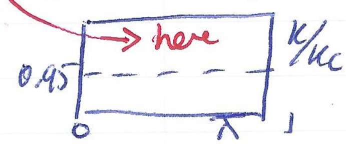
Trivialitysaga: final stage – RG equations expansion & matching w/ hopping param$_\wedge$

Recall where we are: using P.T. we can calculate

$$\Gamma_R^{(n,e)} = Z^{n/2} Z_\theta^e \, \Gamma_{\text{finite}}^{(n,e)}$$

order by order & express them as finite functions of g_R & m_R up to corrections suppressed by a^2 when $m_R = \overline{m}_R a$, $k_j = \overline{k}_j a$ and $a \to 0$ w/ $\overline{m}_R, \overline{k}_j$ fixed.

This means we are necessarily working (in terms of bare parameters κ, λ) near to the critical line. Called the "scaling region".



We have 3 renorm. conditions

$$-\Gamma_R^{(2)}(k) = m_R^2 + k^2 + O(k^4) \quad \text{(determines } m_R \text{ \& } Z)$$

$$-\Gamma_R^{(4)}(0,0,0) = g_R \quad (\Rightarrow g_R)$$

$$\Gamma_R^{(2,1)}(0,0) = 1 \quad (\text{determines } Z_\theta)$$

These conditions can be used throughout the κ, λ plane, not only in the scaling region.

Thus we can calculate them in the hopping param expansion & use various tricks to obtain accurate results at $\kappa/\kappa_c = 0.95$.

N.B. We defined $\Gamma^{(n,e)}$ using "cont" fields $\tilde{\phi}_M$. However we could just as well use the "lattice" fields $\phi_M = \tilde{\phi}_M/\sqrt{2\kappa}$. The resulting $\Gamma_R^{(n,e)}$ will be the same, since Z & Z_θ absorb extra $\sqrt{2\kappa}$ factors when imposing renorm. conditions.

Aside: relation of $\underbrace{\chi_2, \mu_2 \ \& \ \chi_4}$ to $\underbrace{m_R, Z_R \ \& \ g_R}$. ($Z_\theta$)

convenient to calculate in hopping param expansion (as in HW)

needed to match to renormalized PT

$$G_2(k) = \langle \sum_n e^{ik\cdot n} \phi_n \phi_0 \rangle$$

$$= \underbrace{\langle \sum_n \phi_n \phi_0 \rangle}_{\chi_2} + i \cancel{\langle \sum_n k\cdot n \phi_n \phi_0 \rangle}$$

vanishes by "parity" $n_\mu \to -n_\mu$ for each μ separately

$$- \frac{1}{2} k_\mu k_\nu \underbrace{\langle \sum_n n_\mu n_\nu \phi_n \phi_0 \rangle}_{} + \dots$$

$\propto \delta_{\mu\nu}$ by parities & rot^n symmetry

$$\Rightarrow = \frac{\delta_{\mu\nu}}{4} \langle \sum_n n^2 \phi_n \phi_0 \rangle$$

$$\text{So } G_2(k) = \chi_2 - \frac{k^2}{8} \mu_2 \qquad + O(k^4)$$

$$\Rightarrow \Gamma^{(2)}(k) = - G_2(k)^{-1} = -\frac{1}{\chi_2}\left(1 + \frac{\mu_2 k^2}{8\chi_2} + \dots\right)$$

$$= -\frac{\mu_2}{8\chi_2^2}\left(\frac{8\chi_2}{\mu_2} + k^2 + \dots\right)$$

$$\equiv -\frac{1}{Z}\left(m_R^2 + k^2 + \dots\right)$$

$$\Rightarrow \boxed{Z = \frac{8\chi_2^2}{\mu_2}} \quad \& \quad \boxed{m_R^2 = \frac{8\chi_2}{\mu_2}}$$

(This is for the ϕ_n fields.)

indep. of normalization of fields.

