Effective Field Theories for lattice QCD: Lecture 3

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Outline of Lectures

1. Overview & Introduction to continuum chiral perturbation theory (ChPT)
2. Illustrative results from ChPT; SU(2) ChPT with heavy strange quark; finite volume effects from ChPT and connection to random matrix theory
3. Including discretization effects in ChPT using Symanzik's effective theory
4. Partially quenched ChPT and applications, including a discussion of whether m_u=0 is meaningful

Outline of lecture 3

- Why it is useful to include discretization errors in ChPT
- How one includes discretization errors in ChPT
  - Focus on Wilson and twisted mass fermions
- Examples of results
  - Impact of discretization errors on observables
  - Phase transitions induced by discretization errors

Additional references for lecture 3

- K. Symanzik [Symanzik's effective theory], Nucl. Phys. B 226 (1983) 187 & 205
- S.R. Sharpe & R. L. Singleton, "Spontaneous flavor & parity breaking with Wilson fermions," Phys. Rev. D58 (1998) 074501 [hep-lat/9804028]
- R. Frezzotti *et al.* [Twisted mass fermions], JHEP 0108 (2001) 058 [hep-lat/0101001]
- R. Frezzotti & G.C. Rossi [Automatic O(a) improvement at maximal twist], JHEP 0408 (2004) 007 [hep-lat/0306014]
- M. Luscher & P. Weisz [Improved gluon actions], Commun. Math. Phys. 97 (1985) 59
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- M .Luscher, S. Sint, R. Sommer & P. Weisz [NP improvement of action], Nucl. Phys. B478 (1996) 365 [hep-lat/9605038]
- S. Sharpe & J. Wu [tmChPT @ NLO], Phys. Rev. D 71 (2005) 074501 [hep-lat/0411021]
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- O. Bar, "Chiral logs in twisted-mass lattice QCD with large isospin breaking," Phys. Rev. 82 (2010) 094505 [arXiv:1008.0784 (hep-lat)]
- S. Aoki [Aoki phase], Phys. Rev. D30 (1984) 2653
- M. Creutz [Aoki-regime phase structure from linear sigma model], hep-ph/9608216
- S. Aoki, O. Bar & S. Sharpe [NP renormalized currents in WChPT], Phys. Rev. D80 (2009) 014506 [arXiv:0905:0804 [hep-lat]]
- L. Scorzato [Phase structure from tmChPT], Eur. Phys. J. C 37 (2004) 445 [hep-lat/0407023]
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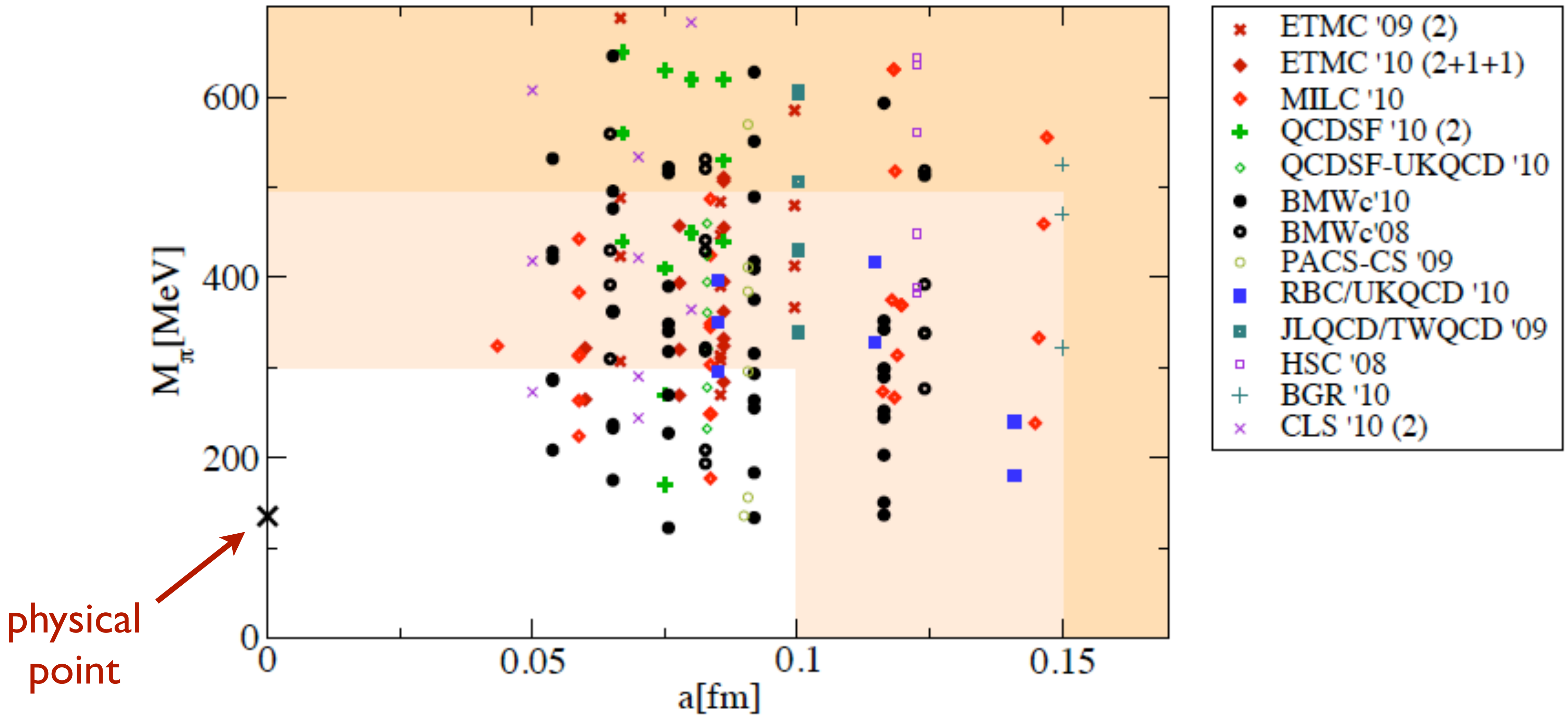
- S. Sharpe and J. Wu [Phase structure from tmChPT], Phys. Rev. D 70 (2004) 094029 [hep-lat/0407025]

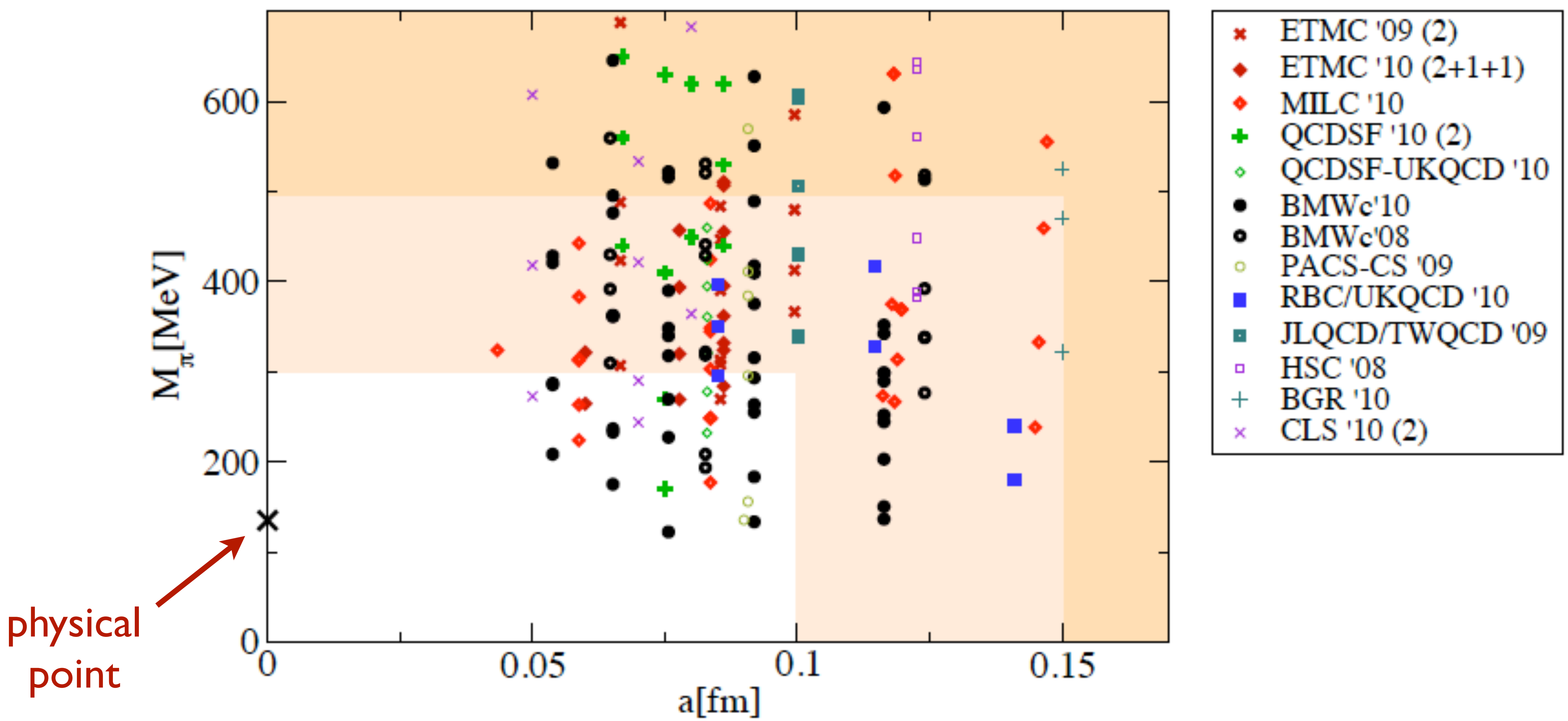
Continuum extrapolation is necessary



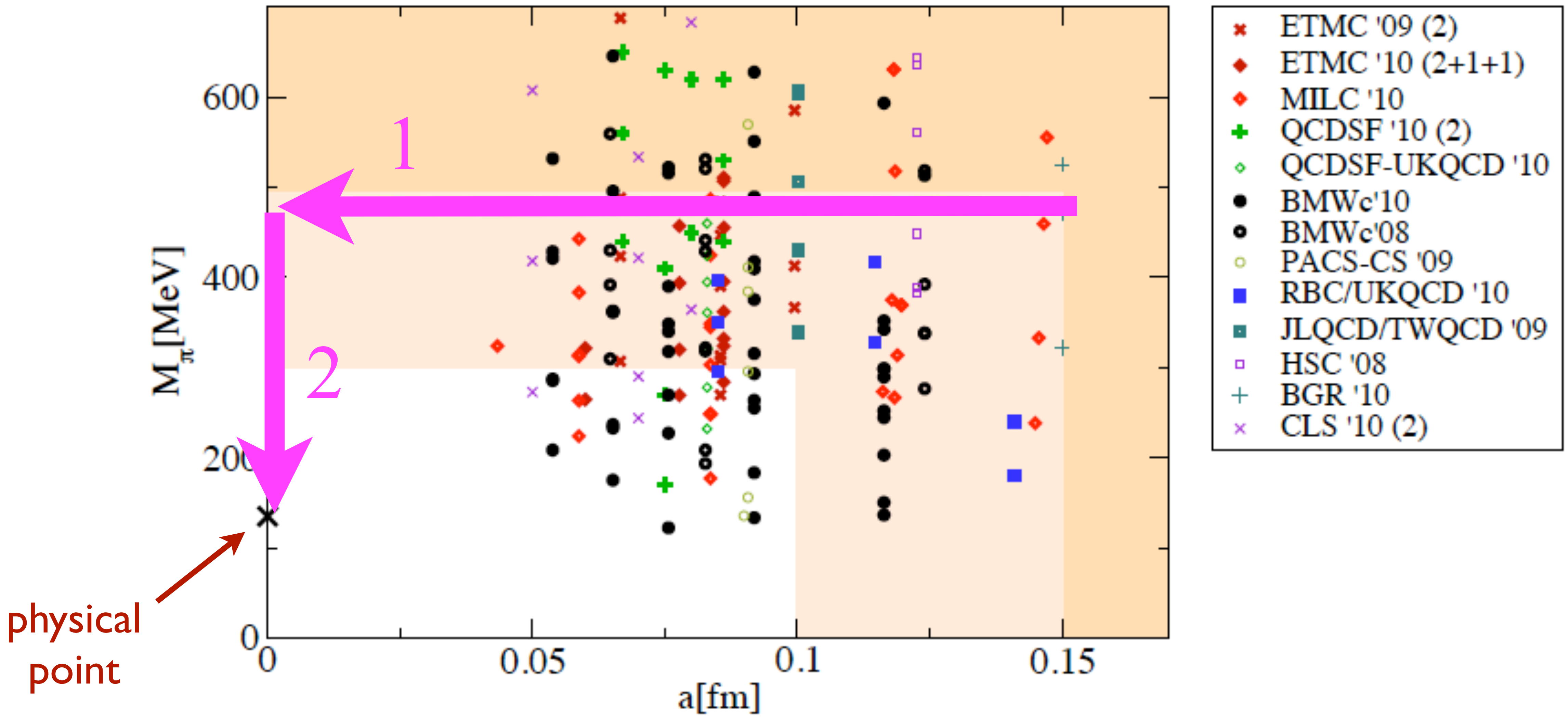
Landscape of recent N_f=2+1 simulations [Fodor & Hoelbling, RMP 2012]

➡ N.B. Leading discretization error is proportional to a^2 with modern actions

Choices of extrapolation



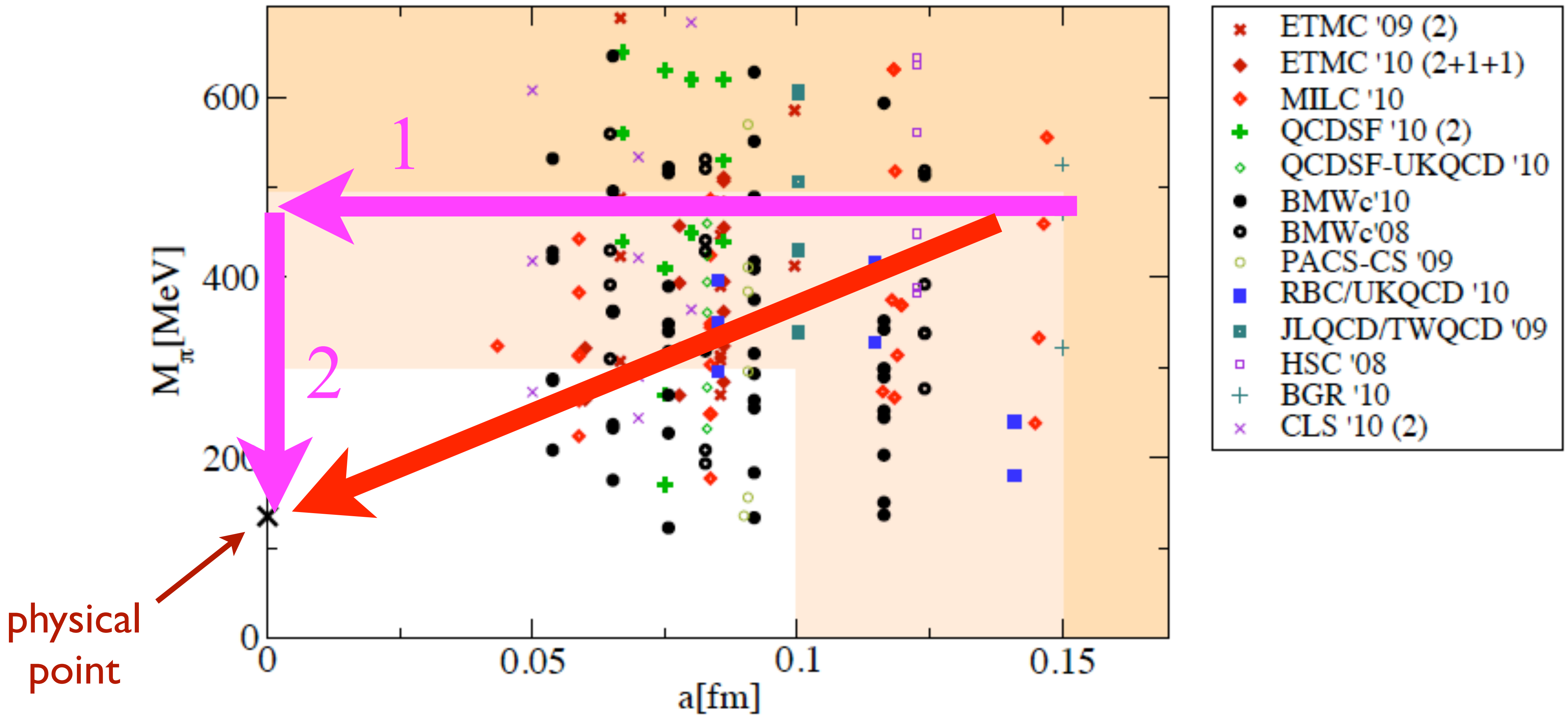
Choices of extrapolation



Two stage extrapolation, e.g.

