Effective Field Theories for lattice QCD

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Outline of Lectures

1. Overview & Introduction to continuum chiral perturbation theory (ChPT)
2. Illustrative results from ChPT; SU(2) ChPT with heavy strange quark; finite volume effects from ChPT and connection to random matrix theory
3. Including discretization effects in ChPT
4. Partially quenched ChPT and applications, including a discussion of whether m_u=0 is meaningful

Outline of lecture 1

- Why LQCD needs effective field theories (EFTs)
  - Focus in these lectures on ChPT, Symanzik's low-energy effective theory for lattice field theories (SET) and their combination
- Examples of uses of ChPT in recent simulations
- Introduction to continuum ChPT
  - Emphasize features & results relevant to LQCD and later lectures

Glossary!

- EFT Effective Field Theory
- χPT: Chiral Perturbation Theory (or ChPT)
- PQQCD: Partially Quenched QCD
- PQχPT: Partially Quenched χPT
- WχPT: Wilson χPT (including lattice spacing effects)
- tmQCD: Twisted mass QCD
- tmχPT: Twisted mass χPT (including lattice spacing effects)
- SχPT: Staggered χPT (including lattice spacing effects)
- (P)GB: (Psuedo) Goldstone Boson
- LEC: Low energy coefficient (in chiral Lagrangian)
- VEV: Vacuum Expectation Value
- LO: leading order
- NLO: next-to-leading order, etc.
- NP: non-perturbative

Some continuum ChPT References

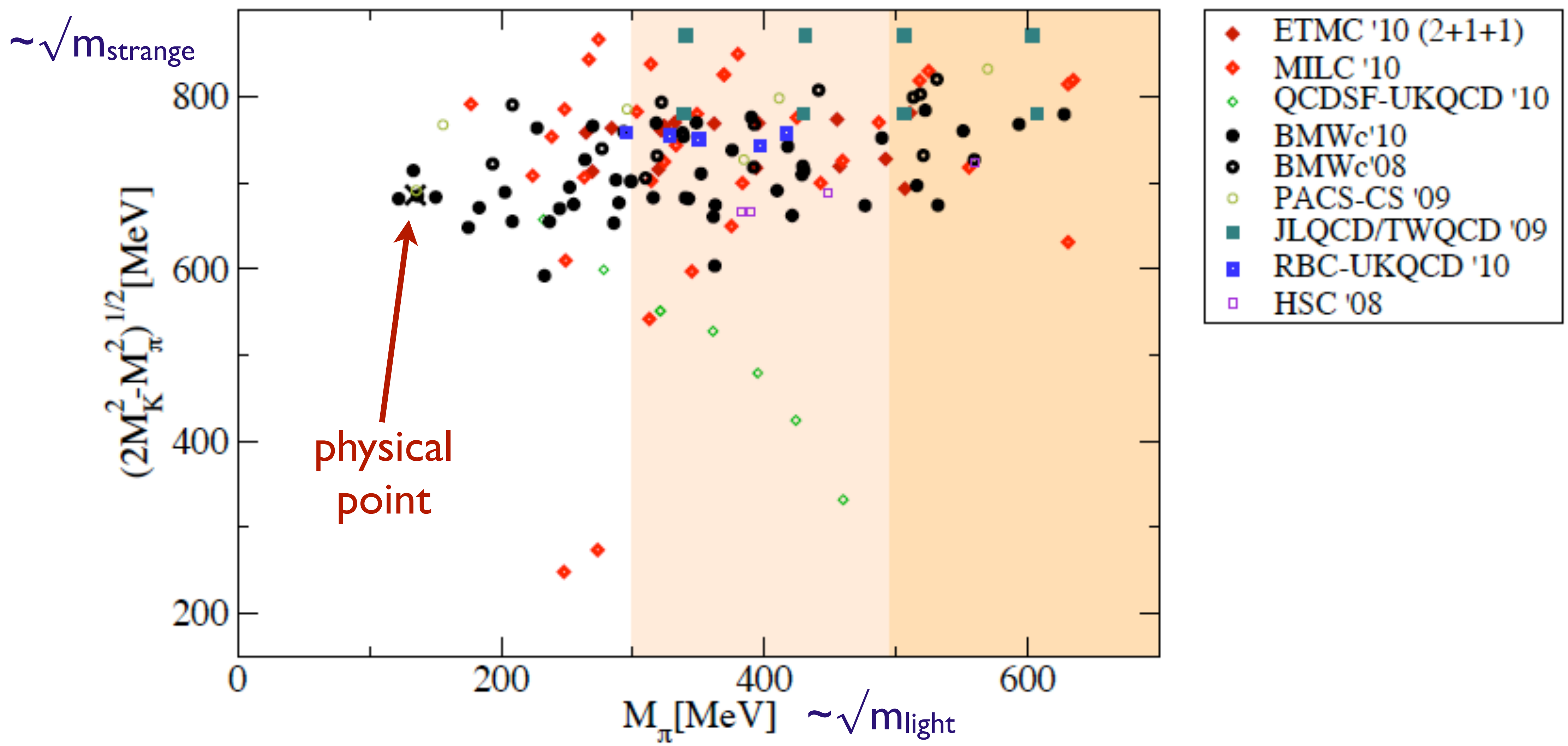
- A selection of books and lecture notes:
  - H. Georgi, "Weak Interactions and Modern Particle Theory"
  - J.F. Donoghue, E. Golowich and B.R. Holstein, "Dynamics of the Standard Model"
  - A.V. Manohar, "Effective Field Theories", hep-ph/9606222
  - G. Ecker, "Chiral Perturbation Theory", hep-ph/9608226,9805300
  - A. Pich, "Introduction to Chiral Perturbation Theory", hep-ph/9502366
  - D.B. Kaplan, "5 lectures on Effective Field Theory", nucl-th/0510023
- Classic papers:
  - S.R. Coleman, J. Wess and B. Zumino, "Structure Of Phenomenological Lagrangians. 1," Phys. Rev. **177**, 2239 (1969).
  - C.G. Callan, S.R. Coleman, J. Wess and B. Zumino, "Structure Of Phenomenological Lagrangians. 2," Phys. Rev. **177**, 2247 (1969).
  - S. Weinberg, "Phenomenological Lagrangians," PhysicaA **96**, 327 (1979).
  - J. Gasser and H. Leutwyler, "Chiral Perturbation Theory To One Loop," Annals Phys. **158**, 142 (1984).
  - J. Gasser and H. Leutwyler, "Chiral Perturbation Theory: Expansions In The Mass Of The Strange Quark," Nucl. Phys. B **250**, 465 (1985).

ChPT for LQCD References

- S.R. Sharpe, "Applications of Chiral Perturbation Theory to Lattice QCD", hep-lat/0607016 (a write-up of an earlier, shorter form of these lectures)
- M.F.L. Golterman, "Applications of chiral perturbation theory to lattice QCD", arXiv:0912.4042 [hep-lat] (from 2009 Les Houches summer school)

Why LQCD needs EFTs

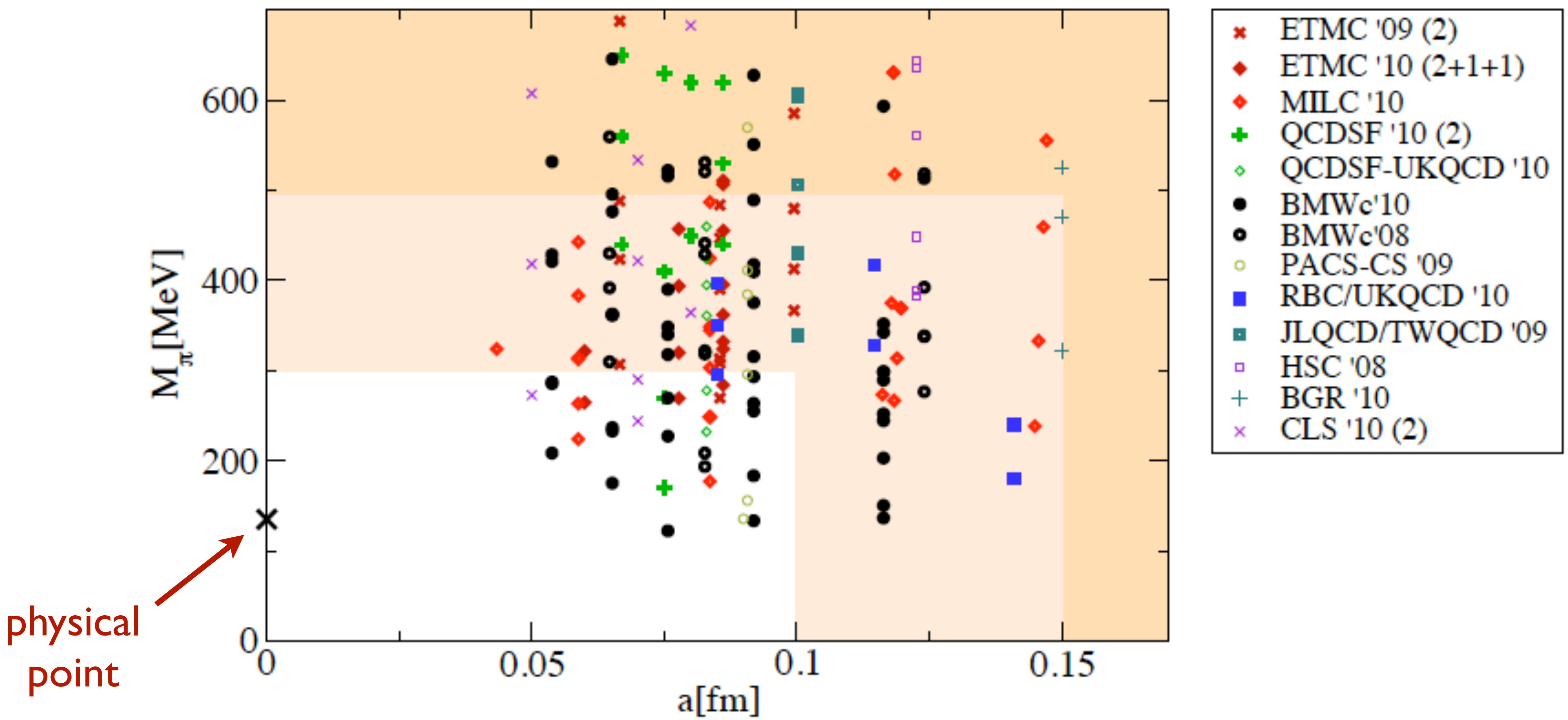
- Most calculations require extrapolation in light (u & d) quark masses