$$\boxed{g_R = -Z^2\,\Gamma^{(4)}_{(0;0,0)}} = -Z^2\left[G_2^{-1}(0)\right]^4 \chi_4$$

connected 4-pt correlator at $k_j = 0$
(No 1PI subtraction needed since Z_2 symm. unbroken.)

$$= -\frac{64\,x_2^4}{\mu_2^2}\,\frac{1}{x_2^4}\,\chi_4$$

$$= \boxed{-\frac{64\,\chi_4}{\mu_2^2}}$$

Finally to obtain Z_θ use

$$\frac{\partial}{\partial K}\underbrace{\Gamma^{(2)}_{(0)}}_{\left(-\frac{1}{x_2}\right)} = \Gamma^{(2,1)}_{(0,0)} = \frac{1}{Z Z_\theta} \quad \text{from (8.13)}$$

$$\Rightarrow \frac{\left(\frac{\partial x_2}{\partial K}\right)}{x_2^2} = \frac{\mu_2}{8x_2^2}\,\frac{1}{Z_\theta} \Rightarrow \boxed{Z_\theta = \frac{\mu_2}{8\,\frac{\partial x_2}{\partial K}}}$$

Again, for ϕ_n fields (not $\tilde{\Phi}_n$ — see (8.12)).

RG eqns, β-fcn, anomalous dims etc

Obtain using Callan–Symanzik eq. (not much discussed in, e.g., Srednicki, but good approach when using mass-dependent renorm.)

– important tool in QFT

$$\frac{\partial}{\partial k}\Gamma_R^{(n,e)}(k_1,\dots k_{n+e-1})\Big|_\lambda = \underbrace{\left[\frac{\partial m_R}{\partial k}\Big|_\lambda \frac{\partial}{\partial m_R} + \frac{\partial g_R}{\partial k}\Big|_\lambda \frac{\partial}{\partial g_R}\right]}_{\text{since } \Gamma_R \text{ only depend on } g_R, m_R}\Gamma_R^{(n,e)}$$

$$= \frac{\partial}{\partial k} Z^{n/2} Z_\theta^e\, \Gamma^{(n,e)}\Big|_\lambda$$

$$= \left[\frac{n}{2}\frac{\partial Z/\partial k|_\lambda}{Z} + e\,\frac{\partial Z_\theta/\partial k|_\lambda}{Z_\theta}\right]\Gamma_R^{(n,e)} + \underbrace{Z^{n/2}Z_\theta^e \underbrace{\frac{\partial \Gamma^{(n,e)}}{\partial k}}_{\Gamma^{(n,e+1)}(k_1,\dots,k_{n+e-1},0)}}_{\frac{1}{Z_\theta}\Gamma_R^{(n,e+1)}}$$

Multiply by $\frac{m_R}{\partial m_R/\partial k}$ & combine

$$\Rightarrow \underbrace{\frac{m_R}{\partial m_R/\partial k}\frac{1}{Z_\theta}}_{\varepsilon m_R^2}\Gamma_R^{(n,e+1)}$$

$$= \left[\frac{\partial}{\partial \ln m_R} + \underbrace{m_R \frac{\partial g_R/\partial k}{\partial m_R/\partial k}}_{\beta}\frac{\partial}{\partial g_R} - n\,\underbrace{\frac{m_R \frac{\partial \ln Z}{\partial k}}{2\,\partial m_R/\partial k}}_{\gamma} - e\,\underbrace{\frac{m_R}{\partial m_R/\partial k}\frac{\partial \ln Z_\theta}{\partial k}}_{\delta}\right]\Gamma_R^{(n,e)}$$

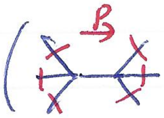
C-S eq

N.B. Except for $n=0$, $e=2$ where there is an extra term due to renorm. condition for $\Gamma_R^{(0,2)}$. Will skip.

to get engineering dim. correct → $\underbrace{a^{(n+2e-4)}\,\Gamma_R^{(n,e)}}_{\Gamma_{as}^{(n,e)}}$