1. $a \to 0$, using $F(a) = f_0 + a^2 f_2 + a^{3 \text{ or } 4} f_{3 \text{ or } 4} + \ldots$
2. $m \to m_{phys}$ using continuum ChPT

Choices of extrapolation



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2. $m \to m_{phys}$ using continuum ChPT

Simultaneous extrapolation in a & m (most common method)

Advantages of simultaneous extrap

- Allows incorporation of constraints on a dependence of chiral fit params
- Constraints can be determined by extending ChPT to a≠0
  - a dependence in different processes is related by chiral symmetry (limited number of new LECs)
  - Incorporates non-analyticities due to PGB loops, e.g.

$$M_\pi^2 \sim m_q \left[1 + (m_q + a^2)\log(m_q + a^2) + \cdots\right]$$

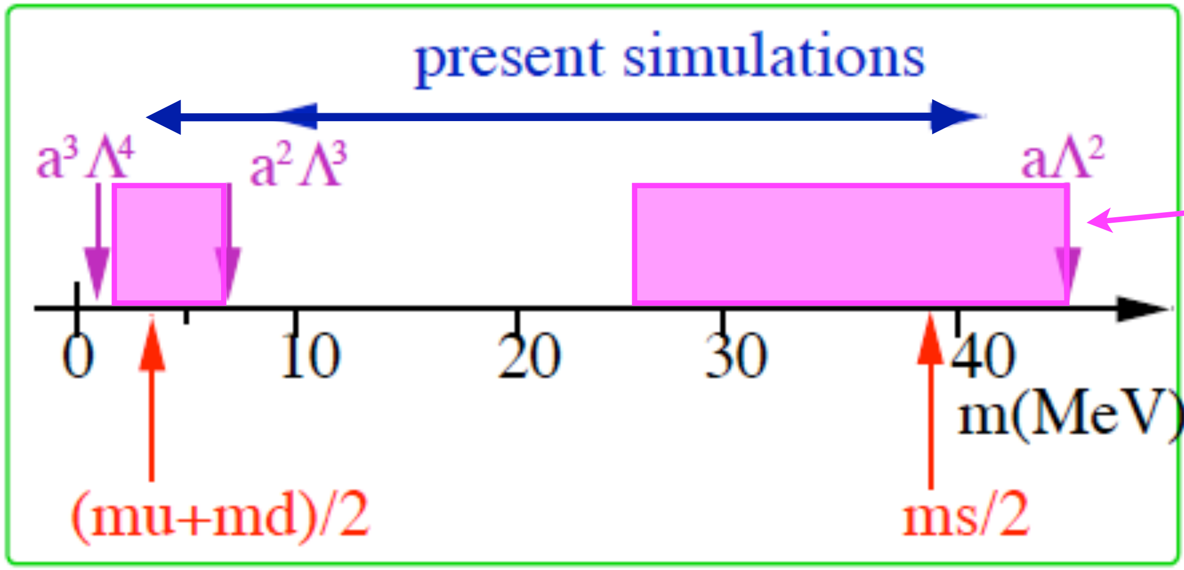
- In practice, used most extensively for overlap/DWF & staggered fermions
  - For exact chiral symmetry, extension of ChPT to a≠0 is almost trivial
  - Highly non-trivial for staggered fermions ⇒ "SChPT"
- Extensive results also available for Wilson and "twisted mass" fermions
  - WChPT and tmChPT (though used less in practice)

Other benefits of ChPT @ a≠0

- Gives detailed understanding of how discretization errors violate continuum symmetries
  - Chiral symmetry breaking with Wilson fermions
  - Chiral & flavor symmetry breaking with twisted-mass fermions
  - Taste symmetry breaking with staggered fermions
- Predicts non-trivial phase structure for $a^2 \Lambda_{QCD}^3 \sim m$
  - E.g. Aoki phase vs. first-order transition for Wilson-like fermions
  - Regions to avoid in numerical simulations
- Predicts discretization errors in eigenvalue distributions in ε-regime
  - Allows simple determination of new LECs introduced by discretization

Extended power counting

- In ChPT we expand in $p^2/\Lambda_\chi^2 \sim M_\pi^2/\Lambda_\chi^2 \sim m/\Lambda_{QCD}$
- Now need to compare to $(a\,\Lambda_{QCD})^n$
  - Equivalently compare m to $a\Lambda_{QCD}^2$, $a^2\Lambda_{QCD}^3$, etc.



Using a=0.05-0.1fm & Λ_{QCD}=300 MeV

- Appropriate power counting is: $a^2\Lambda_{QCD}^3 \lesssim m \lesssim a\Lambda_{QCD}^2$
- Important lessons: O(a) effects must be removed, and O(a^2) understood

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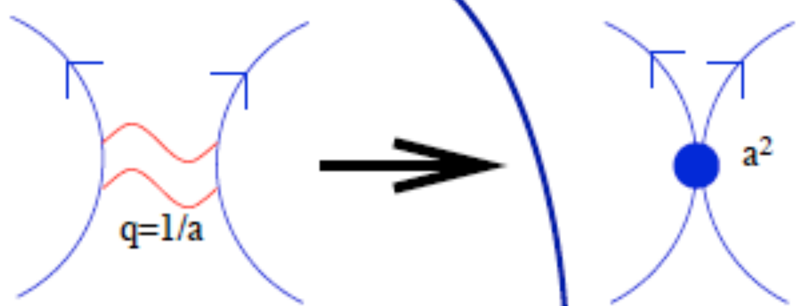
General strategy

Proceed in two steps: [Sharpe & Singleton]

Lattice Lagrangian:
Wilson, tm, staggered



Expansion in ($a\Lambda_{QCD}$) & (am)

Continuum effective Lagrangian:
continuum quark-level theory including explicit nonzero a effects [Symanzik]

Expansion in (m/Λ_{QCD}) & ($a\,\Lambda_{QCD}^2/\Lambda_{QCD}$)

Chiral Lagrangian:
continuum χPT plus effects of additional operators induced by discretization

Apply to "twisted-mass fermions"

- In continuum, twisting the mass means simply QCD with $M \neq M^\dagger$

$$\boxed{\mathcal{L}_{QCD} = \overline{Q}_L \not{D} Q_L + \overline{Q}_R \not{D} Q_R + \overline{Q}_L M Q_R + \overline{Q}_R M^\dagger Q_L}$$

- tmQCD can be obtained from standard QCD with a diagonal mass matrix by an $SU(3)_L \times SU(3)_R$ rotation: $M = U_L M_{\rm diag} U_R^\dagger$
- Physics unchanged by symmetry rotation---expanding about a different point in the vacuum manifold: $\langle \Sigma \rangle = U_L U_R^\dagger$
- Focus on two degenerate flavors, rotated in τ_3 case:

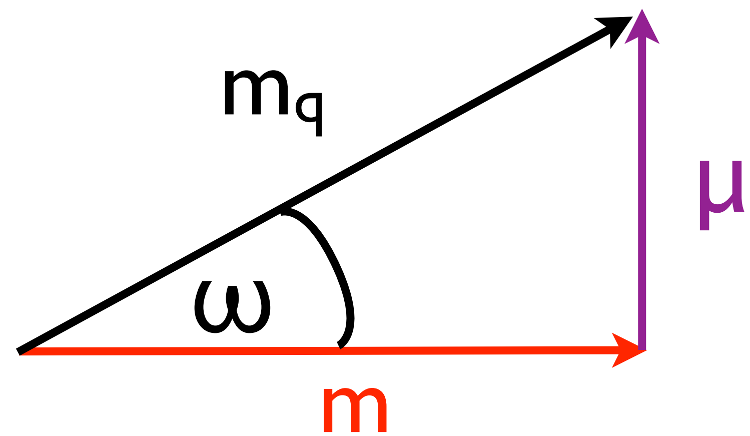
$$M = m_q e^{i\tau_3\omega} \equiv m + i\mu\tau_3 \quad \Rightarrow \quad \underbrace{m = m_q \cos\omega}_{\text{"normal" mass}}, \quad \underbrace{\mu = m_q \sin\omega}_{\text{"twisted" mass}}$$

$$\boxed{\overline{Q}_L M Q_R + \overline{Q}_R M^\dagger Q_L = \overline{Q}(m + i\mu\tau_3\gamma_5)Q}$$

- Apparent breaking of flavor & parity is illusory in continuum

"Geometry" of twisted-mass QCD

$$\overline{Q}_L M Q_R + \overline{Q}_R M^\dagger Q_L = \overline{Q}(m + i\mu\tau_3\gamma_5)Q$$



- ω is redundant in continuum; can use this freedom to pick a better lattice action
  - Maximal twist (ω=±π/2, so that m=0) leads to "automatic improvement", i.e. absence of O(a) terms in physical quantities [Frezzotti & Rossi]

Discretizing twisted-mass QCD

$$S_{\rm tmQCD} = S_{\rm glue} + \int_x \overline{Q} \not{D} Q + \overline{Q}_L M Q_R + \overline{Q}_R M^\dagger Q_L$$

$$\downarrow$$

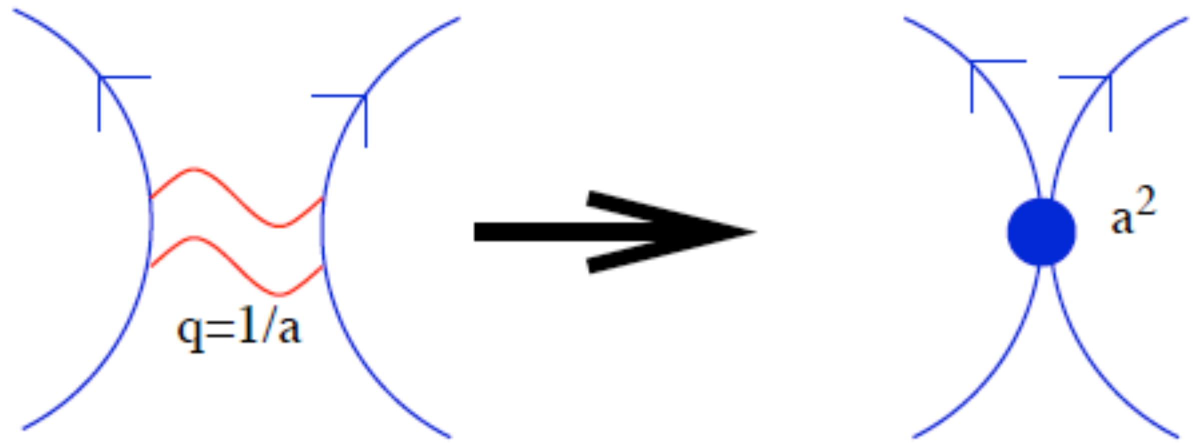
$$S^{\rm lat}_{\rm tmQCD} = S^{\rm lat}_{\rm glue} + a^4 \sum_x \overline{\psi}_l D_W \psi_l + \overline{\psi}_{l,L} M \psi_{l,R} + \overline{\psi}_{l,R} M^\dagger \psi_{l,L}$$

- Uses Wilson's doubler-free derivative:

$$\not{D} \longrightarrow D_W = \frac{1}{2} \sum_\mu \gamma_\mu (\nabla^*_\mu + \nabla_\mu) - \frac{r}{2} \sum_\mu (\nabla^*_\mu \nabla_\mu)$$

- D_W breaks chiral symmetry
  - $\Rightarrow$ M and $U_L M U_R^\dagger$ describe different theories on the lattice
- Full fermion matrix $D_W + M P_R + M^\dagger P_L$ has real positive determinant (and is thus useful in practice) only for special M
  - ▷ e.g. standard twisted mass $M = m + i\mu\tau_3$ for any m, μ

Symanzik EFT ("SET")



- Integrate out high-momentum quarks and gluons ($p \sim 1/a$), obtain a local EFT describing low-momentum modes ($p \ll 1/a$)

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm tmQCD} + a\mathcal{L}^{(5)} + a^2\mathcal{L}^{(6)} + \dots$$

  - Regularize with continuum regulator or finer lattice
  - Factors of a explicit
  - "a" means $\sim a(1 + g[a]^2 \ln a + \dots)$
  - $\mathcal{L}^{(5,6,\dots)}$ contain all operators allowed by *lattice symmetries*
- $\mathcal{L}_{\rm eff}$ gives discretization errors to **all** correlation functions
  - Holds to all orders in PT (where can calculate $\mathcal{L}^{(5,6,\dots)}$) [Symanzik]
  - Demonstrates validity of EFT directly in Euclidean space

Symanzik EFT & improvement

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm tmQCD} + a\mathcal{L}^{(5)} + a^2\mathcal{L}^{(6)} + \ldots$$

- [Symanzik] also showed that can systematically remove $\mathcal{L}^{(5,6,\ldots)}$ by adding corresponding terms to $\mathcal{L}^{\rm lat}$: **IMPROVEMENT**
  - In practice, only $\mathcal{L}^{(5)}$ has been removed
    - **e.g. NP $O(a)$ improved Wilson fermions**
  - Attractive approach—disadvantage for matrix elements is that each operator needs separate $O(a)$ improvement
- We keep both $\mathcal{L}^{(5)}$ and $\mathcal{L}^{(6)}$ because
  - tmLQCD simulations do not always improve the action
    - **Why? Will see that $O(a)$ improvement automatic for $m \approx 0$**
  - Can remove $\mathcal{L}^{(5)}$ by hand to encompass improved Wilson fermions

Symmetries of tm lattice QCD

$$S^{\text{lat}}_{\text{tmQCD}} = S^{\text{lat}}_{\text{glue}} + a^4 \sum_x \overline{\psi}_l \left(D_W + m_0 + i\gamma_5 \tau_3 \mu_0 \right) \psi_l$$

- $\mathcal{L}_{\text{eff}}$ is constrained by the symmetries of tm**L**QCD
- These are the standard symmetries: gauge invariance, lattice rotations and translations, C, fermion number, reflection positivity
- But only a subgroup of flavor $SU(2)$ and parity survive if $\mu_0 \neq 0$:
  - $U(1) \in SU(2)$ with generator τ_3
    - forbids $\bar{\psi}\tau_{1,2}\psi$ terms in $\mathcal{L}_{\text{tmQCD}}$
  - $\mathcal{P}^{1,2}_F$: parity plus discrete flavor rotation
    - $\psi_l(x) \to \gamma_0 (i\tau_{1,2}) \psi_l(x_P)$, $\bar{\psi}_l(x) \to \bar{\psi}_l(x_P)(-i\tau_{1,2})\gamma_0$
    - forbid $\bar{\psi}\gamma_5\psi$, $\widetilde{F}_{\mu\nu}F_{\mu\nu}$, $\bar{\psi}\tau_3\psi$
  - $\widetilde{\mathcal{P}}$: parity combined with $[\mu_0 \to -\mu_0]$
    - requires $\bar{\psi}\tau_3\gamma_5\psi$ to come with factor $\mu_0 \propto \mu$
- Flavor-parity breaking for $a \neq 0$ are price for automatic $O(a)$ improvement

Relating lattice & SET parameters @ LO

LQCD

$$\mathcal{L}^{\rm lat}_{\rm tmQCD} = \mathcal{L}^{\rm lat}_{\rm glue} + \overline{\psi}_l \left(D_W + m_0 + i\gamma_5\tau_3\mu_0 \right) \psi_l$$

SET @ LO

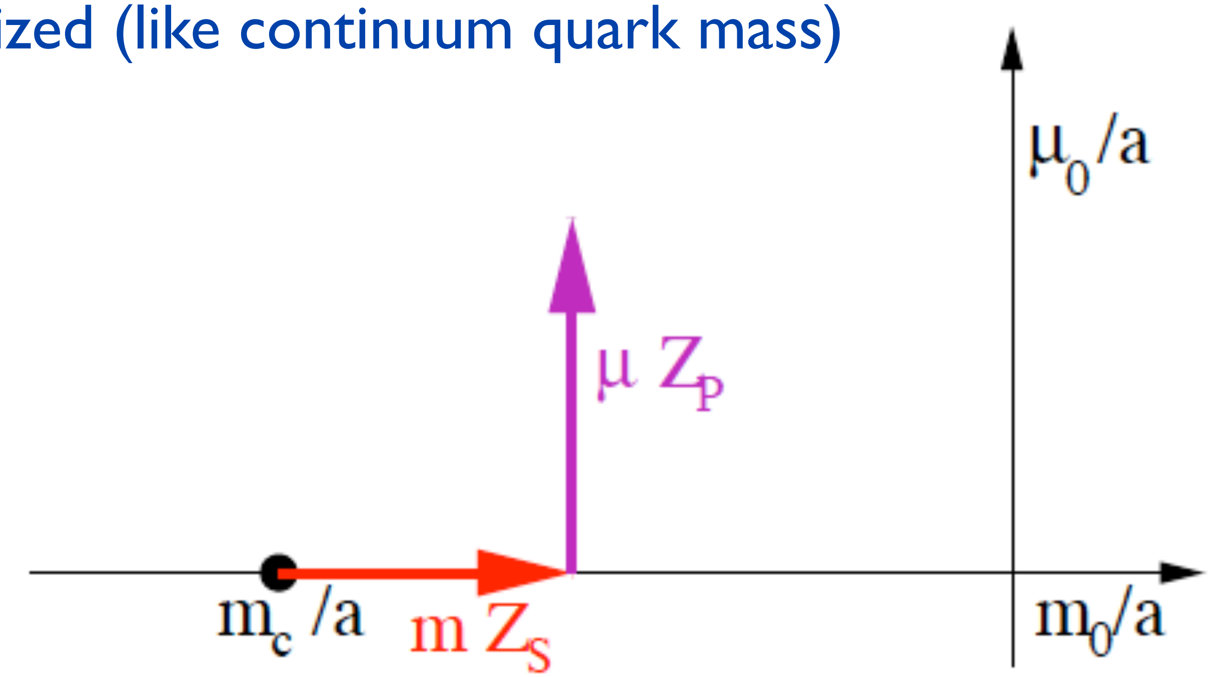
$$\mathcal{L}_{\rm tmQCD} = \mathcal{L}_{\rm glue} + \overline{Q}\not{D}Q + \overline{Q}(m + i\mu\gamma_5\tau_3)Q$$