Landscape of recent N_f=2+1 simulations [Fodor & Hoelbling, RMP 2012]

Why LQCD needs EFTs

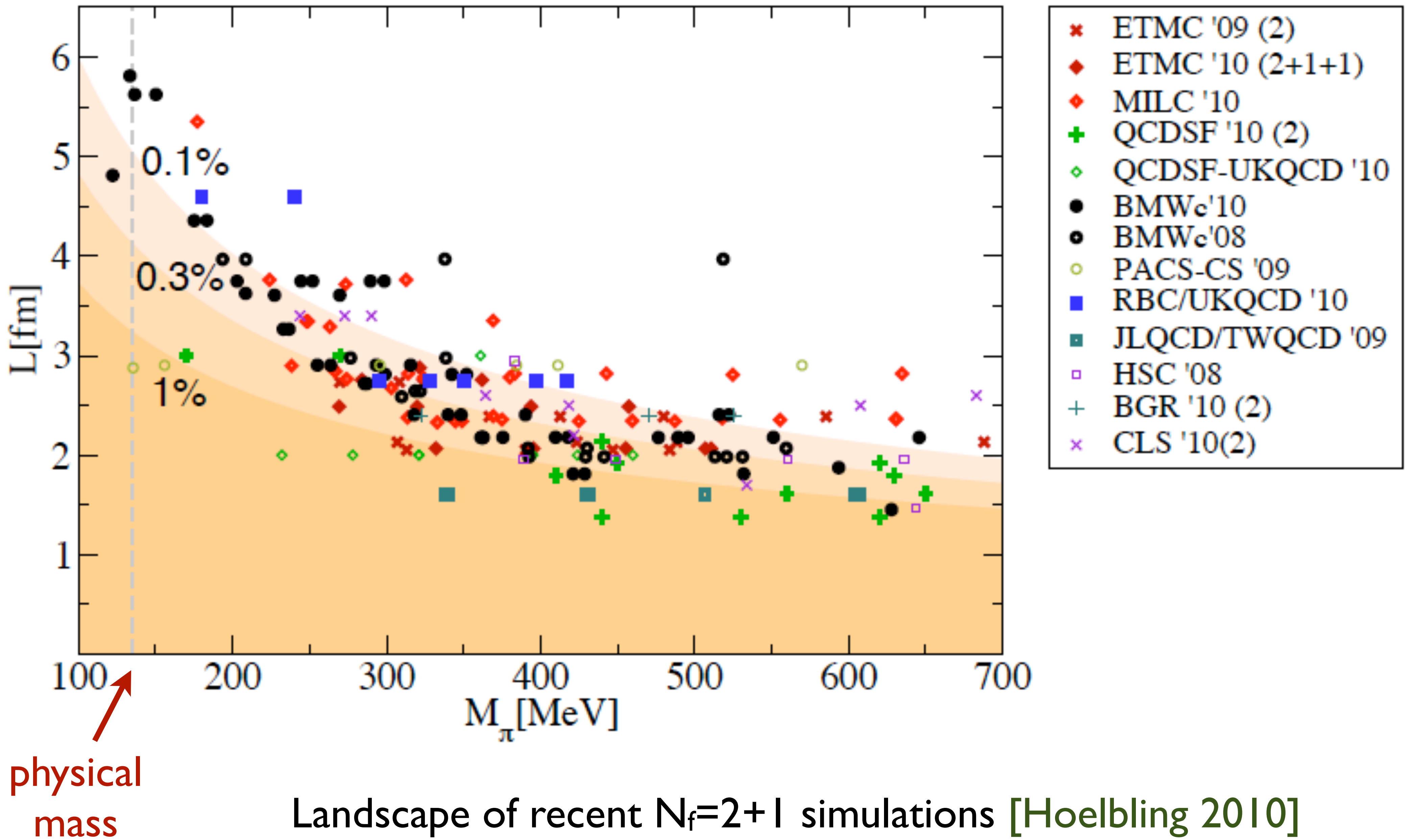
- ALL calculations require extrapolation in lattice spacing a
  - ➡ N.B. Leading discretization error is proportional to a^2 with modern actions



Landscape of recent N_f=2+1 simulations [Fodor & Hoelbling, RMP 2012]

Why LQCD needs EFTs

- Most calculations require extrapolation to infinite box size L
  - ➡ Exception is work in "epsilon-regime" (M_π L < 1)



Landscape of recent N_f=2+1 simulations [Hoelbling 2010]

Why LQCD needs EFTs

- LQCD continues to need extrapolations in m, a & L
- Also needs extrapolation from $m_u=m_d$ (isospin limit) to physical m_u & m_d, and from $\alpha_{EM}=0$ to $\alpha_{EM}=1/137$
  - Increasingly important as percent-level precision is attained (e.g. in f_K/f_π, B_K, ...)
  - First calculations including isospin breaking and EM in measure (either directly or by reweighting) are underway but far from "production" status

- Chiral Perturbation Theory, incorporating discretization effects using Symanzik's effective theory, provides the necessary functional forms for these extrapolations
- ChPT + SET provide predictions for the small-volume epsilon-regime which can be used to determine physical parameters
- These are the topics I aim to introduce in these lectures

- Other EFTs (not discussed here) play a crucial role in lattice calculations
  - Heavy Quark Effective Theory in the calculation of B-meson properties
  - Non-Relativistic QCD for simulating heavy quarks
  - Pion-less (and pion-full) nuclear EFT to extend lattice calculations of multinucleon properties to a broader array of quantities

Why LQCD needs EFTs

- Widespread use of "unphysical" simulations
  - Staggered fermions with the "$\sqrt[4]{\text{Det}}$" trick
    - Theory unitary (at best) in continuum limit
  - "Mixed actions"
    - e.g. Overlap valence fermions on Wilson/tm sea
  - Partially quenched QCD
    - Valence quarks *not degenerate* with sea quarks
    - Gives more information to constrain chiral extrapolations

- ChPT (+ SET) allows one to remove unphysical effects, or (in the case of PQQCD) to see how to use the extra information to learn about physical quantities

Why ChPT needs LQCD!

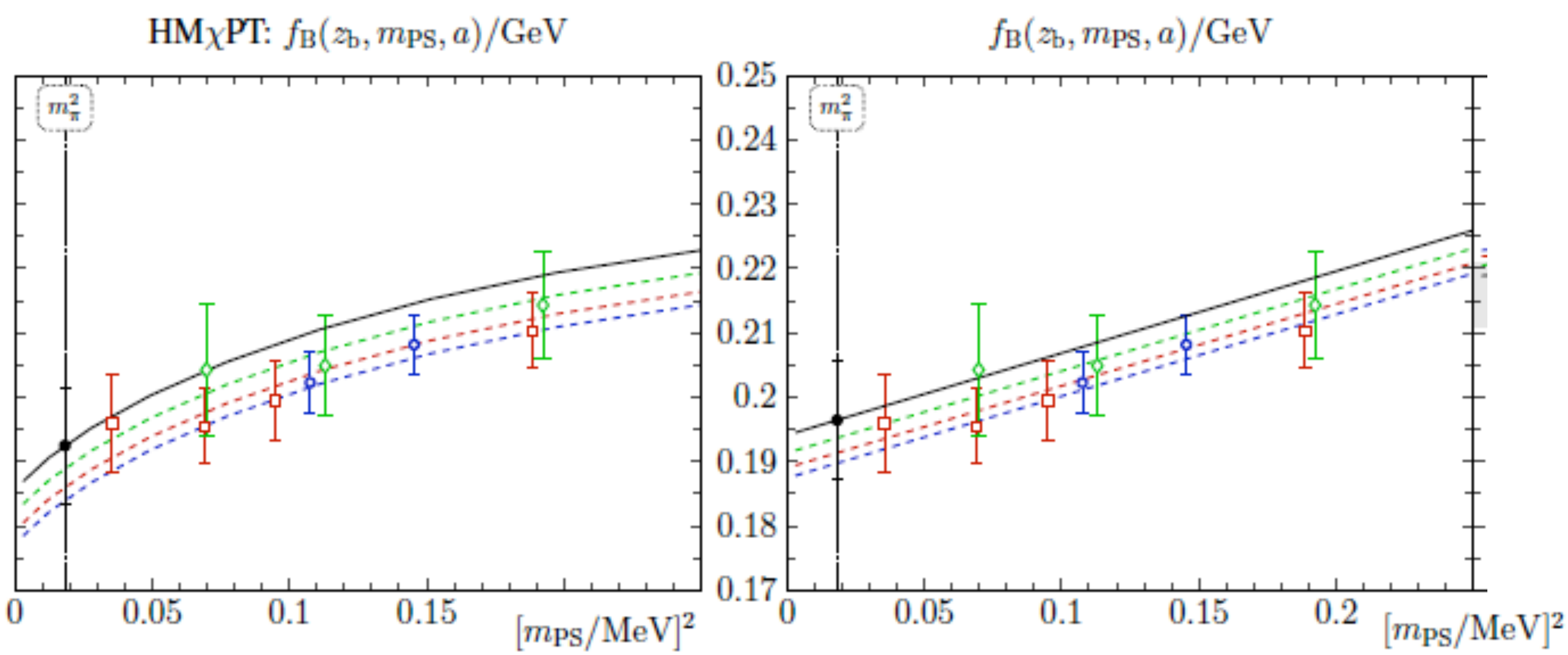
- ChPT contains many *a priori* unknown parameters
  - "Low-energy constants" or LECs
  - Some can be determined by matching with experiment, but many cannot
- LQCD provides additional "dials" that are unavailable in the real world
  - Quark masses can be tuned---making a feature out of a "bug"
  - Allows determination of LECs that are difficult or impossible to obtain otherwise
- ChPT can then be used to extrapolate, approximately, from the simple processes that LQCD can calculate, e.g. $K \longrightarrow \pi\pi$, to those that it cannot, e.g. $K \longrightarrow 3\pi$
- LQCD can test how well ChPT converges in a much more controlled setting than experiment