- In the scaling region, $a^{(n+2e-4)}\,\Gamma_R^{(n,e)}$ depend only on g_R & $\bar{k}_j$ & $\bar{m}_R = m_R/a$ (up to a^2 corrections which we ignore)

  but not on cutoff $1/a$!

  e.g. $n=6, e=0$ $\quad a^2\,\Gamma_R^{(6,0)} = a^2\Big($  $+ \dots \Big)$

$$\sim \frac{a^2 g_R^2 + \dots}{m_R^2 + p^2} \approx \frac{g_R^2}{\bar{m}_R^2 + \bar{p}^2} + \dots$$

- Multiply C-S by a^{n+2e-4}, get eq. for Γ_{as}:

$$\varepsilon\, \bar{m}_R^2\, \Gamma_{as}^{(n,e+1)} = \Big[\frac{\partial}{\partial \ln \bar{m}_R} + \beta \frac{\partial}{\partial g_R} - n\gamma - e\delta\Big]\,\Gamma_{as}^{(n,e)} \leftarrow \text{dimensionfull}$$

- $\Rightarrow \beta, \gamma, \delta$ & ε must themselves depend only g_R & $\bar{m}_R$ (but not on $\bar{k}_j$ from their defns).

But these quantities are all dimensionless
$\Rightarrow$ cannot depend on $\bar{m}_R$ (since a is not allowed)

Thus depend only on g_R. (in scaling region)

[Away from scaling region can still calculate

$$\beta = m_R \frac{\partial g_R/\partial k|_\lambda}{\partial m_R/\partial k|_\lambda} = \beta(g_R, m_R) = \beta(k,\lambda)$$

e.g. using hopping param. exp & then check that answer agrees with that obtained using P.T.]

Thus find that

$$m_R \frac{\left.\frac{\partial g_R}{\partial K}\right|_\lambda}{\left.\frac{\partial m_R}{\partial K}\right|_\lambda} = \left.\frac{\partial g_R}{\partial \ln m_R}\right|_\lambda = \beta(g_R)$$

indep of m_R

can show that holding λ constant is the same as holding g_0 constant to all orders in PT

We found (8.11) that

$$g_R = g_0 + \frac{3}{2}\frac{g_0^2}{16\pi^2}\left[\ln m_0^2 + \text{const.}\right]$$

$= m_R^2$ at this order

$$\Rightarrow \frac{\partial g_R}{\partial \ln m_R} = 3\frac{g_R^2}{16\pi^2} + O(g_R^3)$$

can replace g_0 with g_R at this order.

This is a version of the usual β-fcn
[usually defined using dim reg & $\frac{\partial}{\partial \ln \mu}$]

LW find

$$\beta(g_R) = g_R\left[3\frac{g_R}{16\pi^2} - \frac{17}{3}\left(\frac{g_R}{16\pi^2}\right)^2 + 26.9\left(\frac{g_R}{16\pi^2}\right)^3\right]$$

first two terms are scheme indep
(Exercise: show this)

value for LW scheme

Similarly $\gamma(g_R) = \frac{1}{2} \left.\frac{\partial \ln Z}{\partial \ln M_R}\right|_{\lambda_0 \text{ or } g_0}$

$$= \frac{1}{12}\left(\frac{g_R}{16\pi^2}\right)^2 + 0.1407\left(\frac{g_R}{16\pi^2}\right)^3 + \ldots$$

No g_R term (noted above)

universal

scheme dep.

& $\delta(g_R) = \left.\frac{\partial \ln Z_\theta}{\partial \ln M_R}\right|_{\lambda_0 \text{ or } g_0}$

$$= -\frac{g_R}{16\pi^2} + \frac{5}{6}\left(\frac{g_R}{16\pi^2}\right)^2 - 3.77\left(\frac{g_R}{16\pi^2}\right)^3 + \ldots$$

What about $\varepsilon(g_R)$? Not independent.

