order terms are suppressed by inverse powers of μ.
- Note that Hong and Hsu proposed to keep only the Det D term and mimic effects of the rest by either gluon mass term or Hard Thermal Loop functional
- $\mathrm{Tr\,D^{-1}J} \propto \langle q^\dagger \mathrm{J} q\rangle_A$, so in general the "tadpole" term may be complex since J is not hermitean.
- The crucial issue is whether keeping only Exp$|\mathrm{Tr\,D^{-1}J}|$ will not lead to a qualitatively different ground state.
- Recall that dealing with $|\mathrm{Det}[\not{D}+\gamma_0\mu]|$ instead of $\mathrm{Det}[\not{D}+\gamma_0\mu]$ in the 2 flavor case corresponds to changing baryon number chemical potential into isospin chemical potential case, which does have a qualitatively different phase diagram. Here, however, we only retain parts of fermion determinant coming from low energy excitations about what we expect to be the correct ground state and, so, one may hope that the omitted contribution is not of crucial importance.

The Anti Particle Determinant

Det $\left(i\,\tilde{D}_{v,u} + 2\mu\delta_{v,u}\right)$ is the main piece responsible for the sign problem which also contains most of the equation of state. It is not expected, however, to have significant influence on the low energy properties. One may

- neglect it, i.e. set it to Det $2\mu = const$, as Hong and Hsu did
- try including it into the integrand by taking absolute value etc., again, arguing/hoping that the discarded contributions are unimportant for the the purposes of getting the right ground state.

Summary

- The High Density Effective Theory (Hong 2000) has proven to be a valuable analytical tool in the description of low-energy properties of asymptotically dense quark matter. May the same idea be used to simulate some properties of dense QCD on the lattice?
- Here one only retains parts of the fermion determinant expected to be relevant to the low-energy properties of the matter in the deconfined phase. The "main" piece, the particle-hole determinant, is non-negative. The higher order contributions may introduce complexity but most of them are suppressed by powers of $1/\mu$.
- It needs to be determined if in this setting neglecting contributions responsible for complexity will not lead to qualitatively wrong physical conclusions (i.e. if the sign problem is less severe than in the general case).
- Another issue is whether the approximation may be cast in the form of an EFT expansion, i.e. if it may be systematically improved (without use of coupling expansion).

References

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