Dimension 4 terms allowed by lattice symmetries

- Full Euclidean rotation invariance arises as an "accidental symmetry"
- Wilson term $\nabla_\mu^* \nabla_\mu$ mixes with identity operator ⇒ additive renorm. of m_0
- Twisted mass is multiplicatively renormalized (like continuum quark mass)

$$m = Z_S^{-1}(m_0 - m_c)/a$$

$$\mu = Z_P^{-1}\mu_0/a$$



Dimension 5 terms in SET

[Luscher et al]

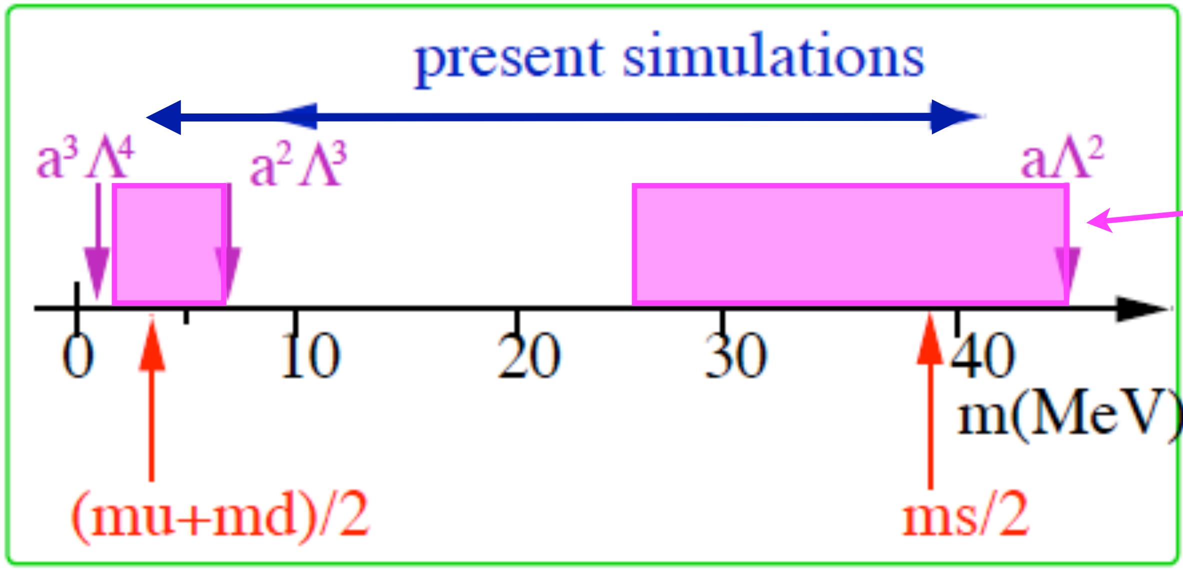
- Straightforward extension of analysis for Wilson fermions [Sharpe & Wu]

$$\begin{aligned}\mathcal{L}^{(5)} &= b_1 \bar\psi i\sigma_{\mu\nu}F_{\mu\nu}\psi + b_2\bar\psi(\not{D}+m+i\gamma_5\tau_3\mu)^2\psi \\ &\quad + b_3 m\bar\psi(\not{D}+m+i\gamma_5\tau_3\mu)\psi + b_4 m\mathcal{L}_{\rm glue} + b_5 m^2\bar\psi\psi \\ &\quad + b_6\mu\bar\psi\{(\not{D}+m+i\gamma_5\tau_3\mu), i\gamma_5\tau_3\}\psi + b_7\mu^2\bar\psi\psi\end{aligned}$$

  - Write in terms of continuum masses m, μ rather than bare masses
  - b_i are real (refl. pos.) and depend on $g^2[a]$ and $\ln a$
  - $b_{6,7}$ are "new" compared to Wilson case (vanish when $\mu \to 0$)
  - Many terms forbidden by lattice symmetries, e.g.
    - $\widetilde{\mathcal{P}}$ forbids: $m\mu\bar\psi\psi$, $m^2\bar\psi i\gamma_5\tau_3\psi$
    - $\widetilde{\mathcal{P}}$ requires twisted Pauli term $\bar\psi\sigma_{\mu\nu}F_{\mu\nu}\tau_3\psi$ to have factor of μ and thus appear in $\mathcal{L}^{(6)}$

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

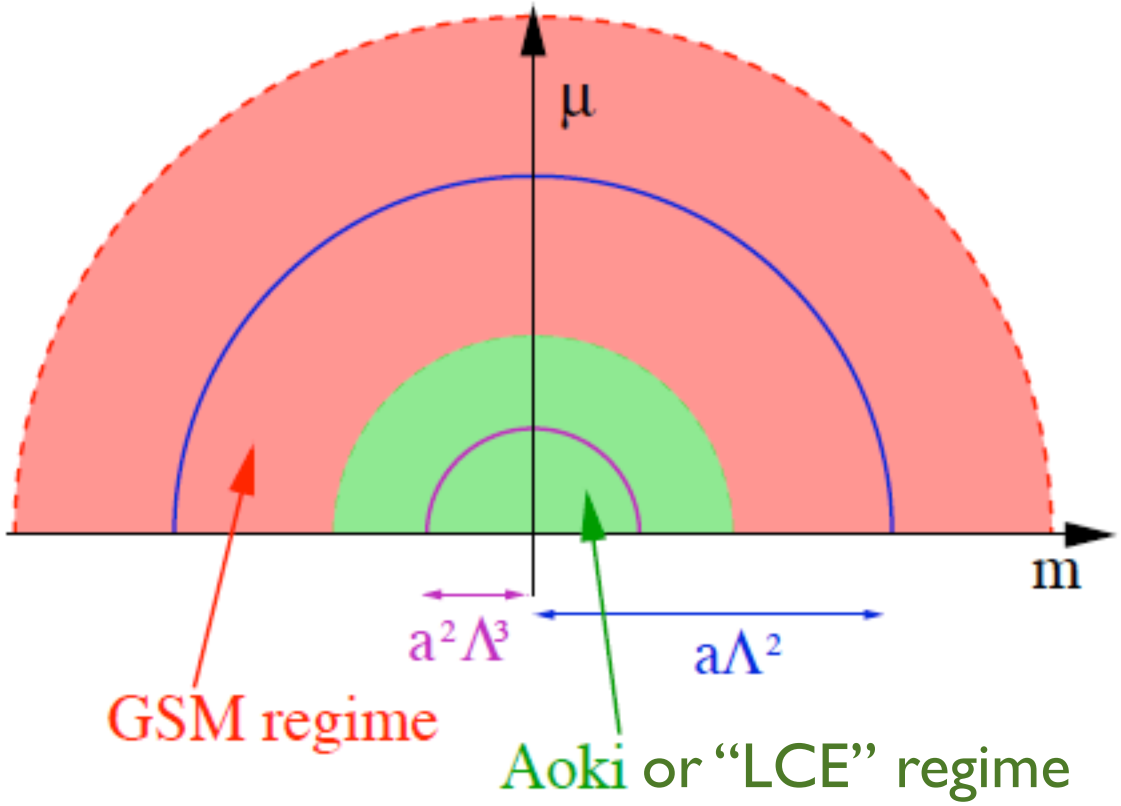
Power-counting redux



Using a=0.05-0.1fm
& Λ_{QCD}=300 MeV

- **Generic Small Mass (GSM) regime**: $a\Lambda_{QCD}^2 \lesssim m_q \ll \Lambda_{QCD}$
  - Includes $a\Lambda_{QCD}^2 \ll m_q$ but *not* $m_q \ll a\Lambda_{QCD}^2$
- **Aoki regime**: $m_q \lesssim a^2\Lambda_{QCD}^3$
  - Includes $m_q \ll a^2\Lambda_{QCD}^3$

- Begin by considering GSM regime



Simplifying dimension 5 terms in SET

- Simplify using change of variables (equivalent to using LO eqns. of mtn.)
  - e.g. $\psi \to [1 + O(a)\not{D} + O(a)m + O(a)i\gamma_5\tau_3\mu]\psi$
  - Convenient but not essential (so don't have to worry about what happens to sources)

$$\begin{aligned}\mathcal{L}^{(5)} \quad = \quad & b_1\bar{\psi}i\sigma_{\mu\nu}F_{\mu\nu}\psi + b_2\bar{\psi}(\not{D} + m + i\gamma_5\tau_3\mu)^2\psi \\ & + b_3 m\bar{\psi}(\not{D} + m + i\gamma_5\tau_3\mu)\psi + b_4 m\mathcal{L}_{\rm glue} + b_5 m^2\bar{\psi}\psi \\ & + b_6\mu\bar{\psi}\{(\not{D} + m + i\gamma_5\tau_3\mu), i\gamma_5\tau_3\}\psi + b_7\mu^2\bar{\psi}\psi\end{aligned}$$

- b_4 leads to am dependence of $g^2_{\rm eff}$ and thus of a
- $b_{5,7}$ imply $m_{\rm phys} = m[1 + O(am)] + O(a\mu^2)$
- These effects are present, but are NNLO if use GSM power counting:

$$m/\Lambda_{\rm QCD} \sim \mu/\Lambda_{\rm QCD} \sim a\Lambda_{\rm QCD}$$

- LO in ChPT is linear in these parameters
  - We will work to quadratic order, i.e. at NLO in GSM regime

Final form of $\mathcal{L}^{(5)}$

- At NLO in our power counting, only need Pauli term

$$\begin{aligned}\mathcal{L}^{(5)}_{\rm NLO} &= b_1\bar\psi i\sigma_{\mu\nu}F_{\mu\nu}\psi + b_2\bar\psi(\not D + m + i\gamma_5\tau_3\mu)^2\psi \\ &+ b_3 m\bar\psi(\not D + m + i\gamma_5\tau_3\mu)\psi + b_4 m\mathcal{L}_{\rm glue} + b_5 m^2\bar\psi\psi \\ &+ b_6\mu\bar\psi\{(\not D + m + i\gamma_5\tau_3\mu), i\gamma_5\tau_3\}\psi + b_7\mu^2\bar\psi\psi\end{aligned}$$

Means "up to NLO" so includes LO

  - Same $\mathcal{L}^{(5)}$ as for Wilson fermions
  - Breaks chiral symmetry even when $m, \mu \to 0$
  - In GSM regime Pauli term contributes at LO in tmChPT as does mass term

Form of $\mathcal{L}^{(6)}$

- Gluonic terms [Lüscher & Wiesz]

$$\mathcal{L}^{(6)}_{\rm glue} \sim \mathrm{Tr}(D_\mu F_{\rho\sigma} D_\mu F_{\rho\sigma}) + \mathrm{Tr}(D_\mu F_{\mu\sigma} D_\rho F_{\rho\sigma}) + \underbrace{\mathrm{Tr}(D_\mu F_{\mu\sigma} D_\mu F_{\mu\sigma})}_{\textbf{Lorentz violating}} + \underbrace{(m^2, \mu^2)\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu})}_{O(a^2m^2, a^2\mu^2)\ \textbf{so NNLO}}$$

- Fermionic terms (generalizing Wilson result [Sheikholeslami & Wohlert])

$$\mathcal{L}^{(6)}_q \sim \underbrace{\bar\psi D^3_\mu \gamma_\mu \psi}_{\textbf{Lorentz violating}} + \underbrace{\bar\psi D_\mu \not{D} D_\mu \gamma_\mu \psi}_{O(a^2)\ \textbf{so NLO}} + \underbrace{(\bar\psi\psi)^2 + (\bar\psi\gamma_\mu\psi)^2 + \dots}_{O(a^2)\ \textbf{so NLO}}$$

$$+ \underbrace{m\bar\psi\not{D}^2\psi + \mu\bar\psi\not{D}^2 i\gamma_5\tau_3\psi}_{O(a^2m, a^2\mu)\ \textbf{so NNLO}} + \underbrace{m\bar\psi i\sigma_{\mu\nu}F_{\mu\nu}\psi + \mu\bar\psi\sigma_{\mu\nu}F_{\mu\nu}\gamma_5\tau_3\psi}_{O(a^2m, a^2\mu)\ \textbf{so NNLO}}$$

$$+ \underbrace{(m^2, \mu^2)\bar\psi\not{D}\psi + m\mu\bar\psi\not{D} i\gamma_5\tau_3\psi}_{O(a^2m^2),\ \textbf{etc. so NNNLO}}$$

$$+ \underbrace{(m^3, m\mu^2)\bar\psi\psi + (\mu^3, \mu m^2) i\gamma_5\tau_3\psi}_{O(a^2m^3),\ \textbf{etc. so NNNNLO}}$$

$\mathcal{L}^{(5)} + \mathcal{L}^{(6)}$ through NLO

- Final NLO result is the same as for Wilson fermions:

$$\mathcal{L}^{(5)}_{\rm NLO} \sim \bar\psi i\sigma_{\mu\nu}F_{\mu\nu}\psi$$

(in fact, of LO)

$$\begin{aligned}\mathcal{L}^{(6)}_{\rm NLO} \sim\ & \mathrm{Tr}(D_\mu F_{\rho\sigma} D_\mu F_{\rho\sigma}) + \mathrm{Tr}(D_\mu F_{\mu\sigma} D_\rho F_{\rho\sigma}) \\ & + \bar\psi D_\mu \not{D} D_\mu \gamma_\mu \psi + \cdots + (\bar\psi\psi)^2 + (\bar\psi\gamma_\mu\psi)^2 + \ldots \\ & + \underbrace{\mathrm{Tr}(D_\mu F_{\mu\sigma} D_\mu F_{\mu\sigma}) + \bar\psi D_\mu^3 \gamma_\mu \psi}_{\textbf{Lorentz violating}}\end{aligned}$$

(really NLO)

  - ▷ No "twisted Pauli term" (since factor of μ makes NNLO)
  - ▷ No flavor or parity breaking in four-fermion terms (requires factors of μ)

⇒ Aside from Lorentz violation, $\mathcal{L}^{(6)}_{\rm NLO}$ breaks no more symmetries than $\mathcal{L}^{(5)}_{\rm NLO}$, i.e. both break chiral symmetry

$\mathcal{L}^{(5)} + \mathcal{L}^{(6)}$ through NLO

- Final NLO result is the same as for Wilson fermions:

$$\mathcal{L}^{(5)}_{\rm NLO} \sim \bar\psi i\sigma_{\mu\nu}F_{\mu\nu}\psi \qquad \text{(in fact, of LO)}$$

$$\begin{aligned}\mathcal{L}^{(6)}_{\rm NLO} \sim\ & {\rm Tr}(D_\mu F_{\rho\sigma}D_\mu F_{\rho\sigma}) + {\rm Tr}(D_\mu F_{\mu\sigma}D_\rho F_{\rho\sigma}) \\ & +\bar\psi D_\mu \not{D} D_\mu\gamma_\mu\psi + \cdots + (\bar\psi\psi)^2 + (\bar\psi\gamma_\mu\psi)^2 + \ldots \qquad \text{(really NLO)} \\ & + \underbrace{{\rm Tr}(D_\mu F_{\mu\sigma}D_\mu F_{\mu\sigma}) + \bar\psi D^3_\mu\gamma_\mu\psi}_{\textbf{Lorentz violating}}\end{aligned}$$

  - No "twisted Pauli term" (since factor of μ makes NNLO)
  - No flavor or parity breaking in four-fermion terms (requires factors of μ)

$\Rightarrow$ Aside from Lorentz violation, $\mathcal{L}^{(6)}_{\rm NLO}$ breaks no more symmetries than $\mathcal{L}^{(5)}_{\rm NLO}$, i.e. both break chiral symmetry

Finally, we are ready for the second step: matching onto ChPT

Matching to ChPT @ LO in GSM regime

[Sharpe & Singleton]

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm glue} + \overline{Q}\not{D}Q + \overline{Q}(m + i\mu\gamma_5\tau_3)Q + ab_1\overline{Q}i\sigma\cdot FQ$$

Matching to ChPT @ LO in GSM regime

[Sharpe & Singleton]