C-S eq for $n=2$, $\ell=0$ & $K_j=0$

$$\varepsilon M_R^2 \xcancel{\Gamma_R^{(2,1)}(0,0)}^{\,1} = \left[\frac{\partial}{\partial \ln M_R} + \cancel{\beta\frac{\partial}{\partial g_R}}^{\,0} - 2\gamma\right](-M_R^2)$$

$$= -(M_R^2)\, 2(1-\gamma)$$

$$\Rightarrow \boxed{\varepsilon = 2(\gamma - 1) = -2 + O(g_R^2)\ldots}$$

Since $\varepsilon = \frac{1}{M_R\, \partial M_R/\partial K}$ this gives the useful relation

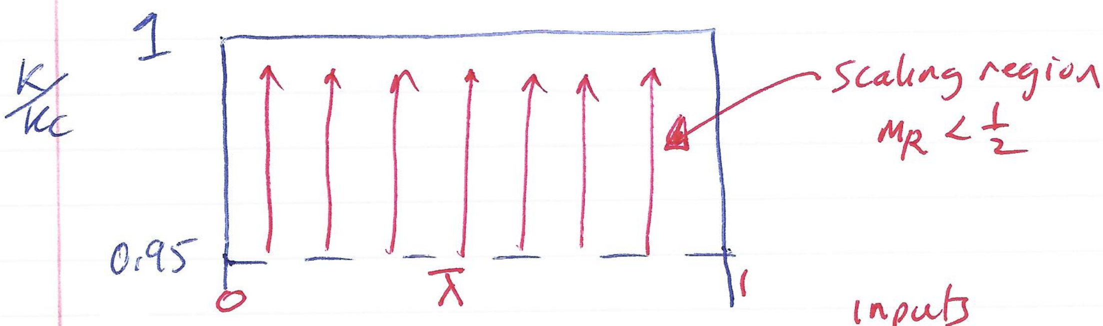
$$\boxed{\left.\frac{\partial M_R}{\partial K}\right|_\lambda = \frac{1}{2(\gamma-1) M_R Z_\theta}}$$

N.B. Since $\gamma \ll 1$, $\frac{\partial M_R}{\partial K} < 0$

since can calculate the RHS in hopping param. exp (and extend in the scaling region using above eqs) can determine $M_R(K)$ on lines of constant λ

Final approach:

Recall: use hopping param exp. to determine κ_c and then m_R, g_R, Z & Z_θ as functions of λ for $\kappa/\kappa_c = 0.95$



all at $\kappa/\kappa_c = 0.95$

0	λ	∞	
0.65	m_R (monotonic)	0.49	outputs of hopping param expansion ↓
0	g_R (monotonic)	41	
3.5	Z (not monotonic)	6.9	
0.0144	Z_θ (not monotonic)	0.21	

Use these as inputs for running m_R, g_R, Z & Z_θ vertically (fixed λ) towards κ_c

Key points:
- In scaling region, since $m_R = \bar{m}_R a < \frac{1}{2}$ cutoff effects are relatively small
- Since g_R is "small", $\frac{g_R}{16\pi^2} \lesssim \frac{1}{3}$, 3 loop running is accurate

LW quantify errors from these statements to be percent level.

Solutions to RG eqs:

$$\left.\frac{\partial g_R}{\partial (\ln m_R)}\right|_\lambda \approx \beta_1 \, g_R^2 \qquad \left(\beta_1 = \frac{3}{16\pi^2}\right)$$

Same form as QCD but opposite sign

(From now on leave $|_\lambda$ implicit)

$$\frac{d g_R}{g_R^2} = \beta_1 \ln m_R = -d\left(\frac{1}{g_R}\right)$$

$$\Rightarrow \frac{1}{g_R} = -\beta_1 \ln m_R + \text{const.}$$

or $$\frac{1}{g_R} - \frac{1}{g_{Ri}} = -\beta_1 \ln\left(\frac{m_R}{m_{Ri}}\right)$$

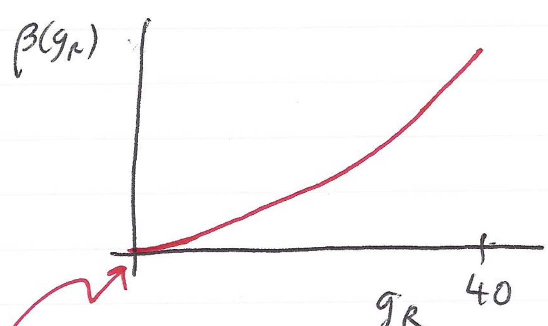
$\Rightarrow$ as $m_R \to 0$, $g_R \to 0$

Triviality!

initial conditions.