Outline of today's lecture

- Why LQCD needs effective field theories (EFTs)
  - Focus in these lectures on ChPT, Symanzik's low-energy effective theory for lattice field theories (SET) and their combination
- **Examples of uses of ChPT in recent simulations**
- Introduction to continuum ChPT
  - Emphasize features & results relevant to LQCD and later lectures

Examples of ChPT in use today

- Chiral-continuum extrapolation of f_B using improved Wilson fermions
  - Left: fit to form using heavy-meson ChPT + a^2 corrections. Curvature from "chiral logs"
  - Right: fit to linear ansatz + a^2 (used to estimate extrapolation error)
  - a = 0.045, 0.065 & 0. 078 fm



Bahr et al. [ALPHA collaboration], arXiv:1211.6327

Examples of ChPT in use today

- Chiral-continuum extrapolation of D⟶K form factor using improved staggered fermions and "staggered ChPT" (SChPT)
    - Left: using SU(3) SChPT; Right: using SU(2) SChPT
    - m_{light} / m_s ranges from 0.4 (blue) to 0.1 (red) [compared to ~0.04 for physical case]
    - Lattice spacings range from 0.045 fm (triangles) to 0.12 fm (squares)

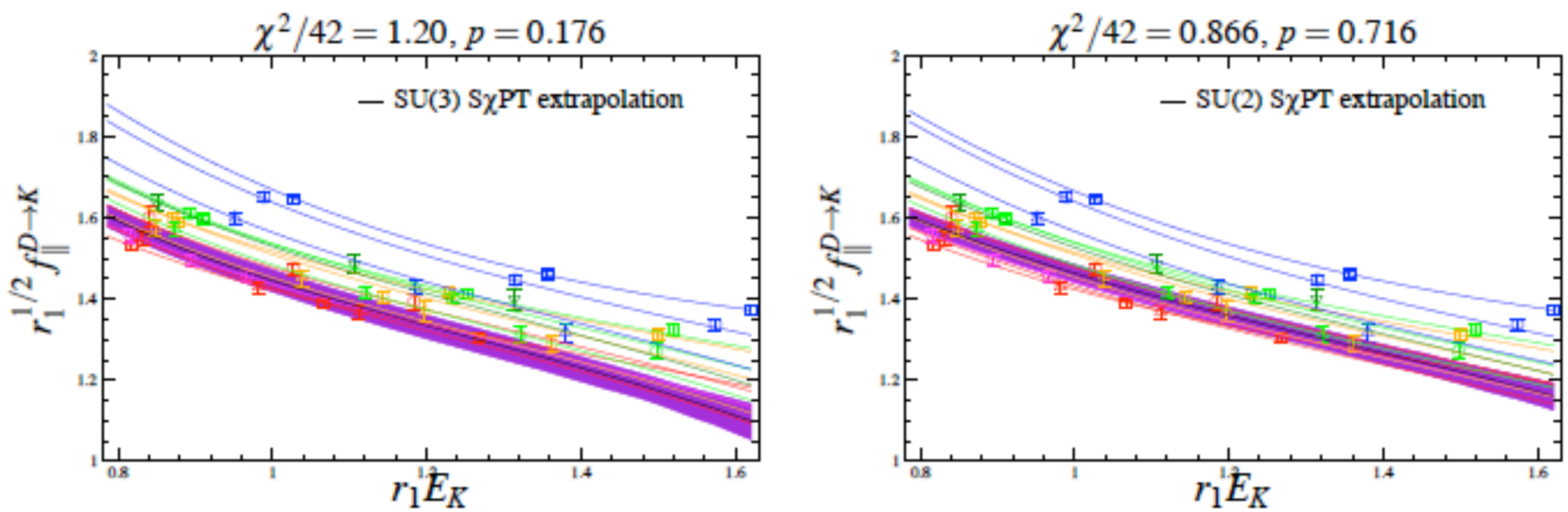


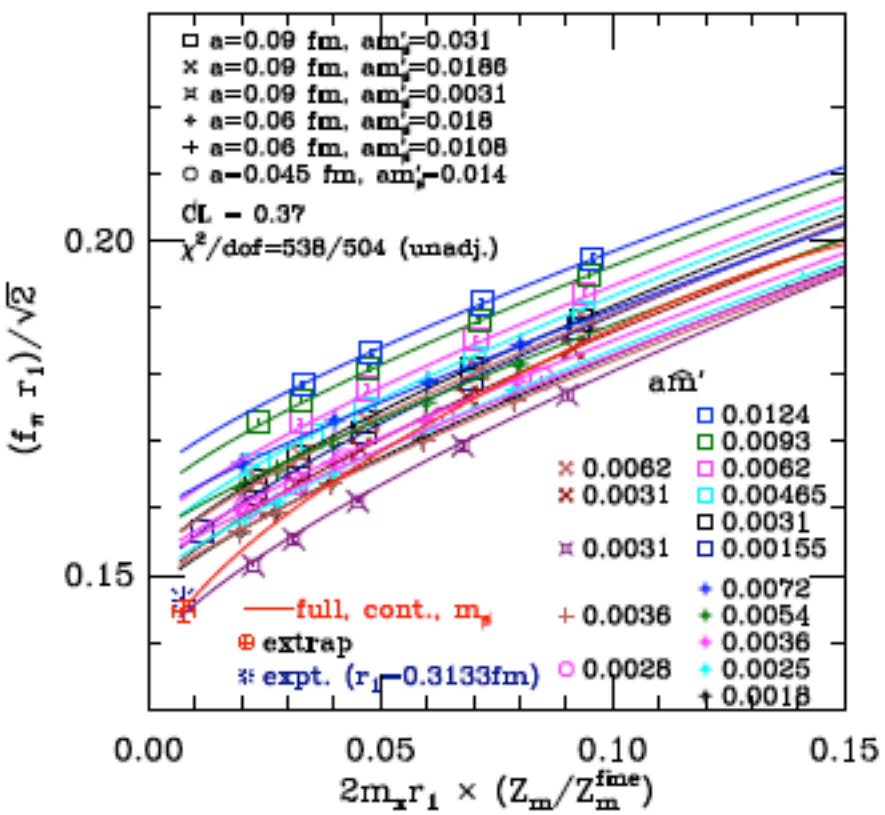
Figure 4: Fits of $f_\parallel^{D\to K}$ data to SU(3) (left) and SU(2) (right) SχPT. The fit functions include the chiral logarithms and analytic terms at NLO and analytic terms at NNLO. Errors are statistical, from bootstrap ensembles.

Bailey et al. [FNAL-MILC collaboration], arXiv:1211.4964

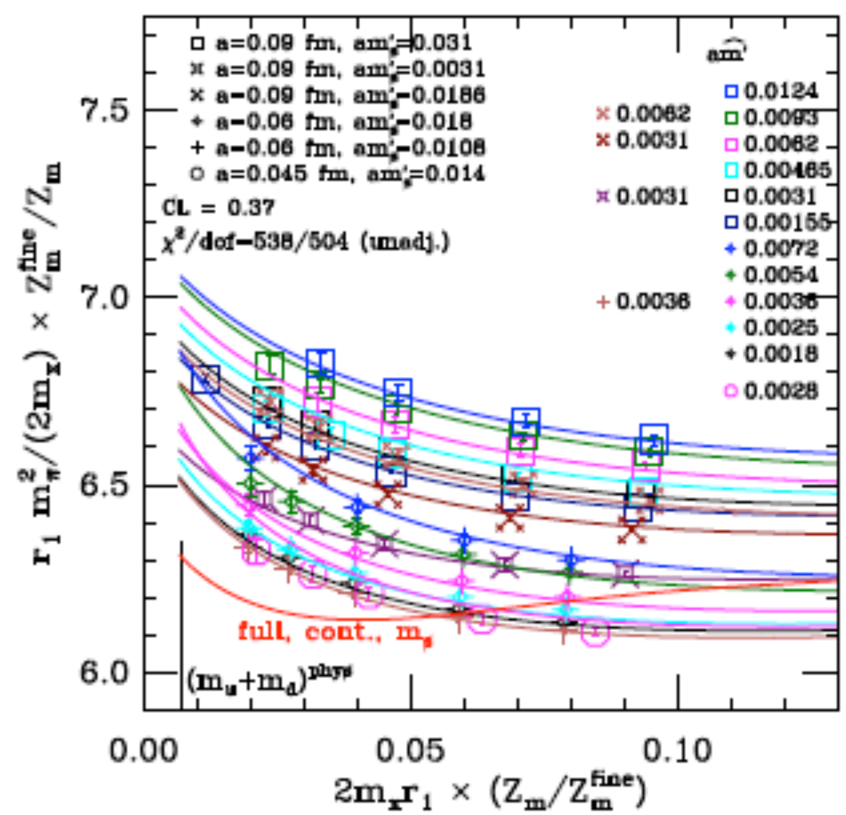
Examples of ChPT in use today

- Chiral-continuum extrapolation of pion decay constant and mass using improved staggered fermions and "staggered ChPT" (SChPT)
  - Includes partially quenched data ($m_{valence} < m_{sea}$)
  - curvature due to NLO chiral logs (which include discretization corrections--- "taste breaking"--- essential for good fits)
  - fits include continuum NNLO chiral logs