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm glue} + \overline{Q}\not{D}Q + \overline{Q}(m + i\mu\gamma_5\tau_3)Q + ab_1\overline{Q}i\sigma\cdot FQ$$

$$\frac{f^2}{4}\mathrm{tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right)$$

Matching to ChPT @ LO in GSM regime

[Sharpe & Singleton]

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm glue} + \overline{Q}\not{D}Q + \overline{Q}(m + i\mu\gamma_5\tau_3)Q + a b_1 \overline{Q} i\sigma\cdot F Q$$

$$=$$

$$\overline{Q}_L M Q_R + \overline{Q}_R M^\dagger Q_L$$

$$(M = m + i\mu\tau_3)$$

$$\frac{f^2}{4}\mathrm{tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right)$$

Spurion

$M \longrightarrow U_L M U_R^\dagger$

$$-\frac{f^2 B_0}{2}\mathrm{tr}\left(M\Sigma^\dagger + M^\dagger\Sigma\right)$$

Matching to ChPT @ LO in GSM regime

[Sharpe & Singleton]

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm glue} + \overline{Q}\not{D}Q + \overline{Q}(m + i\mu\gamma_5\tau_3)Q + ab_1\overline{Q}i\sigma\cdot FQ$$

$$\overline{Q}(m + i\mu\gamma_5\tau_3)Q = \overline{Q}_L M Q_R + \overline{Q}_R M^\dagger Q_L \quad (M = m + i\mu\tau_3)$$

$$ab_1\overline{Q}i\sigma\cdot FQ = \overline{Q}_L\widetilde{A}i\sigma\cdot FQ_R + \overline{Q}_R\widetilde{A}^\dagger i\sigma\cdot FQ_L \quad (\widetilde{A} = ab_1)$$

$$\mathcal{L}_{\rm glue} + \overline{Q}\not{D}Q \longrightarrow \frac{f^2}{4}\mathrm{tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right)$$

Spurion $M \longrightarrow U_L M U_R^\dagger$

$$\overline{Q}_L M Q_R + \overline{Q}_R M^\dagger Q_L \longrightarrow -\frac{f^2B_0}{2}\mathrm{tr}\left(M\Sigma^\dagger + M^\dagger\Sigma\right)$$

Spurion $\widetilde{A} \longrightarrow U_L\widetilde{A}U_R^\dagger$

$$\overline{Q}_L\widetilde{A}i\sigma\cdot FQ_R + \overline{Q}_R\widetilde{A}^\dagger i\sigma\cdot FQ_L \longrightarrow -\frac{f^2W_0}{2}\mathrm{tr}\left(\widetilde{A}\Sigma^\dagger + \widetilde{A}^\dagger\Sigma\right)$$

Matching to ChPT @ LO in GSM regime

[Sharpe & Singleton]

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm glue} + \overline{Q}\not{D}Q + \overline{Q}(m + i\mu\gamma_5\tau_3)Q + ab_1\overline{Q}i\sigma\cdot FQ$$

$$\overline{Q}(m + i\mu\gamma_5\tau_3)Q = \overline{Q}_L M Q_R + \overline{Q}_R M^\dagger Q_L \quad (M = m + i\mu\tau_3)$$

$$ab_1\overline{Q}i\sigma\cdot FQ = \overline{Q}_L\widetilde{A}i\sigma\cdot FQ_R + \overline{Q}_R\widetilde{A}^\dagger i\sigma\cdot FQ_L \quad (\widetilde{A} = ab_1)$$

$$\frac{f^2}{4}\mathrm{tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right)$$

Spurion $M \longrightarrow U_L M U_R^\dagger$

$$-\frac{f^2B_0}{2}\mathrm{tr}\left(M\Sigma^\dagger + M^\dagger\Sigma\right)$$

Spurion $\widetilde{A} \longrightarrow U_L\widetilde{A}U_R^\dagger$

$$-\frac{f^2W_0}{2}\mathrm{tr}\left(\widetilde{A}\Sigma^\dagger + \widetilde{A}^\dagger\Sigma\right)$$

New LEC related to discretization errors

$$\frac{W_0}{B_0} \sim \frac{\langle\pi|\overline{Q}\sigma\cdot FQ|\pi\rangle}{\langle\pi|\overline{Q}Q|\pi\rangle} \sim \Lambda^2_{\rm QCD}$$

LO tm Chiral Lagrangian

$$\mathcal{L}_\chi^{(2)} = \frac{f^2}{4}\mathrm{tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - \frac{f^2}{4}\mathrm{tr}\left(\chi\Sigma^\dagger + \chi^\dagger\Sigma\right) - \frac{f^2}{4}\mathrm{tr}\left(\hat{A}\Sigma^\dagger + \hat{A}^\dagger\Sigma\right)$$

- We introduced useful parameters:

$$\chi = 2B_0 M = 2B_0(m + i\mu\tau_3)$$

$$\hat{A} = 2W_0\widetilde{A} = 2W_0 a b_1$$

- Power counting in GSM regime now very clear:

$$\partial^2 \sim \chi \sim \hat{A}$$

Matching @ NLO including $\mathcal{L}^{(5)}$

[Sharpe & Singleton; Bar, Rupak & Shoresh]

$$
\begin{aligned}
\mathcal{L}_\chi^{(4)} \quad = \quad & -L_2 \mathrm{tr}(D_\mu \Sigma D_\nu \Sigma^\dagger) \mathrm{tr}(D_\mu \Sigma D_\nu \Sigma^\dagger) + L_{45} \mathrm{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma) \mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) \\
& + L_5 \left\{ \mathrm{tr} \left[(D_\mu \Sigma^\dagger D_\mu \Sigma)(\chi^\dagger \Sigma + \Sigma^\dagger \chi) \right] - \mathrm{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma) \mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi)/2 \right\} \\
& - L_{68} \left[\mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) \right]^2 - L_8 \left\{ \mathrm{tr}[(\chi^\dagger \Sigma + \Sigma^\dagger \chi)^2] - \left[\mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) \right]^2 /2 \right\} \\
& - L_7 \left[\mathrm{tr}(\chi^\dagger \Sigma - \Sigma^\dagger \chi) \right]^2 + i L_{12} \mathrm{tr}(L_{\mu\nu} D_\mu \Sigma D_\nu \Sigma^\dagger + p.c.) + L_{13} \mathrm{tr}(L_{\mu\nu} \Sigma R_{\mu\nu} \Sigma) \\
& + W_{45} \mathrm{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma) \mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) - W_{68} \mathrm{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) \mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) \\
& - W'_{68} \left[\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) \right]^2 + W_{10} \mathrm{tr}(D_\mu \hat{A}^\dagger D_\mu \Sigma + D_\mu \Sigma^\dagger D_\mu \hat{A})
\end{aligned}
$$

- Simplified using SU(2) relations; included sources; dropped HECs
- Four new (dimensionless) LECs @ NLO, but one is redundant
  - Expect, as for continuum LECs, that $W_i \sim 1/(4\pi)^2$

Matching @ NLO including $\mathcal{L}^{(6)}$

$$\begin{aligned}\mathcal{L}^{(6)}_{\rm NLO} \quad \sim \quad & \mathrm{Tr}(D_\mu F_{\rho\sigma} D_\mu F_{\rho\sigma}) + \mathrm{Tr}(D_\mu F_{\mu\sigma} D_\rho F_{\rho\sigma}) \\ & + \bar\psi D_\mu \not{D} D_\mu \gamma_\mu \psi + \cdots + (\bar\psi\psi)^2 + (\bar\psi\gamma_\mu\psi)^2 + \ldots \\ & + \mathrm{Tr}(D_\mu F_{\mu\sigma} D_\mu F_{\mu\sigma}) + \bar\psi D_\mu^3 \gamma_\mu \psi\end{aligned}$$

- Lorentz and chiral invariant terms give multiplicative a^2 corrections, which are of NNLO:

$$a^2\mathrm{Tr}(D_\mu F_{\rho\sigma} D_\mu F_{\rho\sigma}) + \cdots + a^2 \bar\psi D_\mu \not{D} D_\mu \gamma_\mu \psi + \ldots \longrightarrow a^2 \mathrm{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger)$$

- Four-fermion operators violate chiral symmetry, but lead to no new $O(a^2)$ terms in $\mathcal{L}_\chi$ [Sharpe & Singleton; Bar, Rupak & Shoresh]

$$(\bar\psi\psi)^2 + (\bar\psi\gamma_\mu\psi)^2 + \ldots \longrightarrow \mathrm{tr}(\hat{A}^\dagger\Sigma + p.c.)^2$$

- Lorentz violating terms lead to Lorentz violating, chirally symmetric terms:

$$a^2\mathrm{Tr}(D_\mu F_{\mu\sigma} D_\mu F_{\mu\sigma}) + a^2\bar\psi D_\mu^3 \gamma_\mu \psi \longrightarrow a^2 \mathrm{tr}(D_\mu^2 \Sigma D_\mu^2 \Sigma^\dagger)$$

  but these are of NNLO

- **CONCLUSION: $\mathcal{L}^{(6)}_{\rm NLO}$ leads to no new terms at NLO**

What if we NP improve the action?

$$\begin{aligned}\mathcal{L}_\chi \;=\;& \frac{f^2}{4}\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger) - \frac{f^2}{4}\mathrm{tr}(\chi^\dagger\Sigma+\Sigma^\dagger\chi) - \frac{f^2}{4}\mathrm{tr}(\hat{A}^\dagger\Sigma+\Sigma^\dagger\hat{A}) \\ & -L_1\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger)^2 - L_2\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger)\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger) \\ & +L_{45}\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{tr}(\chi^\dagger\Sigma+\Sigma^\dagger\chi) - L_{68}\left[\mathrm{tr}(\chi^\dagger\Sigma+\Sigma^\dagger\chi)\right]^2 \\ & +W_{45}\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{tr}(\hat{A}^\dagger\Sigma+\Sigma^\dagger\hat{A}) - W_{68}\mathrm{tr}(\chi^\dagger\Sigma+\Sigma^\dagger\chi)\mathrm{tr}(\hat{A}^\dagger\Sigma+\Sigma^\dagger\hat{A}) \\ & -W'_{68}\left[\mathrm{tr}(\hat{A}^\dagger\Sigma+\Sigma^\dagger\hat{A})\right]^2 + W_{10}\mathrm{tr}(D_\mu\hat{A}^\dagger D_\mu\Sigma + D_\mu\Sigma^\dagger D_\mu\hat{A})\end{aligned}$$

- Terms linear in A are removed
- Exception: W_{10}, which describes pionic matrix elements of A_μ and V_μ
  - Can set $W_{10} \to 0$ if NP improve axial current (vector current discretization errors are automatically improved)
- Term quadratic in A remains, though the value of W'_{68} will change

Summary so far

- Combining Symanzik's EFT with standard ChPT techniques, and introducing GSM power counting (m~a), we have obtained a relatively simple effective Lagrangian for PGBs @ NLO ($m^2 \sim p^2 m \sim p^4 \sim am \sim ap^2 \sim a^2$)
- Valid throughout the "twisted mass plane" (with m & μ dependence explicit)
- At LO, 2 continuum LECs augmented by 1 new "lattice LEC", but we will shortly see that the latter is unphysical !
- At NLO, 8 continuum LECs augmented by 3 new lattice LECs
- Thus there is hope of using tmChPT to provide constraints on continuum-chiral extrapolations
- Generalization to heavy sources (baryons, B-mesons, etc.) straightforward, and of course introduces new LECs

Outline of lecture 3

- Why it is useful to include discretization errors in ChPT
- How one includes discretization errors in ChPT
  - Focus on Wilson and twisted mass fermions
- Examples of results
  - Impact of discretization errors on observables
  - Phase transitions induced by discretization errors

tmChPT @ LO

$$\mathcal{L}_\chi^{(2)} = \frac{f^2}{4}\mathrm{tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - \frac{f^2}{4}\mathrm{tr}\left(\chi\Sigma^\dagger + \chi^\dagger\Sigma\right) - \frac{f^2}{4}\mathrm{tr}\left(\hat{A}\Sigma^\dagger + \hat{A}^\dagger\Sigma\right)$$

$$\chi = 2B_0(m + i\mu\tau_3)\,, \quad \hat{A} = 2W_0 a b_1$$

- Recall additive renorm. of lattice bare m_0: $m = Z_S^{-1}(m_0 - m_c)/a \quad \mu = Z_P^{-1}\mu_0/a$
- m_c is determined non-perturbatively in simulation (e.g. by where $M_\pi \to 0$ if $a=0$)
- m & a terms have same form, so can combine using: $\chi' = \chi + \hat{A} = 2B_0(m' + i\mu\tau_3)$
- Corresponds to additional additive shift in m: $m \longrightarrow m' = m + ab_1 Z_S W_0/B_0$
- NP determination of m_c (e.g. using $M_\pi \to 0$) automatically includes this shift
- ⇒ W_0 is not measurable
- ⇒ There are no O(a) errors in PGB interactions (for any m & μ)!

tmChPT @ LO

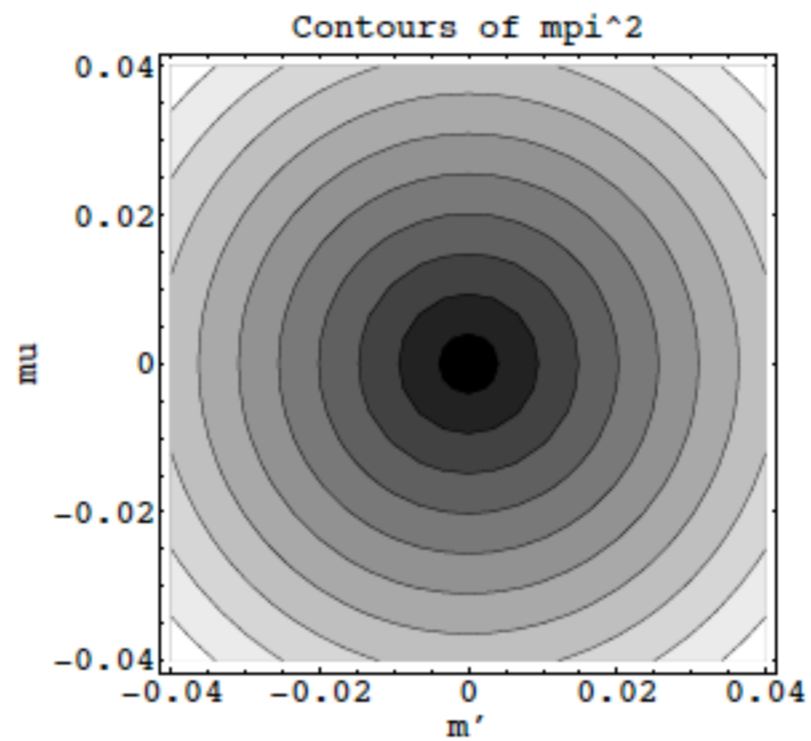
$$\mathcal{L}_\chi^{(2)} = \frac{f^2}{4}\mathrm{tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - \frac{f^2}{4}\mathrm{tr}\left(\chi'\Sigma^\dagger + \chi'^\dagger\Sigma\right)$$

- VEV tracks mass term

$$\left.\begin{array}{l}\chi' = 2B_0(m' + i\mu\tau_3) \equiv |\chi'|e^{i\omega_0\tau_3} \\ |\chi'| = 2B_0\sqrt{m'^2+\mu^2}\,, \quad \tan\omega_0 = \mu/m'\end{array}\right\} \Rightarrow \langle\Sigma\rangle = e^{i\omega_0\tau_3}$$

- Pion mass depends only on $|\chi'|$, with ω_0 redundant

$$M_\pi^2 = |\chi'|$$



tmChPT @ NLO

Rewrite $\mathcal{L}_\chi$ in terms of χ' [Sharpe & Wu]

$$
\boxed{\begin{aligned}
\mathcal{L}_\chi \;=\; & \frac{f^2}{4}\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger) - \frac{f^2}{4}\mathrm{tr}(\chi'^\dagger\Sigma + \Sigma^\dagger\chi') \\
& -L_1\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger)^2 - L_2\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger)\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger) \\
& +L_{45}\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{tr}(\chi'^\dagger\Sigma + \Sigma^\dagger\chi') - L_{68}\left[\mathrm{tr}(\chi'^\dagger\Sigma + \Sigma^\dagger\chi')\right]^2 \\
& +\widetilde{W}\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A}) - W\mathrm{tr}(\chi'^\dagger\Sigma + \Sigma^\dagger\chi')\mathrm{tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A}) \\
& -W'\left[\mathrm{tr}(\hat{A}^\dagger\Sigma + \Sigma^\dagger\hat{A})\right]^2
\end{aligned}}
$$

- Shifted LECs (scale invariant):
  $\widetilde{W} = W_{45} - L_{45}\,, \quad W = W_{68} - 2L_{68}\,, \quad W' = W'_{68} - W_{68} + L_{68}$
- W, W' cause small misalignment of vacuum with χ'
- Skip details, and give examples of results