At 2-loop $$\frac{m_R}{m_{Ri}} = e^{\frac{1}{\beta_1}\left(\frac{1}{g_{Ri}} - \frac{1}{g_R}\right)} \left(\frac{g_{Ri}}{g_R}\right)^{\frac{\beta_2}{\beta_1^2}} \underbrace{(1 + O(g_R))}_{\text{3-loop}}$$

Sign of $\beta(g_R)$ stays positive for $g_R \leq 41$ even though $\beta_2 < 0$.



Increases monotonically.

$g_R = 0$ is IR fixed pt.

IR (not UV) since g_R is a physical long-distance quantity.

Other RG eqs.

$$2\gamma \approx 2\gamma_2 g_R^2 = \frac{\partial \ln Z}{\partial \ln M_R}$$

Since $M_R = M_R(g_R)$ can write $Z = Z(g_R, M_R) = Z(g_R)$

$$\frac{\partial \ln Z}{\partial g_R} = \frac{2\gamma}{\beta} \simeq \frac{2\gamma_2}{\beta_1} \frac{\cancel{g_R^2}}{\cancel{g_R^2}}$$

$$\Rightarrow \ln Z = \ln Z_i + \frac{2\gamma_2}{\beta_1}(g_R - g_{Ri})$$

$$\Rightarrow Z = Z_i \Big(1 + \underbrace{\frac{2\gamma_2}{\beta_1}(g_R - g_{Ri})}_{\text{small and slowly varying.}} \Big)$$

so $Z \sim$ constant

$$\delta \approx \delta_1 g_R = \frac{\partial \ln Z_\theta}{\partial \ln M_R}$$

$$\Rightarrow \frac{\partial \ln Z_\theta}{\partial g_R} \simeq \frac{\delta_1}{\beta_1 g_R} = -\frac{1}{3 g_R}$$

$$\Rightarrow \ln \frac{Z_\theta}{Z_{\theta i}} = -\frac{1}{3} \ln \frac{g_R}{g_{Ri}}$$

$$\Rightarrow Z_\theta = Z_{\theta i} \left(\frac{g_{Ri}}{g_R} \right)^{1/3} (1 + O(g_R))$$

diverges as $g_R \to 0$

negative

Finally, (from 9.7) $\frac{\partial m_R^2}{\partial K} = \frac{-1}{(1-\gamma) Z_\theta} \approx \frac{-1}{Z_{\theta_i}} \left(\frac{g_R}{g_{Ri}}\right)^{\frac{1}{3}}$

small correction

$\to 0$

$g_R \to 0$

Can show that the solution is

integration const.

$$m_R^2 = \frac{(K_c - K)}{Z_{\theta_i}} \left(\frac{g_R}{g_{Ri}}\right)^{1/3} \left[1 + O(g_R)\right]$$

i.e. treat $g_R^{1/3}$ as a constant at leading order

Mean field + logarithmic corrections!

[This form is used to help determine K_c from hopping param. series.]

In practice, LW integrate 3 loop eqs numerically to map out m_R, g_R vs K, λ in scaling region.