f_π



m_{light}



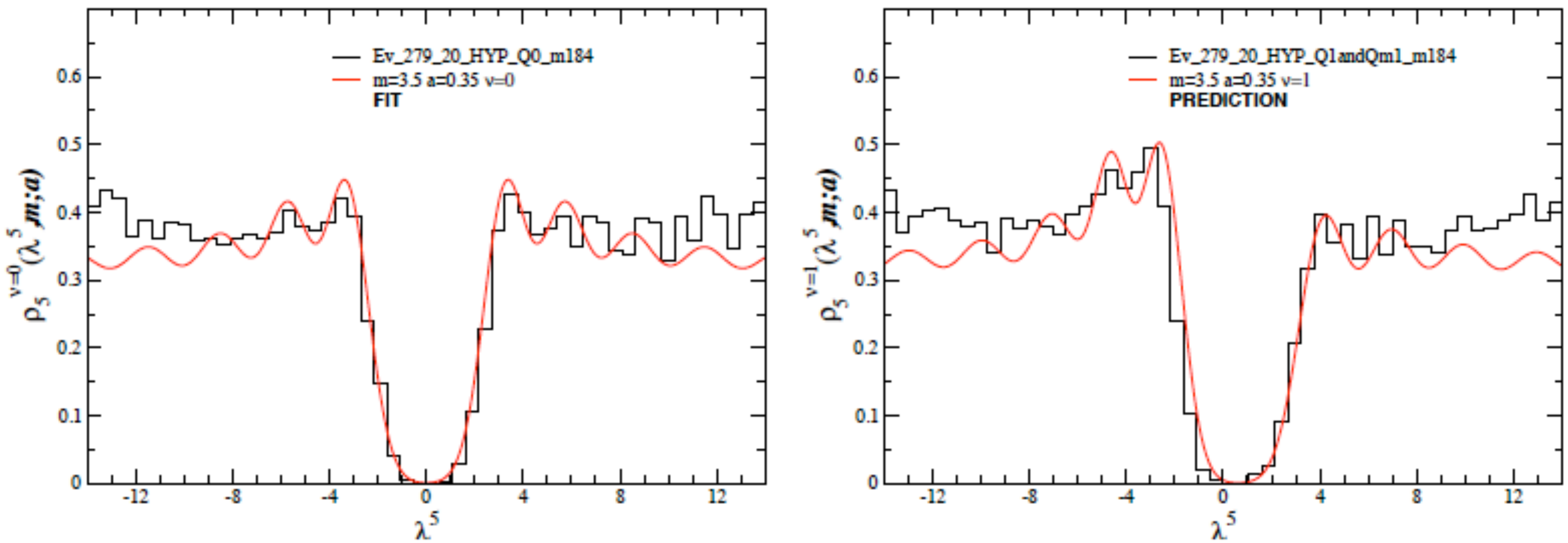
m_π^2/m

m_{light}

Bernard et al. [MILC collaboration], arXiv:1012.0868

Examples of ChPT in use today

- Distribution of small eigenvalues of the Hermitian Wilson-Dirac operator in small (1.5 fm) box compared to "Wilson ChPT" in the epsilon-regime
  - Left: fit to top. charge 0 configs (determines leading order LECs in ChPT)
  - Right: corresponding prediction for top. charge 1 configs (note asymmetry)



Damgaard et al., arXiv:1301.3099

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Essence of EFTs

- Consider only low energy degrees of freedom
  - "Integrate out" high energy or massive degrees of freedom
  - Requires a separation of scales
- Use symmetries alone to constrain terms in effective Lagrangian
  - Systematically order terms using an appropriate power counting in (low scale)/(high scale)

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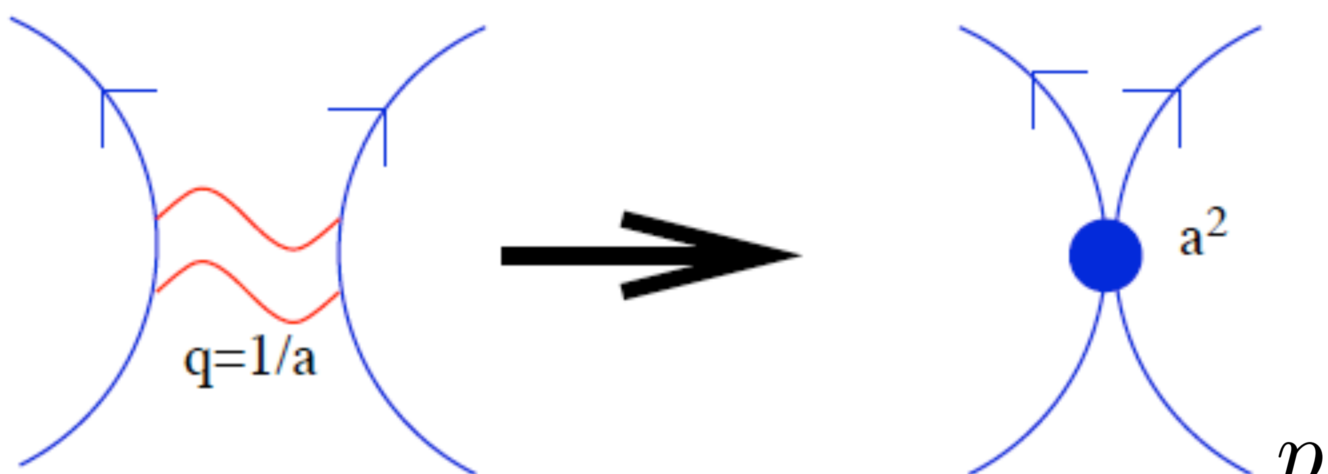
- Examples:

Symanzik: Integrating out $q \sim 1/a$ gluons for staggered fermions leads to four-fermion operators



$$p \sim \Lambda_{\mathrm{QCD}} \ll 1/a$$

Analogous to Fermi theory of weak interactions

Essence of EFTs

- Consider only low energy degrees of freedom
  - "Integrate out" high energy or massive degrees of freedom
  - Requires a separation of scales
- Use symmetries alone to constrain terms in effective Lagrangian
  - Systematically order terms using an appropriate power counting in (low scale)/(high scale)
- Examples:

χPT: QCD correlation functions can be represented by PGB contributions

$$m_\pi \ll m_\rho, m_N \sim 1 \text{ GeV}$$

Conceptually more complicated than SET, since no perturbative treatment

General recipe for an EFT

- The only non-analyticities in correlation functions are due to "light" degrees of freedom (d.o.f.)
  - χPT: PGBs; Symanzik: low momentum quarks and gluons
- Write low-energy effective Lagrangian, $\mathcal{L}_{\rm eff}$, in terms of light d.o.f.
  - **LOCAL** QFT
  - Corresponds to integrating out heavy d.o.f.
  - Constrained by symmetries of underlying theory
  - **Not renormalizable**—valid in limited energy range
  - Terms ordered by power counting in ratio of scales
- Justification: gives most general unitary S-matrix consistent with symmetries [Weinberg]
  - Gives results valid up to truncation error: $(m_\pi/m_p)^n$, $(a\Lambda_{\rm QCD})^n$
  - Unknown LECs (sometimes perturbatively calculable)

Chiral perturbation theory

- I will consider ChPT for pseudo-Goldstone bosons (PGBs): pions, kaons and eta
  - I will only touch briefly on extension to heavy meson or baryon sources, in the context of SU(2) ChPT with heavy kaons (next lecture)
- The story goes back to Nambu's realization that the lightness of pions compared to other mesons and baryons can be understood if they are PGBs of chiral symmetry, which brings us to.....

Chiral symmetry in QCD

- Fermionic part of Euclidean Lagrangian in matrix notation:

$$\boxed{\mathcal{L}_{QCD} = \overline{Q}_L \not{D} Q_L + \overline{Q}_R \not{D} Q_R + \overline{Q}_L M Q_R + \overline{Q}_R M^\dagger Q_L}$$

  - $Q^{tr} = (u, d, s)$, $\overline{Q}_{L,R} = \overline{Q}_{L,R}(1 \pm \gamma_5)/2$, $Q_{L,R} = [(1 \mp \gamma_5)/2] Q_{L,R}$
- In the massless limit, have $\mathcal{G} = SU(3)_L \times SU(3)_R$ symmetry:
  - $Q_{L,R} \to U_{L,R} Q_{L,R}$ and $\overline{Q}_{L,R} \to \overline{Q}_{L,R} U^\dagger_{L,R}$, with $U_{L,R} \in SU(3)_{L,R}$
- Also have Vector $U(1)$ (quark number) symmetry, but trivial
- Axial $U(1)$ broken by anomaly
- Add in mass term, e.g. $M = \mathrm{diag}(m_u, m_d, m_s)$, $m_q \neq 0$
  - axial transformations $U_L = U_R^\dagger$ broken
  - vector $SU(3)$ subgroup $(U_L = U_R)$ also broken, except if masses degenerate
- If treat M as complex "spurion" field then maintain full chiral symmetry
  - $M \to U_L M U_R^\dagger$, $M^\dagger \to U_R M^\dagger U_L$

Explicit breaking of Chiral symmetry

- Chiral symmetry is useful if M is **small**:
  - What is small? $m_q \ll \Lambda_{QCD} \sim 300$ MeV
  - More precise criterion in χPT: $m_{\pi,K,\eta} \ll \Lambda_\chi \equiv 4\pi f_\pi \approx 1200$ MeV
  - $(m_u + m_d)/2 \approx 4$ MeV $\Rightarrow SU(2)_L \times SU(2)_R$ is a good approximate symmetry
  - $m_s \approx 100$ MeV or $m_{K,\eta} \approx \Lambda_\chi/2 \Rightarrow SU(3)_L \times SU(3)_R$ is much less good
- Important question for lattice applications of chiral perturbation theory
  - Is m_s small enough that approximate chiral symmetry is useful to determine the quark mass dependence when $m_s^{\rm lat} \approx m_s$?
  - If not, then can only use chiral symmetry to guide extrapolations in m_u and m_d.