Charged pion mass @ NLO in tmChPT

$$\boxed{\begin{aligned} m^2_{\pi_\pm} &= |\chi'| + \textbf{cont. 1-loop chiral logs} \\ &\quad + \frac{16}{f^2}\left[|\chi'|^2(2L_{68} - L_{45}) + |\chi'|\hat{a}\cos\omega_0(2W - \widetilde{W}) + 2\hat{a}^2(\cos\omega_0)^2 W'\right] \end{aligned}}$$

Same as $\hat{A}$

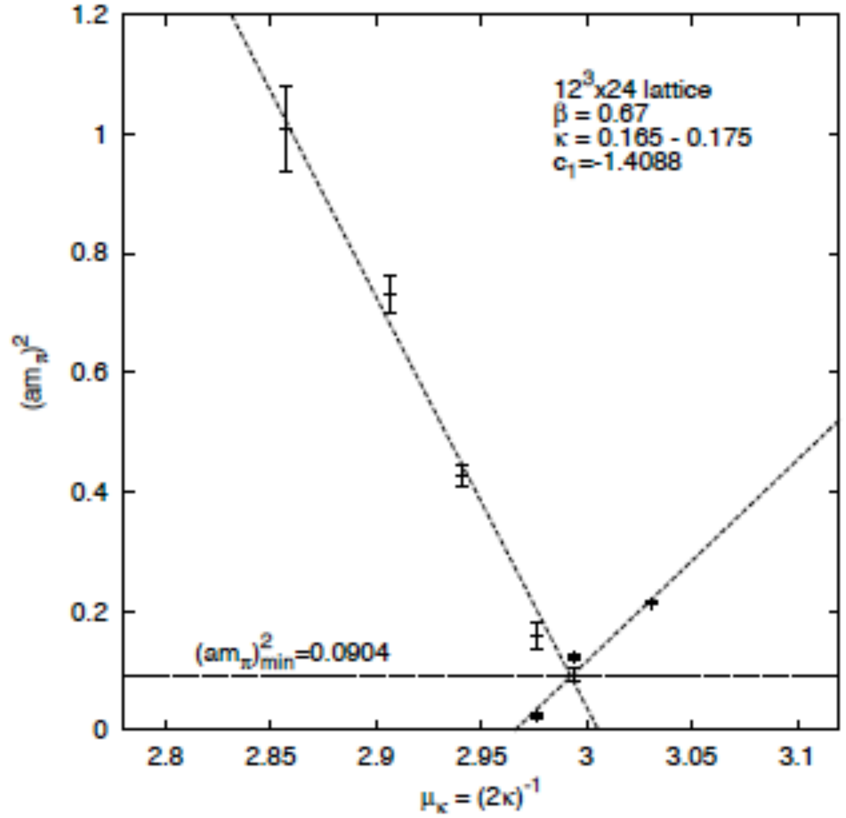
[Scorzato; Sharpe & Wu]

- Compared to lecture 2, this is for SU(2) (not SU(3)) and with twisted mass
- M_π now depends on ω_0 and on a
- Linear dependence on a removed by setting $\omega_0 = \pm\pi/2 + O(a)$
  - Automatic O(a) improvement at maximal twist [Frezzotti & Rossi]
- In this case, $O(a^2)$ term also vanishes at maximal twist, but not true in general

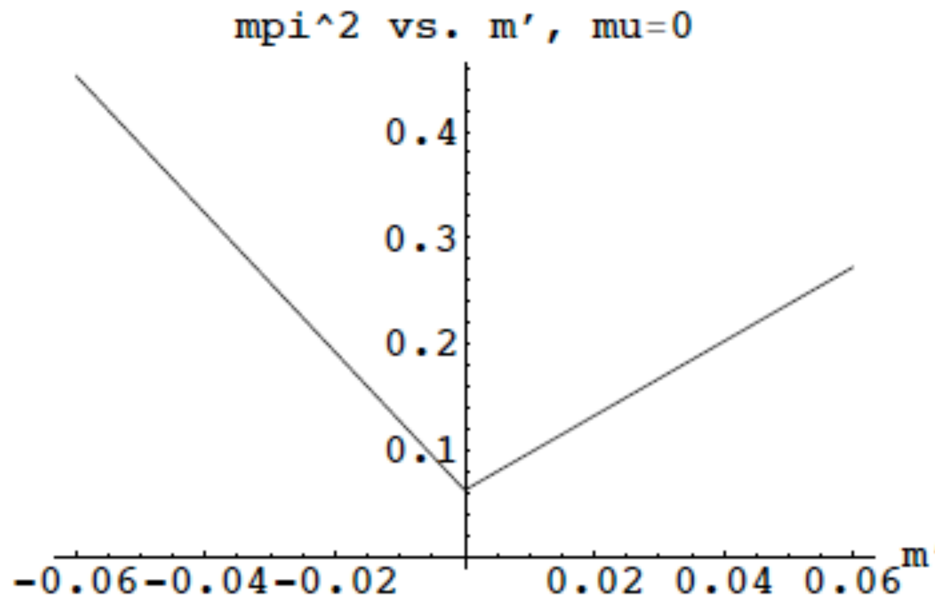
Charged pion mass @ NLO in tmChPT

$$m^2_{\pi_\pm} = |\chi'| + \textbf{cont. 1-loop chiral logs}$$
$$+\frac{16}{f^2}\left[|\chi'|^2(2L_{68}-L_{45}) + |\chi'|\hat{a}\cos\omega_0(2W-\widetilde{W}) + 2\hat{a}^2(\cos\omega_0)^2 W'\right]$$

Results with no twist (ω_0=0 or π)



[Farchioni *et al*,hep-lat/0410031]

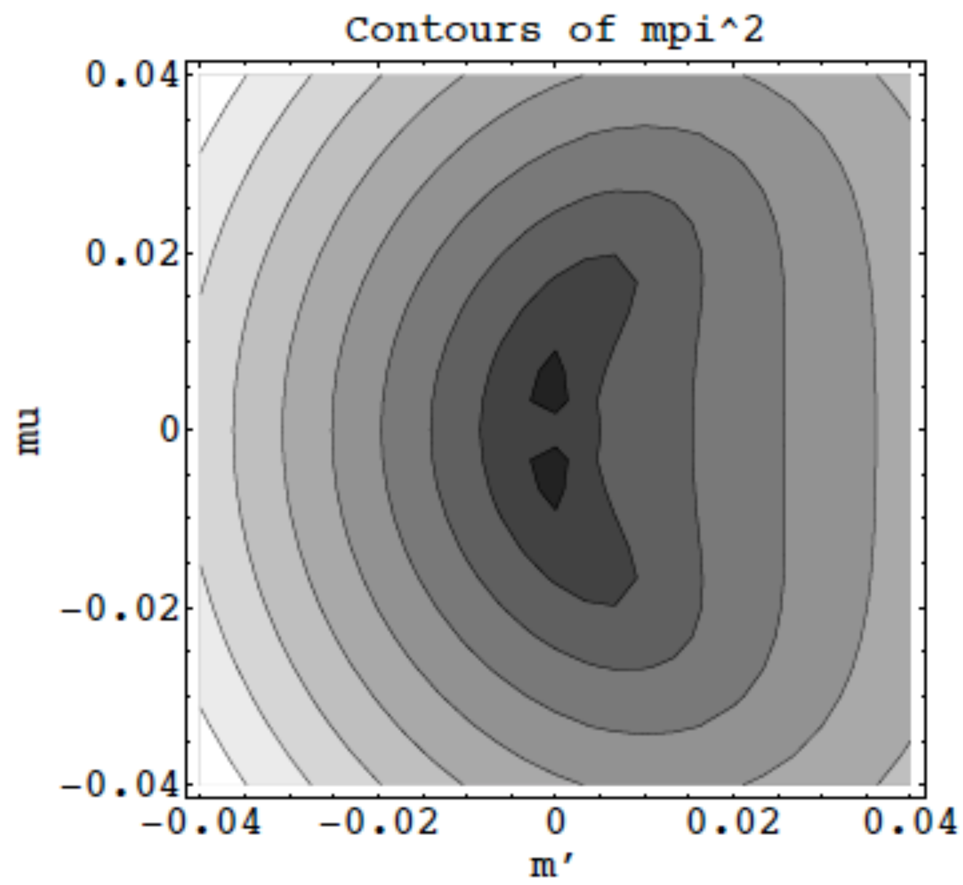


tmChPT with no chiral logs

- Clear antisymmetry of $\approx 30\% \sim a\Lambda^2$ with $\Lambda \approx 300$ MeV
- Non-vanishing minimum pion mass due to W'

ω_0 no longer redundant

$$m^2_{\pi\pm} = |\chi'| + \textbf{cont. 1-loop chiral logs}$$
$$+\frac{16}{f^2}\left[|\chi'|^2(2L_{68}-L_{45}) + |\chi'|\hat{a}\cos\omega_0(2W-\widetilde{W}) + 2\hat{a}^2(\cos\omega_0)^2 W'\right]$$



- LECs chosen to roughly fit data of [Farchioni04]

Isospin breaking @ NLO in tmChPT

$$m^2_{\pi^0} - m^2_{\pi^\pm} = -\frac{32W'\hat{a}^2}{f^2}(\sin\omega_0)^2 + O(a^3)$$
$$= -\frac{32W'\hat{a}^2}{f^2}\frac{\mu^2}{m'^2+\mu^2} + O(a^3)$$

- Splitting is $O(a^2)$ throughout twisted-mass plane, though maximal at maximum twist
  - Splitting vanishes for $\mu=0$ as expected since isospin then a good symmetry
- To calculate $M_{\pi 0}$ numerically, must include quark disconnected contractions
  - ETMC simulations find $m_{\pi 0} < m_{\pi\pm}$ (so $W'>0$) [e.g. Herdoiza *et al.*, arXiv:1303.3516]
- Numerical values imply that we are on the border of the "Aoki" or LCE regime

$$\frac{M^2_{\pi_0} - M^2_{\pi_\pm}}{M^2_{\pi_\pm}} \sim \frac{a^2\Lambda^4_{\rm QCD}}{m\Lambda_{\rm QCD}} \approx -0.32$$

a=0.08 fm, $M_{\pi+}$=330 MeV

Practical utility of tm/Wilson ChPT?

- For (untwisted) Wilson fermions, simulations are O(a) improved and WChPT calculations have not been done to requisite order to control a^2 errors (NNLO in GSM regime)
  - Potential relations between discretization errors not being used (but lots of new LECs, so not clear how useful these relations would be in practice)
- Same holds for tm fermions at maximal twist (automatically O(a) improved)
- For tm fermions, large isospin splitting suggests using Aoki counting $m \sim a^2$
  - Same power-counting as for staggered fermions, where it is found that including taste-splittings in the chiral logs is essential for obtaining good fits
  - [Bar] has done this for maximal twist, and finds significant effects, e.g.

$$M_{\pi_\pm}^2 = 2B_0\mu\left[1 + \frac{M_{\pi_0}^2}{2\Lambda_\chi^2}\log(M_{\pi_0}/\Lambda_3) + O(\mu) + O(a^2)\right]$$

Enhanced chiral log and FV effects

- [Frezzotti, Rossi & ETMC] argue that large a^2 effects are restricted to pion splitting, but this is hard to understand from tmChPT

No time for...

- Extensions to higher order using different power counting [Aoki, Bar, *et al.*]
- Understanding automatic O(a) improvement at maximal twist using tmChPT
- Subtleties in obtaining prediction for quantities requiring NP renormalization (e.g. vector and axial current matrix elements) [Aoki, Bar & Sharpe]
- tmChPT results for baryons, operator matrix elements,...
- Predictions for parity non-invariant quantities that are NOT automatically O(a) improved [Sharpe & Wu]
- Methods for determining maximal twist non-perturbatively (a subject now well understood)
- ...

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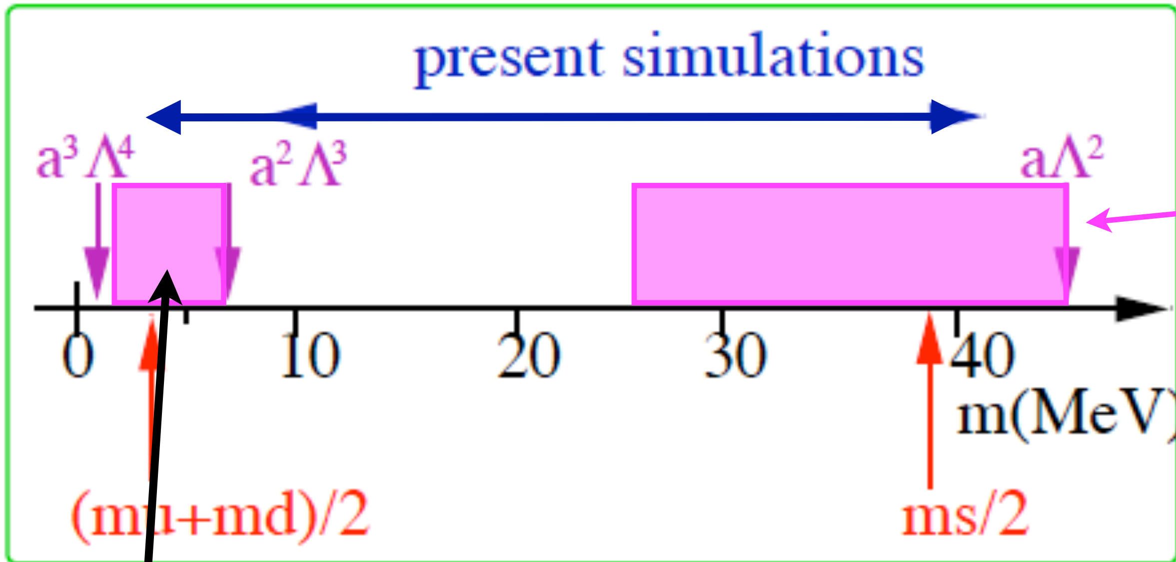
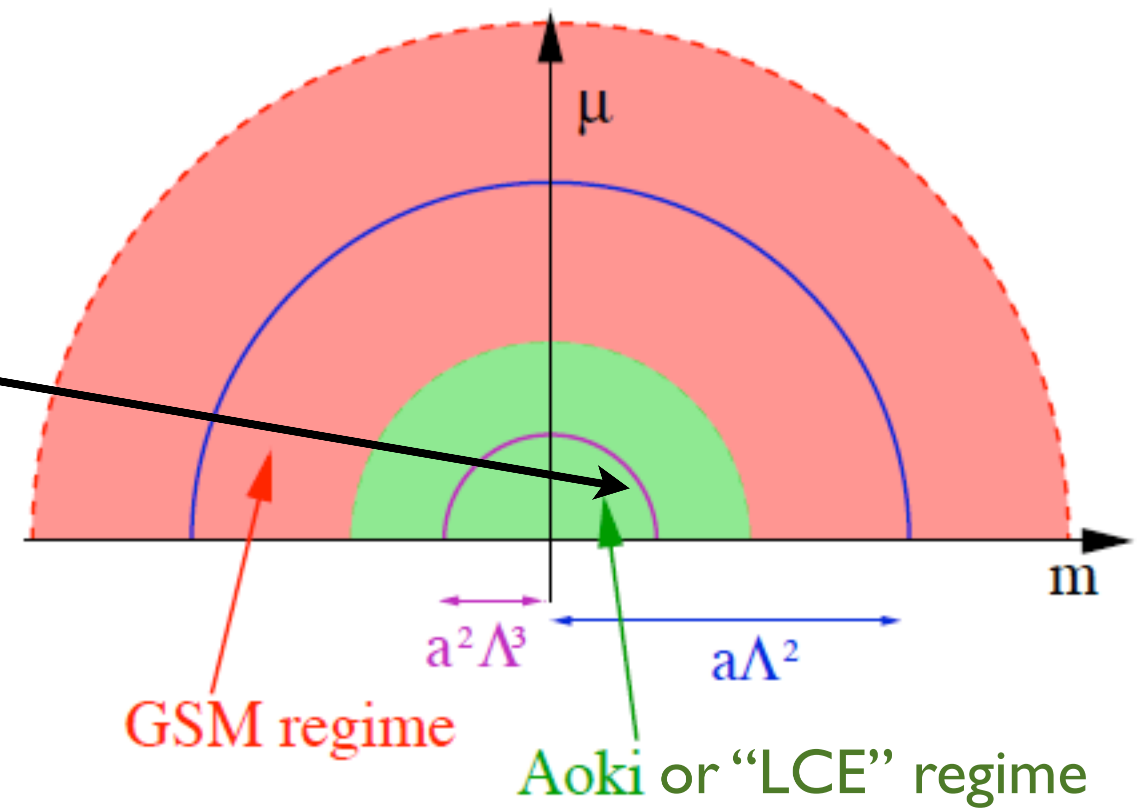
Power-counting in Aoki regime