Collecting the scaling results

Let $\tau = \frac{\kappa - \kappa_c}{\kappa} = 1 - \frac{\kappa_c}{\kappa}$; scaling region is $0 < \tau < \frac{1}{20}$

$$g_R \underset{M_R \to 0}{\approx} \frac{1}{\beta(-\ln M_R^2)} \; ; \qquad m_R^2 \propto \tau \, g_R^{1/3}$$

$$\Rightarrow -\ln m_R^2 \approx -\ln \tau$$

Keeping dominant term.

$$\Rightarrow \; g_R \propto (-\ln \tau)^{-1}$$

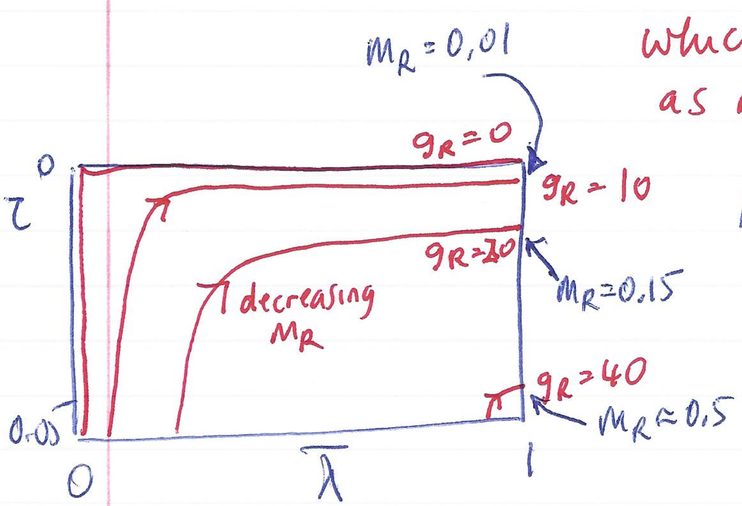
$$m_R^2 \propto \tau (-\ln \tau)^{-\frac{1}{3}}$$

$$Z \propto \text{constant}$$

$$Z_\phi \propto (-\ln \tau)^{1/3}$$

Bottom line : even for $\lambda, g_0 \to \infty$ renormalized coupling g_R remains finite & perturbatively small at boundary of scaling region

$\Rightarrow$ can use as input for RG eqs which tell us that $g_R \to 0$ as $M_R \to 0$.

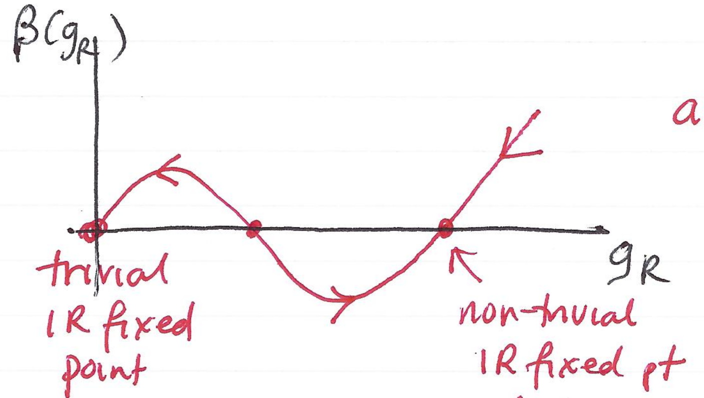


Lines of constant $g_R > 0$ ("constant physics") do not reach the critical line, but end for $M_R > 0$

$\Rightarrow \bar{m}_R a = \frac{\bar{m}_R}{\Lambda}$ finite.

Post mortem

We might have hoped that



arrows are flows for decreasing M_R

at large g_R outside perturbative domain.

Would have led to a more complicated phase diagram.

But it didn't happen.

LW II extended the analysis to the broken phase in the scaling region, i.e. $\kappa \gtrsim \kappa_c$, using renormalized PT. In the broken phase one defines $g_R = 3\frac{M_R^2}{V_R^2}$ where V_R is the renormalized VEV.

One finds $g_R \to 0$ when $M_R \to 0$

& that, for $M_R \lesssim \frac{1}{2}$, $g_R^{max} \approx 31 \Rightarrow M_R \lesssim 3V_R$

This is the upper bound on the "Higgs" mass — if the cutoff is lowered further then $M_R \sim 1$ ($\bar{M}_R \sim 1/a$) & there are huge (unphysical) cut-off effects, requiring new physics to cancel.