I.E. Can one use SU(3) ChPT or is one forced to use only SU(2) ChPT?

Spontaneous breaking of Chiral symmetry

- Vacuum breaks chiral symmetry
  - No parity doubling in spectrum: $m_N(P=+) \neq m_N(P=-)$
  - Lightness of π, K and η consistent with their being pseudo-Goldstone bosons (PGBs)
- Order parameter

$$\langle \bar{q} q \rangle = \langle (\bar{q}_L q_R + \bar{q}_R q_L) \rangle \sim \Lambda^3_{\rm QCD} \neq 0 \,, \quad q = u, d, s$$

  - Lattice simulations $\Rightarrow \langle \bar{q} q \rangle \neq 0$
  - Success of chiral perturbation theory
- Vector symmetry not spontaneously broken
  - If $m_u = m_d = m_s$ then $\langle \bar{u} u \rangle = \langle \bar{d} d \rangle = \langle \bar{s} s \rangle$
  - Based on experiment, and [Vafa-Witten] theorem

Symmetry breaking if M=0

- Condensate is LR flavor matrix:

$$\Omega_{ij} = \langle Q_{L,i,\alpha,c} \overline{Q}_{R,j,\alpha,c} \rangle \xrightarrow[\mathcal{G}]{} U_L \, \Omega \, U_R^\dagger$$

  - All choices of Ω_{ij} are equivalent: "vacuum manifold"
  - Unbroken vector symmetry $\Rightarrow$ $\Omega_{ij} = \omega \, \delta_{ij}$ is in manifold
  - $\omega \neq 0$ implies chiral symmetry breaking: $\omega = -\langle \bar{q} q \rangle > 0$

$$\underbrace{SU(3)_L \times SU(3)_R}_{\mathcal{G}} \longrightarrow \underbrace{SU(3)}_{\mathcal{H}}$$

- Goldstone's theorem: 8 broken generators $\Rightarrow$ 8 GBs (π, K, η)

$$\boxed{\text{This (massless QCD) is the theory around which chiral } \textit{perturbation} \text{ theory is developed}}$$

Turning on a (small) mass matrix M

- VEV of mass-term in Lagrangian leads to a potential for $\Omega_{ij} = \langle Q_{L,i,\alpha,c} \overline{Q}_{R,j,\alpha,c} \rangle$

$$\overline{Q}_L M Q_R + \overline{Q}_R M^\dagger Q_L \longrightarrow \mathcal{V} = -\mathrm{tr}(\Omega M^\dagger) - \mathrm{tr}(\Omega^\dagger M)$$

- If M diagonal and positive (conventional choice) minimizing V gives $\Omega_{ij} = \omega\, \delta_{ij}$

- Since $\Omega = \omega \underset{\mathcal{G}}{\longrightarrow} \omega U_L U_R^\dagger$ VEV breaks SU(3)$_A$ while SU(3)$_V$ is unbroken

- Pions, kaons and eta become massive pseudo-Goldstone bosons (PGBs), since symmetry is explicitly broken

- A "twisted mass" leads to equivalent physics (useful for "twisted-mass fermions")

$$M \to U_L M U_L \longrightarrow \Omega = \omega U_L^2$$

Constructing the EFT

- We have the correct ingredients for an EFT:
  - Separated scales $m_{\mathrm{PGB}} \ll \Lambda_\chi \sim m_\rho, m_{\mathrm{nucleon}}$
  - Maintain scale separation by considering $p_{\mathrm{GB}} \ll 1\mathrm{GeV}$
- Build EFT using **only GB fields, static sources, and spurions** (M)
  - Construct most general local Lagrangian consistent with symmetries
  - Non-renormalizable, many unknown LECs (low energy constants)
  - Gives most general unitary S-matrix consistent with symmetries [Weinberg]
- Order terms using power-counting
  - Small parameter is $p^2/\Lambda_\chi^2 \sim M/\Lambda_{\mathrm{QCD}}$

Representing Goldstone Bosons

- Conceptually most non-trivial step of construction:
  - EFT built from GBs (mesons), while QCD built from quarks
  - Choice of GB fields not unique (not discussed here)
- Use complex scalar field theory as guide ("Mexican hat"):

$$V = -\mu^2 \phi^\dagger \phi + \frac{\lambda}{2}(\phi^\dagger \phi)^2$$

  - Classical minimum $|\langle\phi\rangle| = v$, breaks symmetry $\mathcal{G} = U(1) \longrightarrow \mathcal{H} = 1$
  - Spectrum: GB (phase rotations) and massive field (radial excitations):

$$\phi(x) = v \exp[\rho(x)] \exp[i\theta(x)]$$

  - To obtain low-energy EFT (GB fields only) can either:
    - Integrate out $\rho(x)$ by hand, considering only correlators with external $\theta(x)$
  - Or:
    - Write down most general $\mathcal{L}_{\rm eff}$ consistent with symmetries

$$\mathcal{L}_{\rm eff} \sim c_2 \partial_\mu(e^{i\theta})\partial_\mu(e^{-i\theta}) + c_4 \partial_\mu(e^{i\theta})\partial_\mu(e^{-i\theta})\partial_\nu(e^{i\theta})\partial_\nu(e^{-i\theta}) + \ldots$$

      Can determine low-energy constants c_i by perturbative matching
    - No mass term since no invariants without derivatives: $(e^{i\theta}e^{-i\theta})^n = 1$

Exponential parametrization of GBs

- Lesson for EFT: parameterize excitations using full vacuum manifold
  - $U(1)$ theory: $\frac{\langle\phi\rangle}{v} = e^{i\theta} \longrightarrow e^{i\theta(x)}$
  - Note that any choice of $\langle\theta\rangle$ breaks symmetry:
    - Symmetry breaking is built in by use of angular variables, so Goldstones' theorem guarantees correct spectrum
- Corresponding choice for QCD is

$$\frac{\Omega_{ij}}{\omega} \equiv \frac{\langle Q_{L,i,\alpha,c}\overline{Q}_{R,j,\alpha,c}\rangle}{|\langle\bar{q}q\rangle|} \equiv \Sigma_{ij} \longrightarrow \Sigma_{ij}(x) \in SU(3)$$

- Tranforms under $\mathcal{G} = SU(3)_L \times SU(3)_R$ like Ω (i.e. linearly):

$$\Sigma(x) \underset{\mathcal{G}}{\longrightarrow} U_L \Sigma(x) U_R^\dagger$$

- Any VEV of Σ breaks $\mathcal{G}$ to $\mathcal{H} = SU(3) \Rightarrow$ desired symmetry breaking
- Can decompose into GB (pion) fields. Taking $\langle\Sigma\rangle = 1$ have:

$$\Sigma(x) = \exp\left(2i\pi^a(x)T^a/f\right), \qquad a = 1,8$$

- GB fields transform non-linearly

Building blocks for ChPT

- Ingredients are Σ, $\Sigma^\dagger$, M, $M^\dagger$ and external sources (discussed later)

  $$\Sigma \to U_L \Sigma U_R^\dagger\,, \qquad \Sigma^\dagger \to U_R \Sigma^\dagger U_L^\dagger\,, \qquad M \to U_L M U_R^\dagger\,, \qquad M^\dagger \to U_R M^\dagger U_R^\dagger$$

- Useful building blocks (noting $\Sigma\Sigma^\dagger = 1 = \Sigma^\dagger\Sigma$)

  LH: $L_\mu = \Sigma\partial_\mu\Sigma^\dagger = -\partial_\mu\Sigma\Sigma^\dagger = -L_\mu^\dagger \longrightarrow U_L L_\mu U_L^\dagger$

  LH: $M\Sigma^\dagger \to U_L(M\Sigma^\dagger)U_L^\dagger\,, \qquad \Sigma M^\dagger \to U_L(\Sigma M^\dagger)U_L^\dagger$

  RH: $R_\mu = \Sigma^\dagger\partial_\mu\Sigma = -\partial_\mu\Sigma^\dagger\Sigma = -R_\mu^\dagger \longrightarrow U_R R_\mu U_R^\dagger$

  RH: $M^\dagger\Sigma \to U_R(M^\dagger\Sigma)U_R^\dagger\,, \qquad \Sigma^\dagger M \to U_R(\Sigma^\dagger M)U_R^\dagger$

  [Derivatives only act on object immediately to right]

- Important property follows from $\det(\Sigma) = 1$:

  $$0 = \partial_\mu(\det\Sigma) = \partial_\mu(\exp\mathrm{tr}\ln\Sigma) = \det\Sigma\ \mathrm{tr}(\Sigma^{-1}\partial_\mu\Sigma) = -\mathrm{tr}(L_\mu)$$