Using a=0.05-0.1 fm

When approaching physical m_u & m_d one almost inevitably enters the Aoki regime



Power-counting in Aoki regime

- Power counting differs from GSM regime:
  - **No $\mathrm{O(a)}$ since absorbed into m'**
  - LO: $m_q \sim a^2$
  - NLO: $m_q a \sim a^3$
  - NNLO: $m_q^2 \sim m_q a^2 \sim a^4$
- Reorders terms in $\mathcal{L}_\chi$:

$$\mathcal{L}_\chi^{\mathrm{LO}} = \frac{f^2}{4}\mathrm{tr}(D_\mu \Sigma D_\mu \Sigma^\dagger) - \frac{f^2}{4}\mathrm{tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi') - W' \left[\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A})\right]^2$$

$$\mathcal{L}_\chi^{\mathrm{NLO}} = \widetilde{W}\mathrm{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma)\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) - W\mathrm{tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi')\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A})$$
$$- \frac{W_{3,1}}{f^2}\mathrm{tr}(\hat{A}^\dagger \hat{A})\mathrm{Tr}(\hat{A}^\dagger \Sigma + p.c.) - \frac{W_{3,3}}{f^2}\left[\mathrm{tr}(\hat{A}^\dagger \Sigma)^3 + p.c.\right]$$

  - At LO have competition between continuum and "lattice" terms
  - Two extra LECs at NLO, but $W_{3,1}$ can be absorbed by shift in m'

Power-counting in Aoki regime

- Power counting differs from GSM regime:
  - **No $\mathrm{O}(\mathrm{a})$ since absorbed into m'**
  - LO: $m_q \sim a^2$
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$$\mathcal{L}_\chi^{\mathrm{NLO}} = \widetilde{W}\mathrm{tr}(D_\mu \Sigma^\dagger D_\mu \Sigma)\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A}) - W\mathrm{tr}(\chi'^\dagger \Sigma + \Sigma^\dagger \chi')\mathrm{tr}(\hat{A}^\dagger \Sigma + \Sigma^\dagger \hat{A})$$
$$-\frac{W_{3,1}}{f^2}\mathrm{tr}(\hat{A}^\dagger \hat{A})\mathrm{Tr}(\hat{A}^\dagger \Sigma + p.c.) - \frac{W_{3,3}}{f^2}\left[\mathrm{tr}(\hat{A}^\dagger \Sigma)^3 + p.c.\right]$$

  - At LO have competition between continuum and “lattice” terms
  - Two extra LECs at NLO, but $W_{3,1}$ can be absorbed by shift in m'

We work only to LO here

(Untwisted) Wilson fermions

$$\mathcal{L}_\chi^{\rm LO} = \frac{f^2}{4}\mathrm{tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - \frac{f^2}{4}\mathrm{tr}\left(\chi'\Sigma^\dagger + \chi'^\dagger\Sigma\right) - W'\left[\mathrm{tr}(\hat{A}^\dagger\Sigma + \hat{A}\Sigma^\dagger)\right]^2$$

$$\mathcal{L}_\chi^{\rm LO} = \frac{f^2}{4}\mathrm{tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right) - \frac{f^2 2B_0 m}{4}\mathrm{tr}\left(\Sigma^\dagger + \Sigma\right) + \frac{c_2}{16}\left[\mathrm{tr}(\Sigma + \Sigma^\dagger)\right]^2$$

Dropped prime
on m

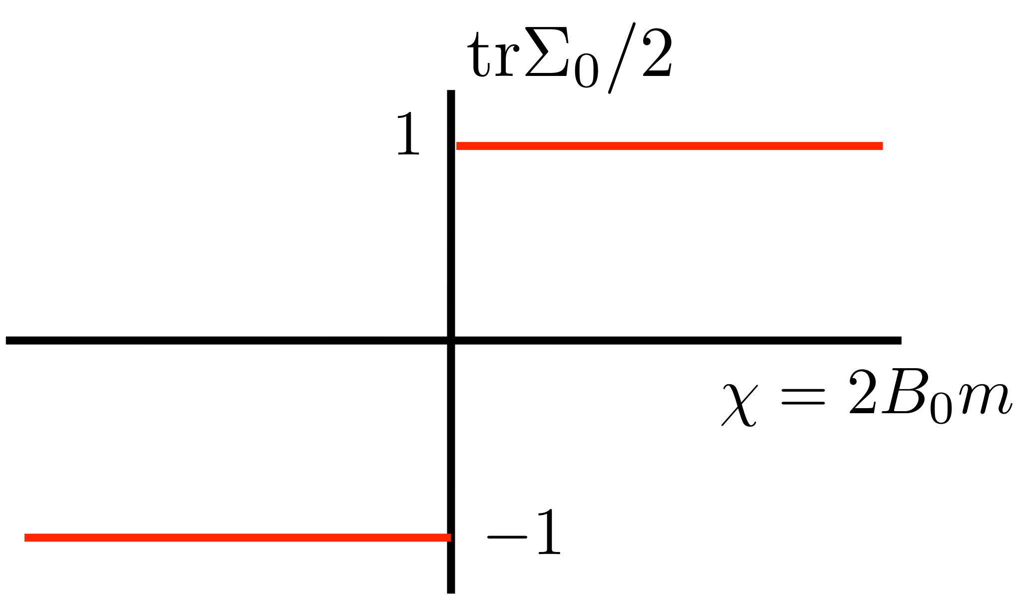
Change of notation
$c_2 \sim a^2$
Opposite sign to W'

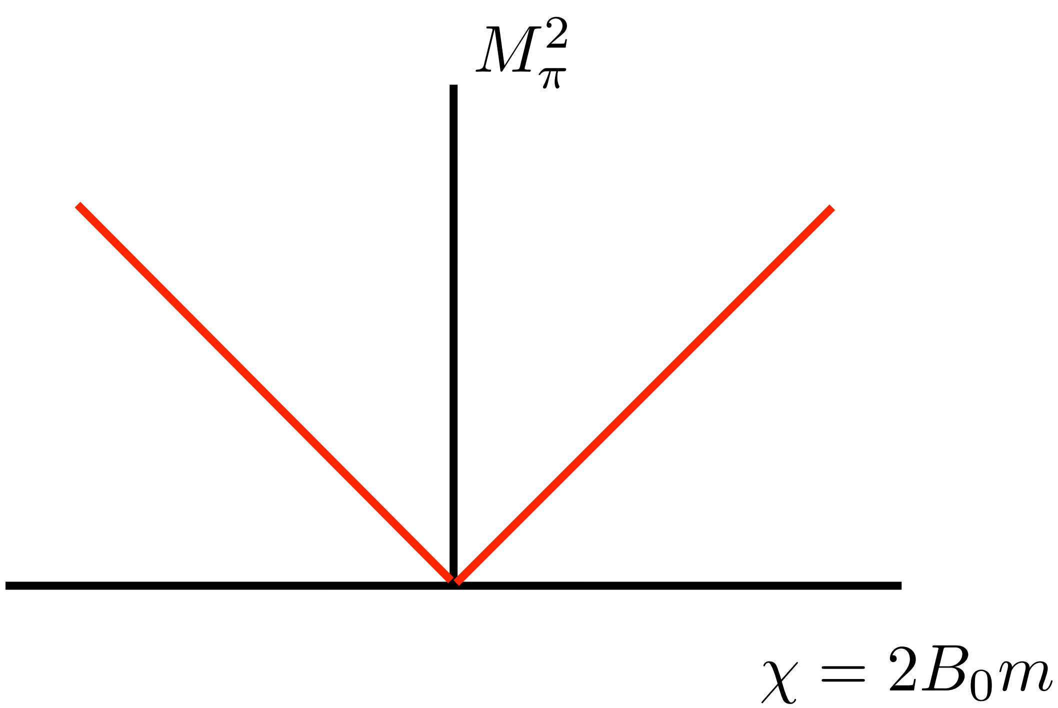
Phase structure: continuum

- In continuum, have "first-order transition" when m passes through zero, though the two sides are related by non-singlet axial SU(2) transformation

$$\mathcal{V} \propto -m\langle\Sigma + \Sigma^\dagger\rangle \quad \Rightarrow \Sigma_0 = \langle 0|\Sigma|0\rangle = \mathrm{sign}(m)\mathbf{1}$$

$$\Rightarrow M_\pi^2 = 2B_0|m|$$





Phase structure: lattice

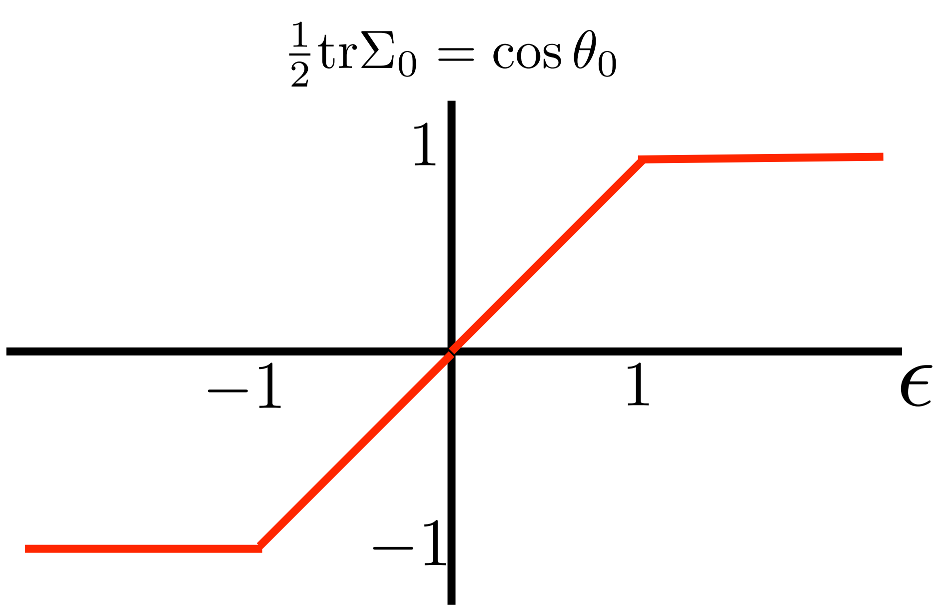
[Creutz 96,
Sharpe & Singleton 98]

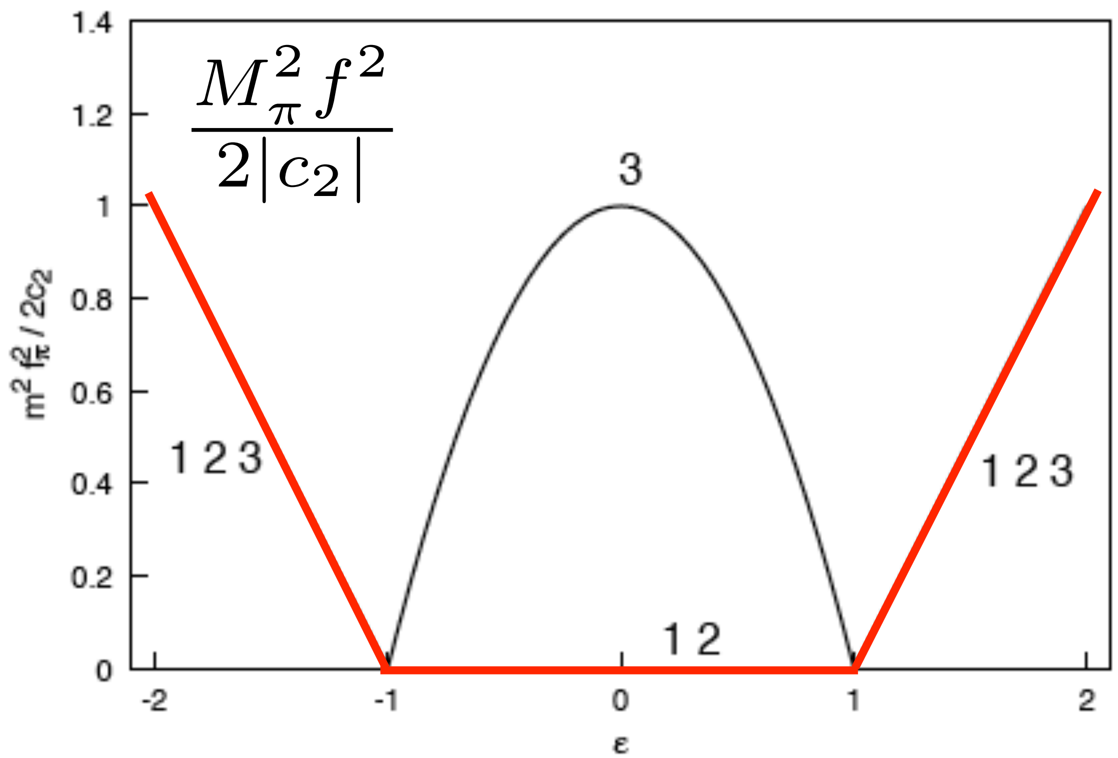
- Competition between two terms when m~a^2

$$\mathcal{V} = -\frac{f^2}{4}\chi\langle\Sigma+\Sigma^\dagger\rangle + \frac{c_2}{16}\langle\Sigma+\Sigma^\dagger\rangle^2 \propto -\epsilon\cos\theta_0 + \frac{1}{2}\frac{|c_2|}{c_2}\cos\theta_0^2$$

$$\epsilon = \frac{2mB_0 f^2}{2|c_2|} \qquad \Sigma_0 = \cos(\theta_0) + i\sin(\theta_0)\vec{n}_0\cdot\sigma$$

- If $c_2 > 0$, then get Aoki phase, flavor spont. broken:





Aoki phase

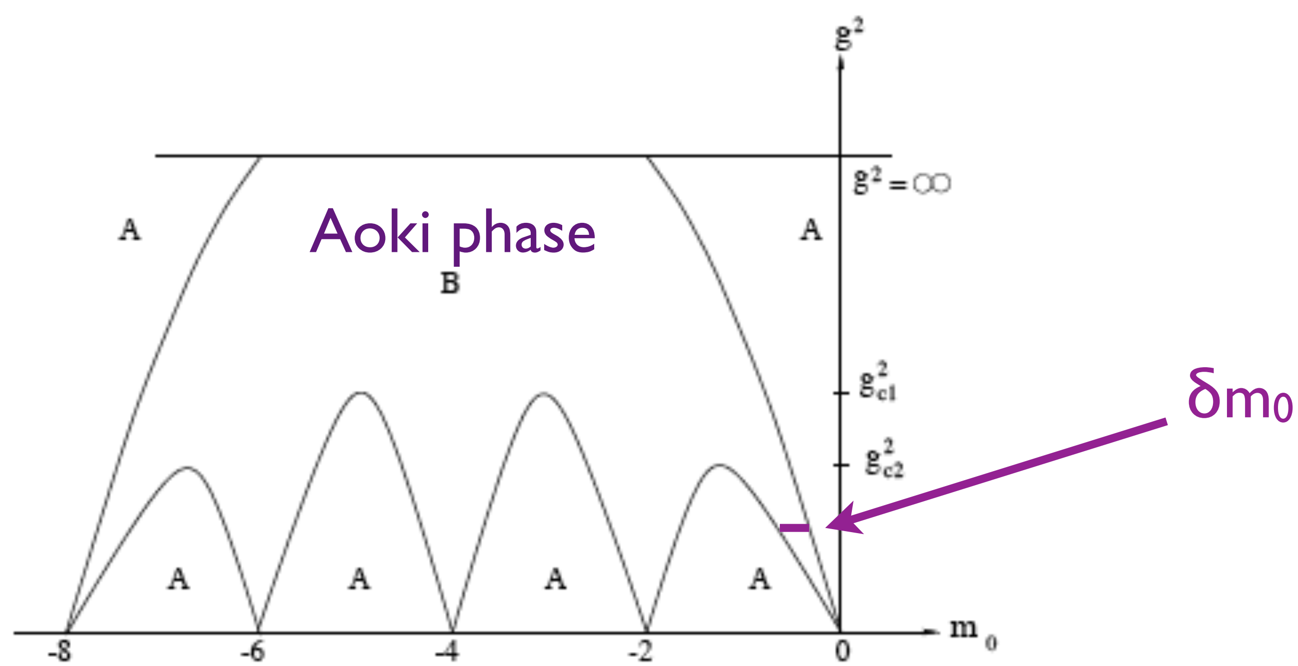
[Aoki 84]

- Explains why M_π=0 on lattice, even though have no chiral symmetry!