  - Thus L_μ, R_μ are elements of Lie algebra $su(3)$

- Can often just use **LH** building blocks and enforce parity at end
  - If $\langle\Sigma\rangle = 1$, parity:

    $\pi(x) \to -\pi(x_P)$, $\Sigma(x) \leftrightarrow \Sigma^\dagger(x_P)$, $L_\mu(x) \leftrightarrow R^\mu(x_P)$, $M \to M^\dagger$

Putting the blocks together

- Remember the rule: construct most general local Lagrangian consistent with symmetries ($SU(3)_L$, $SU(3)_R$, Lorentz/Euclidean, C, P, T)
  - No derivatives: no terms since $\Sigma^\dagger\Sigma = 1$
  - One derivative: no terms by Lorentz invariance
  - Two derivatives:

    1. $\mathrm{tr}(L_\mu L_\mu) = -\mathrm{tr}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger) = \mathrm{tr}(R_\mu R_\mu)$

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  - No derivatives and one mass insertion:

    2. $\mathrm{tr}(M\Sigma^\dagger) + \mathrm{tr}(\Sigma M^\dagger)$

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  - No derivatives and one mass insertion:

2. $\mathrm{tr}(M\Sigma^\dagger) + \mathrm{tr}(\Sigma M^\dagger)$

  - Four derivatives:

3. $[\mathrm{tr}(L_\mu L_\mu)]^2$
4. $\mathrm{tr}(L_\mu L_\nu)\mathrm{tr}(L_\mu L_\nu)$
5. $\mathrm{tr}(L_\mu L_\mu L_\nu L_\nu)$ $\quad$ [not independent in $SU(2)$]
6. $\mathrm{tr}(L_\mu L_\nu L_\mu L_\nu)$ $\quad$ [not independent in $SU(2)$ or $SU(3)$]
7. Wess-Zumino-Witten term involving ε-tensor---gives $U(1)_A$ anomaly mediated processes, e.g. $\pi_0 \longrightarrow \gamma\gamma$. Not discussed here.

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- Remember the rule: construct most general local Lagrangian consistent with symmetries ($SU(3)_L$, $SU(3)_R$, Lorentz/Euclidean, C, P, T)
  - No derivatives: no terms since $\Sigma^\dagger \Sigma = 1$
  - One derivative: no terms by Lorentz invariance
  - Two derivatives:

    1. $\mathrm{tr}(L_\mu L_\mu) = -\mathrm{tr}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) = \mathrm{tr}(R_\mu R_\mu)$
  - No derivatives and one mass insertion:

    2. $\mathrm{tr}(M\Sigma^\dagger) + \mathrm{tr}(\Sigma M^\dagger)$
  - Four derivatives:

    3. $[\mathrm{tr}(L_\mu L_\mu)]^2$
    4. $\mathrm{tr}(L_\mu L_\nu)\mathrm{tr}(L_\mu L_\nu)$
    5. $\mathrm{tr}(L_\mu L_\mu L_\nu L_\nu)$ [not independent in $SU(2)$]
    6. $\mathrm{tr}(L_\mu L_\nu L_\mu L_\nu)$ [not independent in $SU(2)$ or $SU(3)$]
    7. Wess-Zumino-Witten term involving ε-tensor---gives $U(1)_A$ anomaly mediated processes, e.g. $\pi_0 \longrightarrow \gamma\gamma$. Not discussed here.
  - Two derivatives & one mass term (2 terms)
  - Two mass terms (3 terms)

Leading order chiral Lagrangian

- Will find that expanding in powers of $\partial^2 \sim M$ is appropriate
- At leading order have: [recall $\Sigma = \exp(2i\pi^a T^a/f)$]

$$\mathcal{L}^{(2)} = \frac{f^2}{4}\mathrm{tr}\left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right) - \frac{f^2 B_0}{2}\mathrm{tr}(M\Sigma^\dagger + \Sigma M^\dagger)$$

- Two (so far) unknown LECs: f and B_0 (both mass dimension 1)
  - Expect $f \sim B_0 \sim \Lambda_{\mathrm{QCD}}$
- Up to this stage, M is a complex spurion field. Now set to physical value:

$$M = \mathrm{diag}(m_u, m_d, m_s) = M^\dagger$$

  - Allowed us to expand about massless theory with exact chiral symmetry
- Can now determine VEV $\langle\Sigma\rangle$ by minimizing potential:

$$\mathcal{V}^{(2)} = -\frac{f^2 B_0}{2}\mathrm{tr}\left(M[\Sigma^\dagger + \Sigma]\right)$$

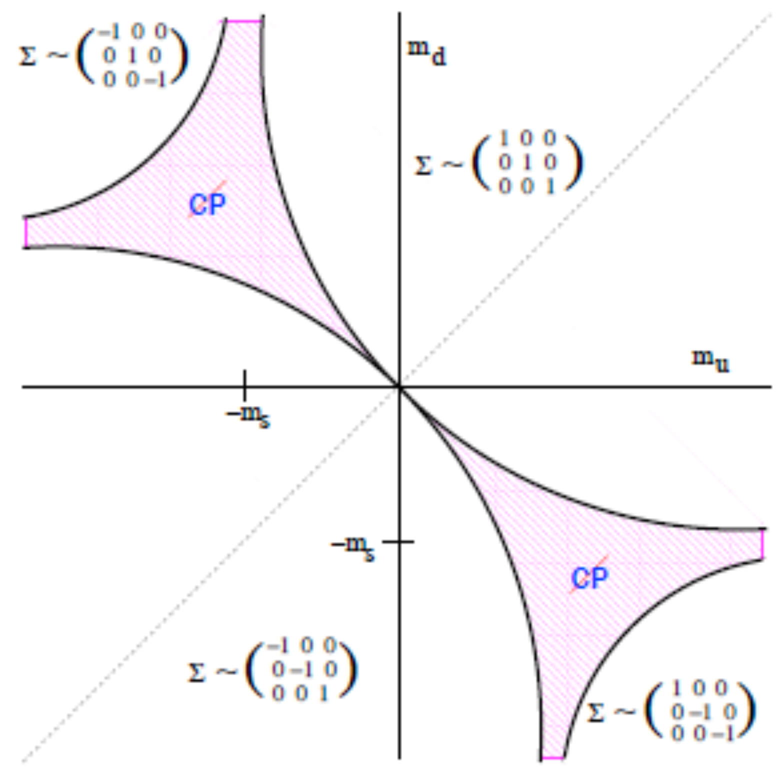
- If all $m_q > 0$, find $\langle\Sigma\rangle = 1$

Aside on vacuum structure

$$\mathcal{V}^{(2)} = -\frac{f^2 B_0}{2} \mathrm{tr}\left(M[\Sigma^\dagger + \Sigma]\right)$$

- For two flavors:
  - If we use $\langle\Sigma\rangle = \exp(i\theta\vec{n}\cdot\vec{\tau})$, then $\langle[\Sigma^\dagger + \Sigma]\rangle = 2\cos\theta \times 1$
  - Thus $\mathcal{V}^{(2)} \propto -\mathrm{tr}(M)\cos\theta$
  - So if $\mathrm{tr}M > 0$, $\langle\Sigma\rangle = 1$, while if $\mathrm{tr}M < 0$, $\langle\Sigma\rangle = -1$

- For three flavors, $\Sigma = -1$ not possible
  - Interesting phase structure if some $m_q < 0$ [Dashen,Creutz]
  - $m_u = 0$ is not special if $m_d \neq 0$: no subgroup of $SU(3)_L \times SU(3)_R$ is restored



$m_s > 0$ fixed

Leading order (P)GB properties

- Insert $\Sigma = \exp(2i\pi/f)$, with $\pi \equiv \pi^a T^a$, into leading order (LO) $\mathcal{L}$

$$
\begin{aligned}
\mathcal{L}^{(2)} &= \frac{f^2}{4}\mathrm{tr}\left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right) - \frac{f^2 B_0}{2}\mathrm{tr}(M[\Sigma^\dagger + \Sigma]) \\
&= \mathrm{tr}(\partial_\mu \pi \partial_\mu \pi) + 2B_0\ \mathrm{tr}(M\pi^2) \\
&\quad + \frac{1}{3f^2}\mathrm{tr}([\pi, \partial_\mu \pi][\pi, \partial_\mu \pi]) - \frac{2B_0}{3f^2}\mathrm{tr}(M\pi^4) + O(\pi^6)
\end{aligned}
$$

  - Choosing $\mathrm{tr}(T^a T^b) = \delta^{ab}/2$, kinetic term normalized correctly (f's cancel)
  - If $M = 0$, GB interactions all involve derivatives
  - In general $m^2_{\mathrm{PGB}} \propto M$
    - For degenerate quarks, $m_\pi^2 = 2B_0 m_q$
  - Sequence of non-renormalizable interactions involving even numbers of PGBs, size determined by f and $B_0 M$
    - $\Rightarrow$ LO χPT predictive: e.g. 6 pion interactions given by 4 pion term
  - WZW term leads to interactions involving odd numbers of pions (5, . . .)

LO ChPT predictions for PGB masses

- Determine physical particles using $U(3)_V$ $(\pi \to U_V\ \pi\ U_V^\dagger)$

$$\pi = \begin{pmatrix} \frac{\pi^0}{2} + \frac{\eta}{\sqrt{12}} & \frac{\pi^+}{\sqrt{2}} & \frac{K^+}{\sqrt{2}} \\ \frac{\pi^-}{\sqrt{2}} & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{12}} & \frac{K^0}{\sqrt{2}} \\ \frac{K^-}{\sqrt{2}} & \frac{\overline{K^0}}{\sqrt{2}} & -\frac{2\eta}{\sqrt{12}} \end{pmatrix}$$

- Inserting into $-2B_0 \mathrm{tr}(M\pi^2)$ find
  - Charged particle masses are simple: $m^2_{q_i q_j} = B_0(m_i + m_j)$, $i \neq j$

$$\Rightarrow \frac{m^2_{K^+} + m^2_{K^0}}{2m^2_{\pi^+}} = \frac{m_\ell + m_s}{2m_\ell} + \mathrm{EM} \approx 13 \qquad \left(m_\ell = \frac{m_u + m_d}{2}\right)$$

  - π^0 and η mix, but with small angle $\theta \sim (m_u - m_d)/m_s \ll 1$

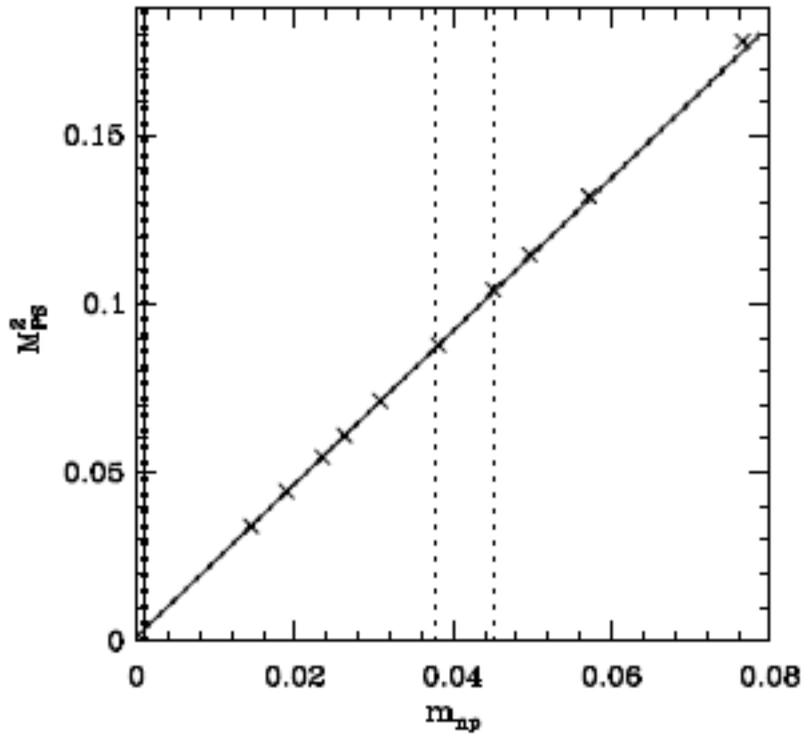
$$\begin{aligned} m^2_{\pi^0} &= m^2_{\pi^+} + O(\theta^2 m_K^2) + \mathrm{EM}\,, \\ \underbrace{m^2_\eta}_{(548\ \mathrm{MeV})^2} &= \underbrace{(2[m^2_{K^+} + m^2_{K^0}] - m^2_{\pi^+})/3}_{(566\ \mathrm{MeV})^2} + O(\theta^2 m_K^2) \end{aligned}$$

[Gell-Mann Okubo relation]

- Cannot determine quark masses from χPT since scale dependent
  - Always appear in combination $\chi \equiv 2B_0 M$ which I use below

Lessons for lattice (1)

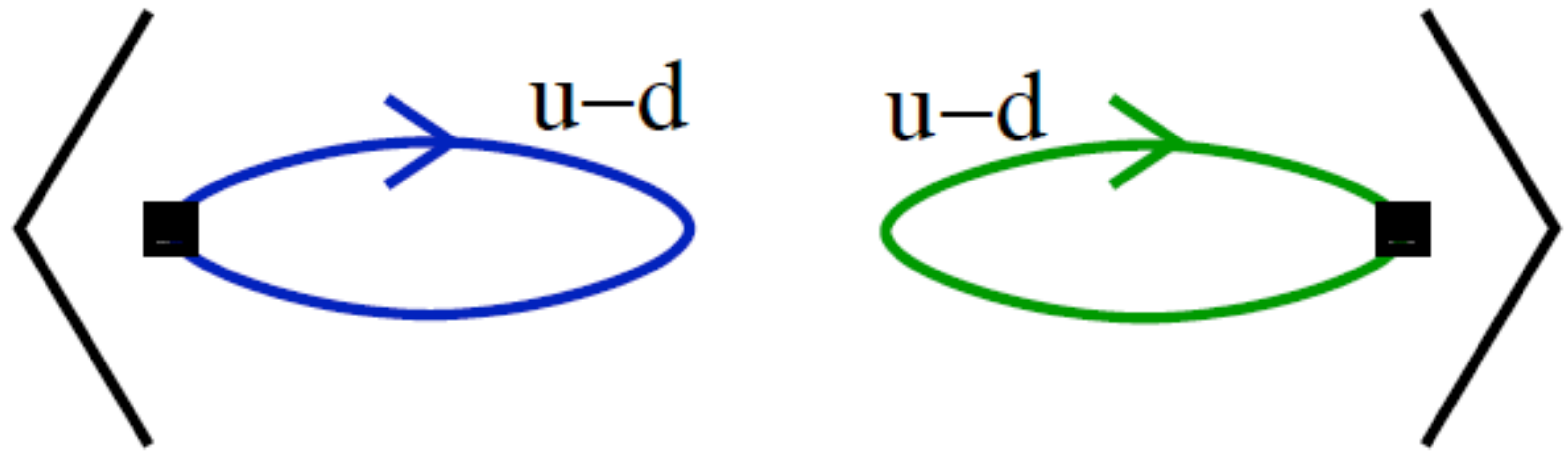
- LO χPT works to $\sim 10\%$ in GMO relation
  - Indeed, $m_{\pi^+}^2/m_q \sim \mathrm{const.}$ seen in all simulations (since 1983)
  - E.g. quenched Wilson fermions [Bhattacharya95]
    [vertical lines indicate $m_s^{\rm phys}$]



- Can vary m_q in simulations (more "knobs to turn" than in real QCD), and χPT describes dependence on quark masses **in terms of the physical LECs**

Lessons for lattice (2)

- **+** If simulate isospin limit $m_u = m_d$ then close to real QCD:
  - ▷ $m_u/m_d \sim 1/2$ does not lead to large isospin violations
  - ▷ Differences are suppressed by $(m_u - m_d)/m_s$ (PGBs) or by $(m_u - m_d)/\Lambda_{\rm QCD}$ (other hadrons)
- **-** Calculating isospin breaking effects (e.g. $m^2_{\pi^+} - m^2_{\pi^0}$) is hard
  - ▷ Quark mass contributions involve disconnected diagrams and are small



  - ▷ EM contributions are comparable and not easy to calculate
  - ▷ Recent progress using "reweighting" of isospin-symmetric configurations [Ishikawa et al., arXiv:1202.6018 & Aoki et al., arXiv:1205.2961]
  - ▷ Much recent progress in reducing errors in disconnected diagrams using "all-to-all" techniques

Power-counting in ChPT (M=0)

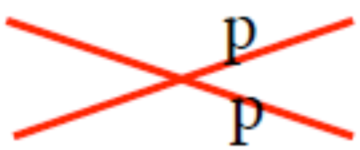
- How can a non-renormalizable theory be predictive?

$$\mathcal{L}^{(2)} \sim f^2 \mathrm{tr}(L_\mu L_\mu) \sim (\partial\pi)^2 + \frac{\pi^2(\partial\pi)^2}{f^2} + \dots$$

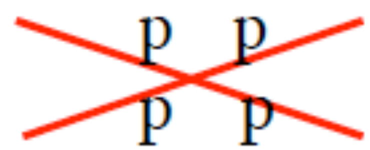
$$\mathcal{L}^{(4)} \sim L_{GL} \mathrm{tr}(L_\mu L_\mu)^2 + \dots \sim L_{GL}\left[\frac{(\partial\pi)^4}{f^4} + \frac{\pi^2(\partial\pi)^4}{f^6}\right]$$

  - L_{GL} are unknown dimensionless Gasser-Leutwyler coeffs

- Consider $\pi\pi$ scattering (with, say, dim. reg. to avoid power divergences):

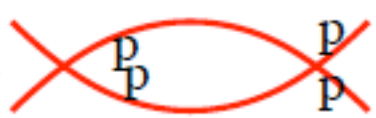
$\mathcal{L}^{(2)}_{\rm tree}$: 

$\sim \frac{p^2}{f^2}$

$\mathcal{L}^{(4)}_{\rm tree}$: 

$\sim L_{GL}\left(\frac{p^2}{f^2}\right)^2$

$\mathcal{L}^{(2)}_{\rm 1-loop}$: 

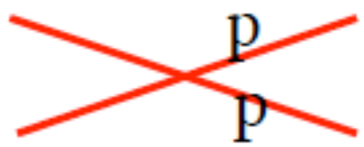
$\sim$ 

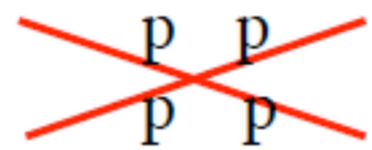
$\sim \left(\frac{p^2}{f^2}\right)^2 \frac{\ln(p^2/\mu^2)}{(4\pi)^2}$