  ✓ (two) pions are PGBs of flavor breaking: $SU(2)_f \rightarrow U(1)_f$

- Parity is also broken (but not in the continuum)

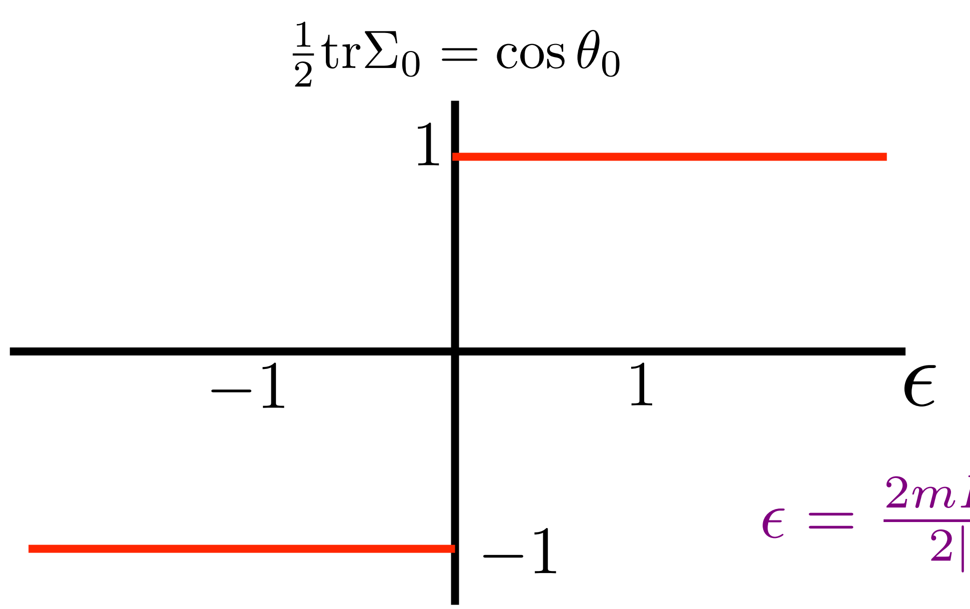
- Width of phase is $\delta m \sim a^2 \Rightarrow \delta m_0 \sim a^3$



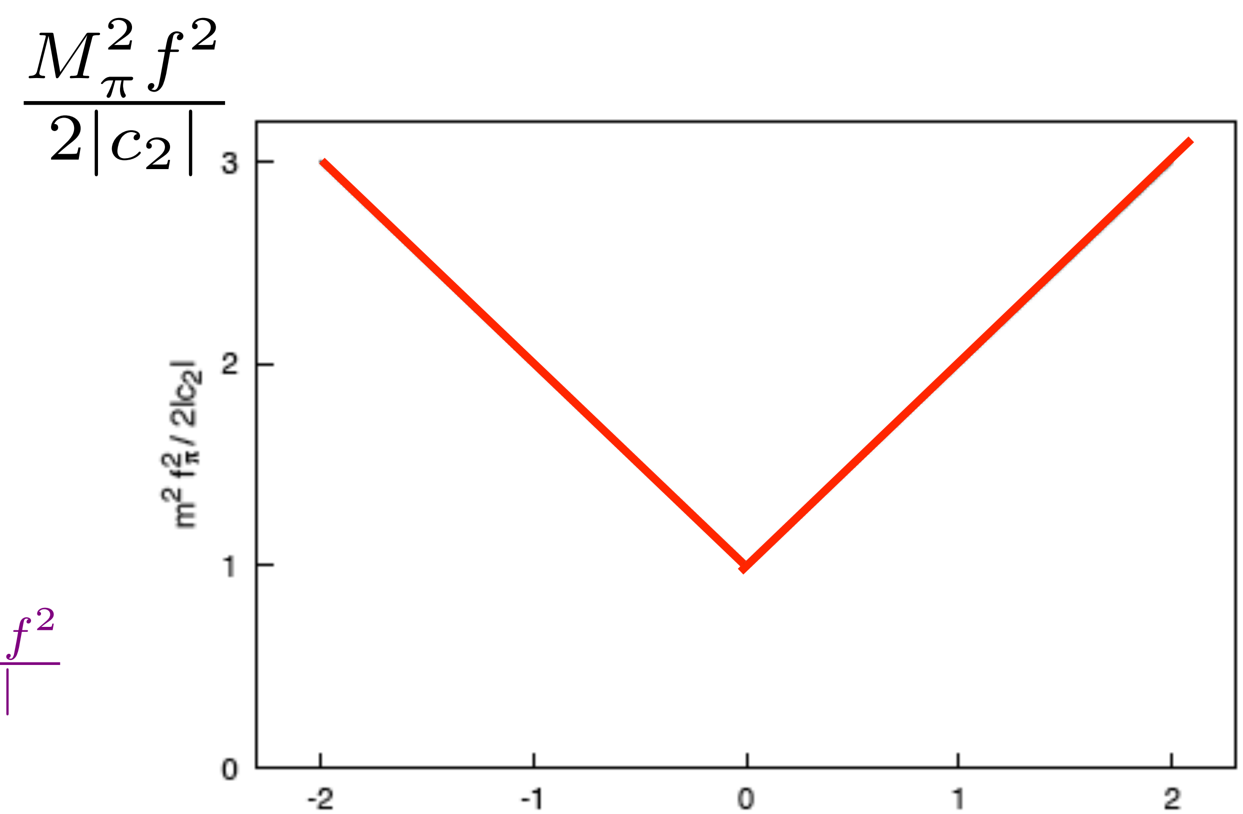
First-order scenario

$$\mathcal{V} = -\frac{f^2}{4}\chi\langle\Sigma+\Sigma^\dagger\rangle + \frac{c_2}{16}\langle\Sigma+\Sigma^\dagger\rangle^2 \propto -\epsilon\cos\theta_0 + \frac{1}{2}\frac{|c_2|}{c_2}\cos\theta_0^2$$

- If $c_2 < 0$, get first-order transition, with minimum pion mass $M_\pi(\text{min})$~a
- Explicit chiral symmetry breaking ⇒ No GB

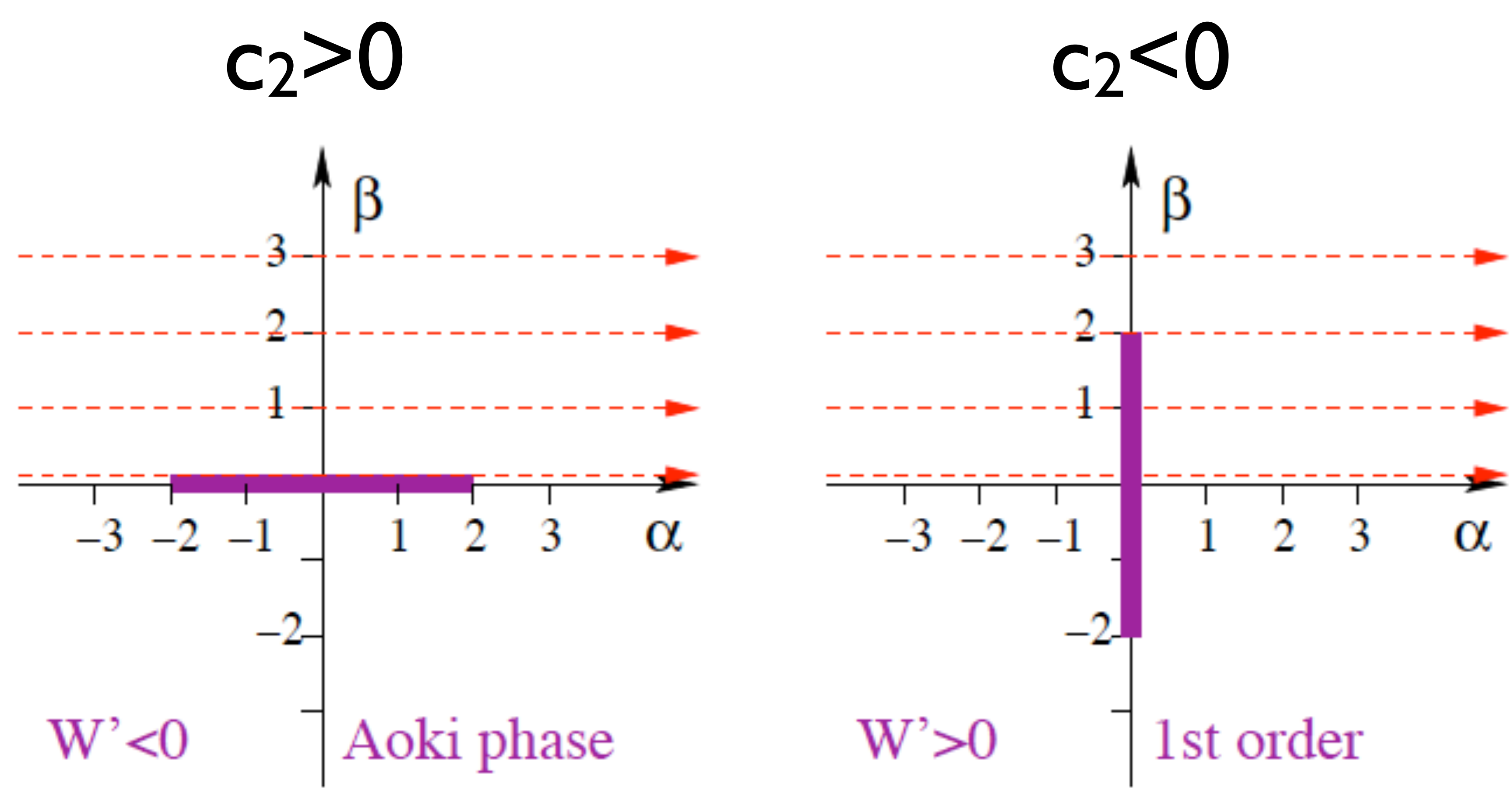


$$\epsilon = \frac{2mB_0 f^2}{2|c_2|}$$



Extend to twisted-mass plane

[Munster; Sharpe & Wu; Scorzato]



$$\alpha = 2B_0 m'/(16|W'|\hat{a}^2/f^2), \quad \beta = 2B_0\mu/(16|W'|\hat{a}^2/f^2)$$

$$M_{\pi^0} \geq M_{\pi^\pm} \qquad M_{\pi^0} \leq M_{\pi^\pm}$$

Equality only on Wilson axis (μ=0) outside Aoki phase

Mass difference determines sign & value of c_2 [Scorzato]

Example with first-order scenario

[Farchioni et al., 05]

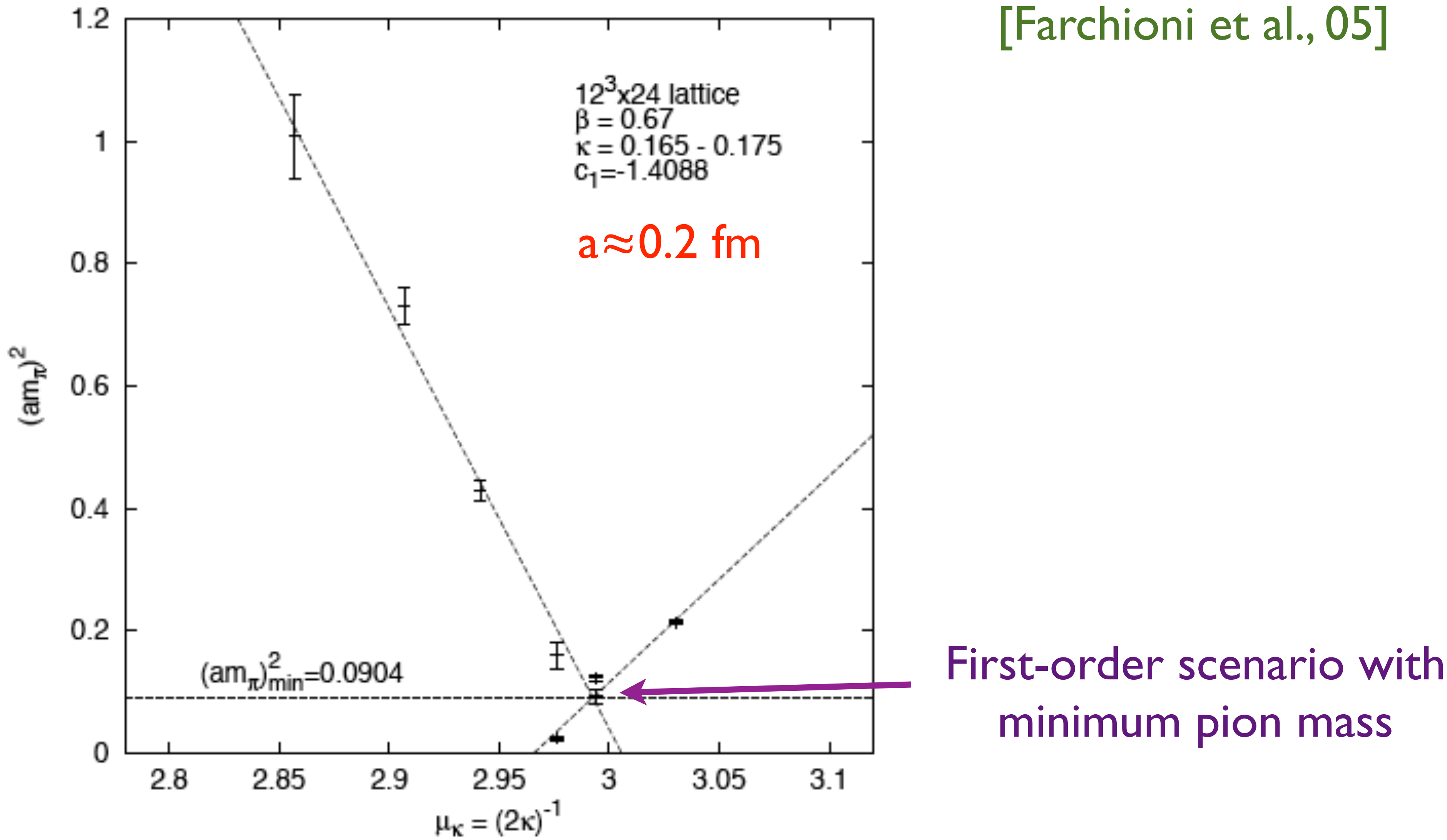
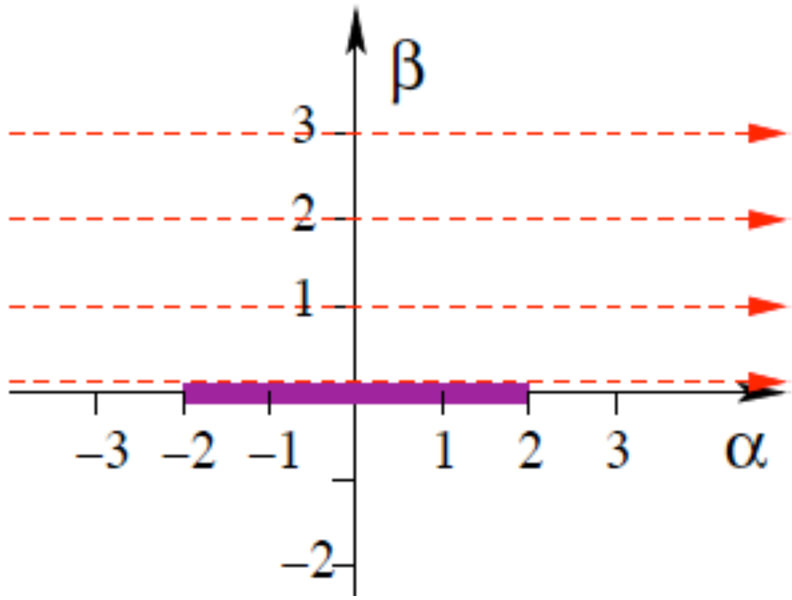


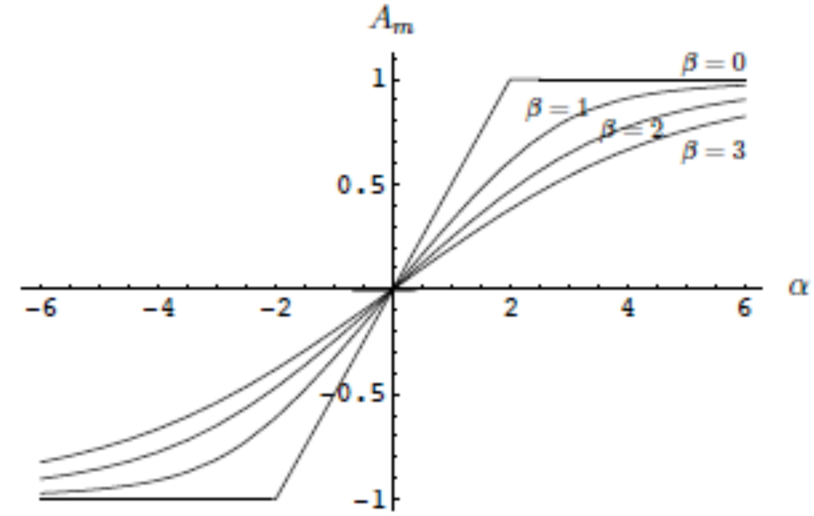
Figure 9. Unquenched results for $(am_\pi)^2$ as a function of $(2\kappa)^{-1} = m_0 + 4$ for $\mu = 0$ and with $a^{-1} \approx 0.2$ fm[60]. Straight lines are to guide the eye.