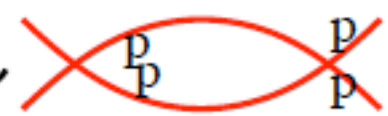
- Straightforward power-counting exercise (counting factors of f) $\Rightarrow$ have expansion in p^2/f^2 up to logs
  - LO: $\mathcal{L}^{(2)}_{\rm tree}$ ("trivial" to calculate)
  - NLO: $\mathcal{L}^{(4)}_{\rm tree} + \mathcal{L}^{(2)}_{\rm 1-loop}$ ("easy" to calculate)
  - NNLO: $\mathcal{L}^{(6)}_{\rm tree} + \mathcal{L}^{(4)}_{\rm 1-loop} + \mathcal{L}^{(2)}_{\rm 2-loop}$ (hard but done)

- For $M \neq 0$, $p^2/f^2 \longrightarrow (p^2 \text{ or } m^2_{\rm PGB})/f^2$

Power-counting in ChPT (M=0)

$\mathcal{L}^{(2)}_{\rm tree}$:  $\sim \frac{p^2}{f^2}$

$\mathcal{L}^{(4)}_{\rm tree}$:  $\sim L_{GL}\left(\frac{p^2}{f^2}\right)^2$

$\mathcal{L}^{(2)}_{1-\rm loop}$:  $\sim$  $\sim \left(\frac{p^2}{f^2}\right)^2 \frac{\ln(p^2/\mu^2)}{(4\pi)^2}$

- Theory is predictive up to truncation errors:
  - E.g. at LO, $\mathcal{A}(\pi\pi \to \pi\pi)$ predicted in terms of $f (= f_\pi)$, up to errors of relative size p^2/f^2
  - Only a finite number of diagrams and LECs at each order, so can always make predictions
  - Non-analytic behavior ("chiral logs") does not involve new LECs
    - ~ Determined by unitarity (2 particle cut)
  - Loops renormalize LECs: $L_{GL} \to L_{GL}(\mu)$

True expansion parameter?

- LEC's run with μ:
  - $dL_{GL}/d\ln(\mu) \approx 1/(4\pi)^2 \Rightarrow L_{GL}(2\mu) - L_{GL}(\mu) \approx 1/(4\pi)^2$
- So guess ("naive dimensional analysis"):
  - $L_{GL}(\mu \approx m_\rho) \approx 1/(4\pi)^2$
- Works well phenomenologically: $-1 \lesssim L_{GL}(4\pi)^2 \lesssim +1$
- Implies expansion parameter is p^2/Λ_χ^2, with $\Lambda_\chi = 4\pi f$
- For $M \neq 0$, $p^2/\Lambda_\chi^2 \longrightarrow (p^2 \text{ or } m_{\rm PGB}^2)/\Lambda_\chi^2$

Lessons for lattice (3)

+ Use χPT to extend reach of lattice to multiparticle processes
  - Calculate LECs from lattice simulations using simple physical quantities (e.g. masses)
  - Use χPT + LECs to determine multiparticle processes (scattering amplitudes, $\pi\pi \to 4\pi$, etc.) that are difficult or impossible to determine directly using simulations
    - Determining $\mathcal{A}(K \to \pi\pi)$ using unphysical, but more accessible, matrix elements [Rome-Southampton, Laiho-Soni] In disrepute(?)

- Always have truncation error when using χPT
  - Need to include NNLO terms (at least approximately) to determine NLO coefficients (L_{GL})
  - Fitting requires (approximate) NNNLO coefficients to work up to m_s^{phys} [MILC]

Technical aside: adding sources

- Matrix elements of V_μ, A_μ, S and P are phenomenologically interesting
- Incorporate in QCD using external sources (hermitian matrices)

  $\mathcal{L}_{\rm QCD} = \overline{Q}_L(i\not{D} - \gamma^\mu l_\mu)Q_L + \overline{Q}_R(i\not{D} - \gamma^\mu r_\mu)Q_R - \overline{Q}_L(s+ip)Q_R - \overline{Q}_R(s-ip)Q_L$

  - Switched to Minkowski space for the moment
  - s, p not new—rewriting of spurions $M = s + ip$, $M^\dagger = s - ip$
  - Obtain correlation functions in QCD by functional derivatives of $Z_{\rm QCD}(l_\mu, r_\mu, s, p)$
- Basic assumption of χPT: $Z_{\rm QCD}(l_\mu, r_\mu, s, p) = Z_\chi(l_\mu, r_\mu, s, p)$ for $p, m_{\rm PGB} \ll \Lambda_\chi$, up to truncation errors
- Functional derivatives of Z_χ give χPT result for correlation functions

$$\text{e.g.}\ \frac{\delta}{\delta l_\mu(x)}\frac{\delta}{\delta p(y)} \ln Z_\chi\Big|_{l=r=p=0,s=M} \sim \langle T[L^\mu(x)P(y)]\rangle$$

$$\Rightarrow \qquad f_\pi \propto \langle 0|L_\mu|\pi\rangle$$

Technical aside: adding sources

- How determine $Z_\chi(l_\mu, r_\mu, s, p)$?
  - Generalize spurion trick to **local** $SU(3)_L \times SU(3)_R$ symmetry
  - $\mathcal{L}_{\rm QCD}$ invariant if l, r_μ transform as gauge fields:
    $l_\mu \to U_L l_\mu U_L^\dagger + i U_L \partial_\mu U_L^\dagger$, $r_\mu \to U_R r_\mu U_R^\dagger + i U_R \partial_\mu U_R^\dagger$
  - s, p transform as before: e.g. $(s+ip) \to U_L (s+ip) U_R^\dagger$
- Z_{QCD} invariant (up to anomalies) $\Rightarrow$ Z_χ invariant (up to anomalies)
- $\Rightarrow \mathcal{L}_\chi$ invariant [Gasser-Leutwyler]
  - Can be accomplished using covariant derivatives: $\partial_\mu \to D_\mu$
    e.g. $D_\mu \Sigma = \partial_\mu \Sigma - i l_\mu \Sigma + i \Sigma r_\mu \to U_L (D_\mu \Sigma) U_L^\dagger$
  - Normalization of l, r_μ terms fixed
  - Remainder of enumeration as before (except $D_\mu M$ now allowed)
  - Convenient to introduce $\chi = 2B_0(s+ip) = 2B_0 M$
    - In general χ is a matrix source
    - But also use notation $\chi_q = 2B_0 m_q$

LO chiral Lagrangian including sources

- At LO (back to Euclidean space):

$$\mathcal{L}^{(2)} = \frac{f^2}{4}\mathrm{tr}\left(D_\mu \Sigma D_\mu \Sigma^\dagger\right) - \frac{f^2}{4}\mathrm{tr}(\chi\Sigma^\dagger + \Sigma\chi^\dagger)$$

  - Using $\delta/\delta l_\mu(x)|_{l=r=p=0,s=M}$ can "match" currents with QCD:

$$\overline{Q}_L \gamma_\mu T^a Q_L \simeq (if^2/2)\mathrm{tr}(T^a \Sigma \partial_\mu \Sigma^\dagger) = -(f/2)\partial_\mu \pi^a + \ldots$$

    $\Rightarrow$ at LO, $f = f_\pi \approx 93$ MeV

  - Using $\delta/\delta s(x)|_{l=r=p=0,s=M}$ can relate condensate to B_0:

$$\overline{Q}Q \simeq -(f^2 B_0/2)\mathrm{tr}(\Sigma + \Sigma^\dagger) = -N_f f^2 B_0 + O(\pi^2)$$

    $\Rightarrow$ at LO, $\langle \overline{q}q \rangle = -f^2 B_0$ [Gell-Mann–Oakes–Renner]

    - Only using lattice can one determine B_0

NLO chiral Lagrangian

- At NLO have 10 LECs and 2 "high-energy coefficients":

$$
\begin{aligned}
\mathcal{L}^{(4)} \quad = \quad & -L_1\,\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger)^2 - L_2\,\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger)\mathrm{tr}(D_\mu\Sigma D_\nu\Sigma^\dagger) \\
& + L_3\,\mathrm{tr}(D_\mu\Sigma D_\mu\Sigma^\dagger D_\nu\Sigma D_\nu\Sigma^\dagger) \\
& + L_4\,\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{tr}(\chi^\dagger\Sigma + \Sigma^\dagger\chi) + L_5\,\mathrm{tr}(D_\mu\Sigma^\dagger D_\mu\Sigma)[\chi^\dagger\Sigma + \Sigma^\dagger\chi]) \\
& - L_6\left[\mathrm{tr}(\chi^\dagger\Sigma + \Sigma^\dagger\chi)\right]^2 - L_7\left[\mathrm{tr}(\chi^\dagger\Sigma - \Sigma^\dagger\chi)\right]^2 - L_8\,\mathrm{tr}(\chi^\dagger\Sigma\chi^\dagger\Sigma + \mathrm{p.c.}) \\
& + L_9\, i\mathrm{tr}(L_{\mu\nu}D_\mu\Sigma D_\nu\Sigma^\dagger + p.c.) + L_{10}\,\mathrm{tr}(L_{\mu\nu}\Sigma R_{\mu\nu}\Sigma^\dagger) \\
& + H_1\,\mathrm{tr}(L_{\mu\nu}L_{\mu\nu} + p.c.) + H_2\,\mathrm{tr}(\chi^\dagger\chi)
\end{aligned}
$$

- L_i are "Gasser-Leutwyler coefficients"
  - Fundamental parameters of QCD, akin to hadron mass ratios
  - A subset can be determined experimentally to good accuracy
  - A different subset is straightforward to determine on the lattice
- $H_{1,2}$ give contact terms in correlation functions
- $L_{\mu\nu} = \partial_\mu\ell_\nu - \partial_\nu\ell_\mu + i[\ell_\mu, \ell_\nu]$
- At NNLO there are 90 LECs and 4 HECs! [Bijnens et al]