<u>Caveat</u>: LO WChPT may not apply for such a coarse lattice

Aoki scenario ($c_2>0$) in detail



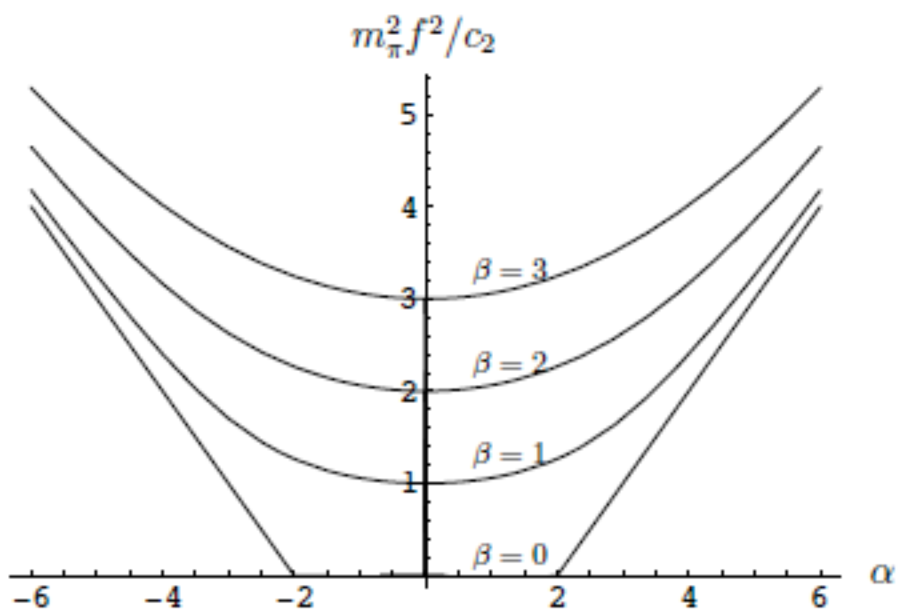
Condensate: $\langle\Sigma\rangle = A_m + iB_m\tau_3$



Aoki phase washed out for

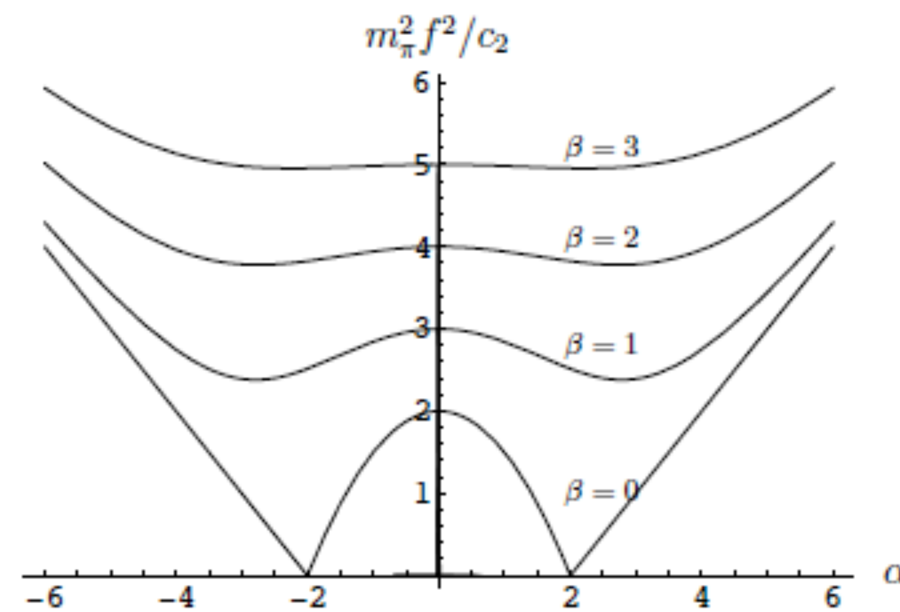
$\mu \propto \beta \neq 0$

$M^2_{\pi^\pm}$



(a) Mass of π_1 and π_2

$M^2_{\pi^0}$

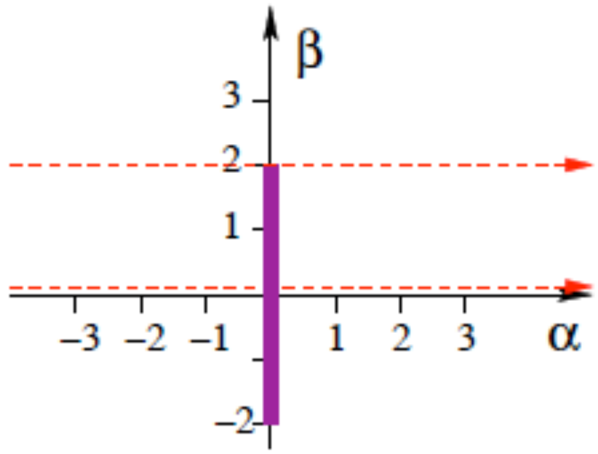


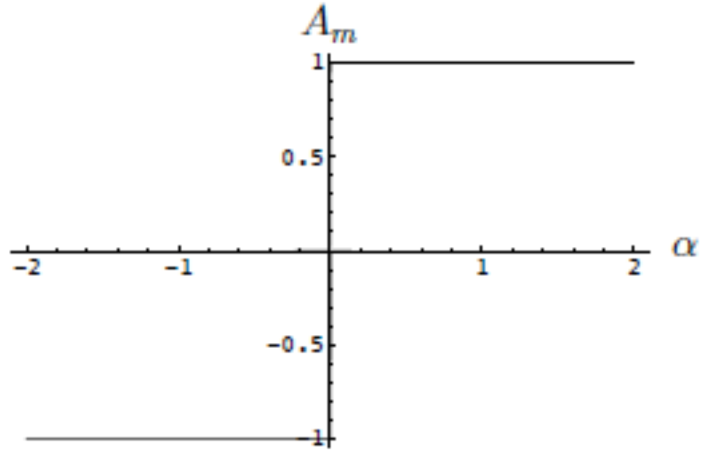
(b) Mass of π_3

$$M_{\pi^\pm} \leq M_{\pi^0}$$

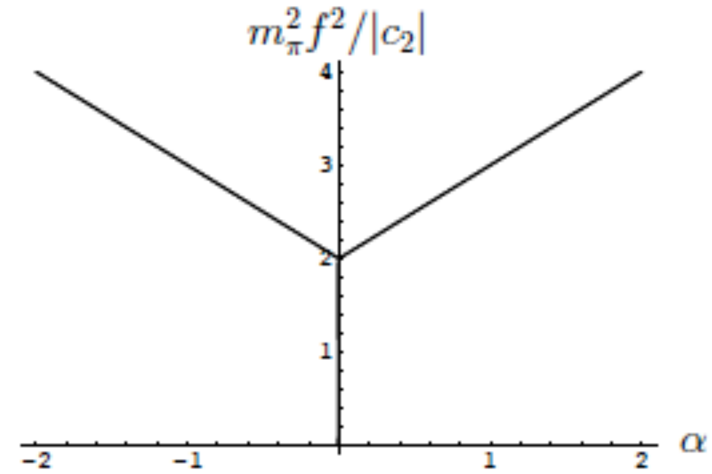
First-order scenario ($c_2<0$) in detail

Along Wilson axis:



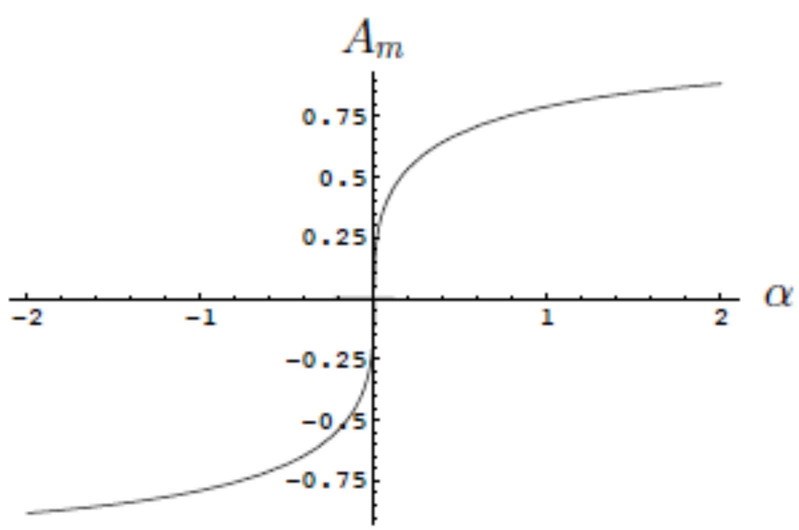


(a) Global minimum, $\beta = 0$

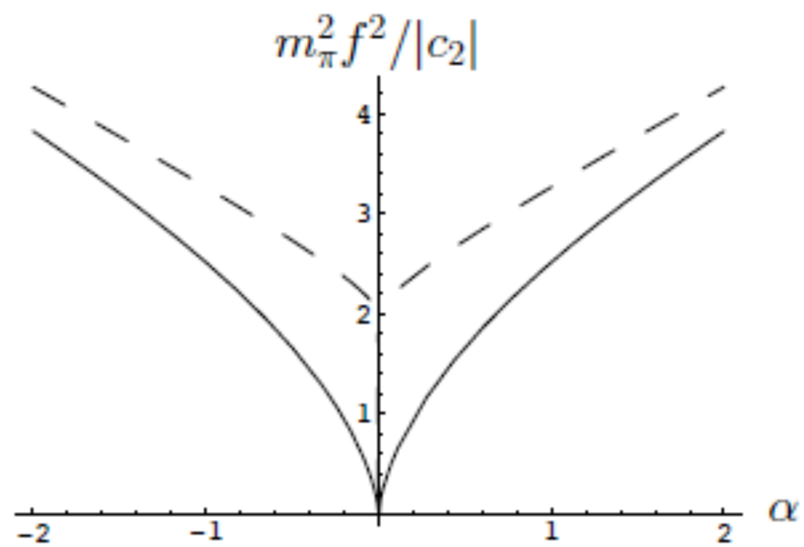


(b) Pion masses, $\beta = 0$

At top of phase transition: (dashed: charged pions; solid: neutral)



(e) Global minimum, $\beta = 2$

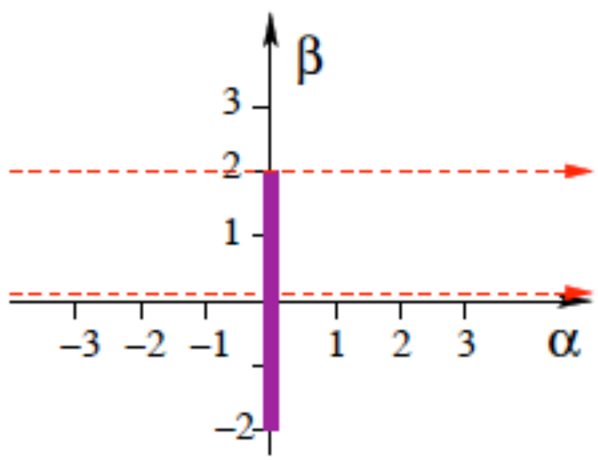


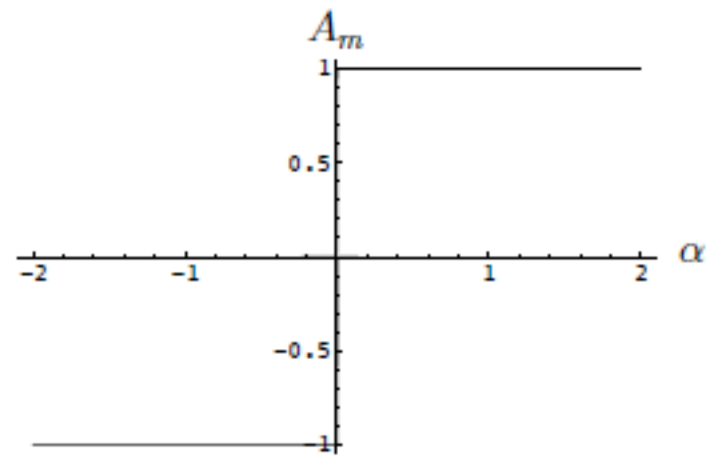
(f) Pion masses, $\beta = 2$

$$M_{\pi^\pm} \geq M_{\pi^0}$$

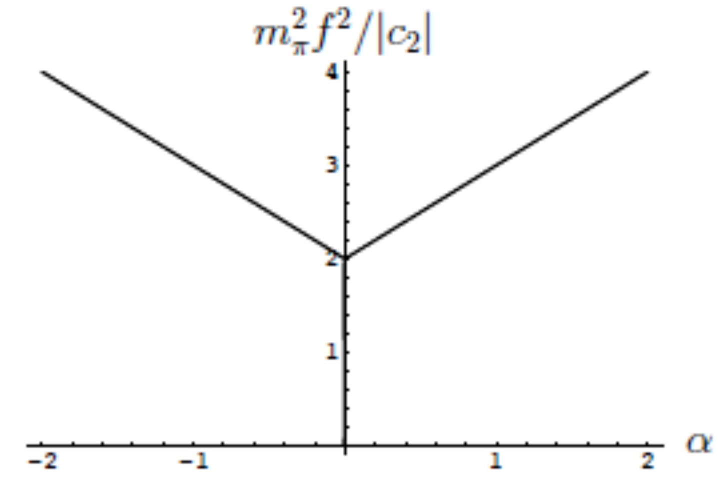
First-order scenario ($c_2 < 0$) in detail

Along Wilson axis:



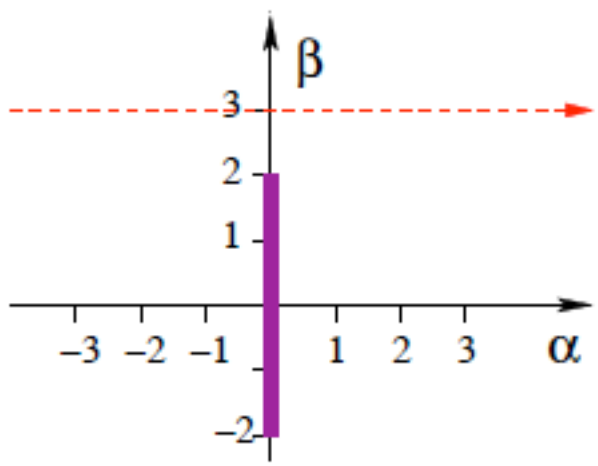


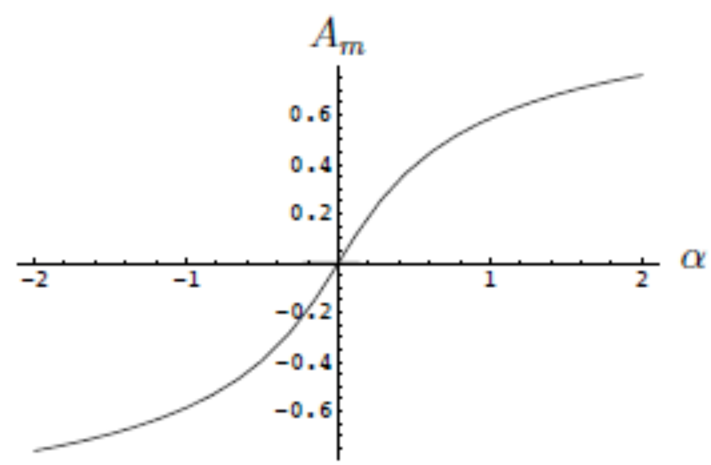
(a) Global minimum, $\beta = 0$



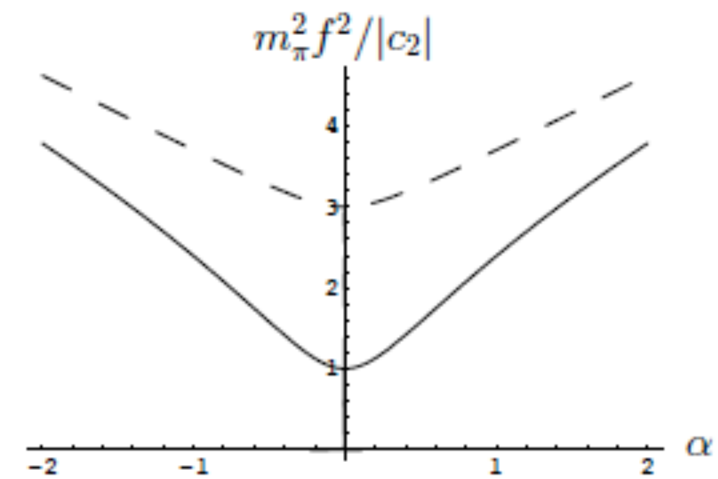
(b) Pion masses, $\beta = 0$

Above phase transition: (dashed: charged pions; solid: neutral)





(g) Global minimum, $\beta = 3$



(h) Pion masses, $\beta = 3$

$$M_{\pi^\pm} \geq M_{\pi^0}$$

Lessons for lattice

- Simulations are either already in or close to the Aoki/LCE regime ($m \sim a^2$)
- Phase structure can lead to large lattice artifacts
  - Metastabilities if first order
  - Distortion of physical quantities near second-order endpoints [Aoki]
  - Spectral gap in hermitian Wilson-Dirac operator can be reduced leading to numerical issues in simulations
- Basic message: understand where the dangers are and STAY AWAY

Summary

- Combining Symanzik's effective theory with chiral effective theory provides a method for analyzing lattice-spacing effects which incorporates all known symmetry constraints
- Applied to Wilson, tm & staggered fermions
- Most important applications to date have been chiral/continuum fits for staggered fermions and unraveling the phase structure for Wilson/tm fermions
- Recent work (not discussed) shows how microscopic eigenvalues of Dirac operator are sensitive to the same LECs that enter into W/tmChPT