Triviality saga – almost final stage: renormalized pert. thy

* Story so far: using 14th order hopping param series (and same techniques/tricks mentioned below) LW could determine $\kappa_c(\lambda)$ to high accuracy.

• Convenient variables to use in plots are $\kappa/\kappa_c(\lambda)$

and
$$\bar{\lambda} = -\frac{1}{2}\left(\frac{\gamma_4 - 3\gamma_2^2}{\gamma_2^2}\right)$$

[Recall $\gamma_n = \langle \phi^n \rangle_{1\text{-site}} = \dfrac{\int d\phi\, e^{-s(\phi)}\, \phi^n}{\int d\phi\, e^{-s(\phi)}}$

where $s(\phi) = \phi^2 + \lambda(\phi^2-1)^2$]

This is proportional to the result for g_R you obtained in the extended Random-walk approx

It has the properties:

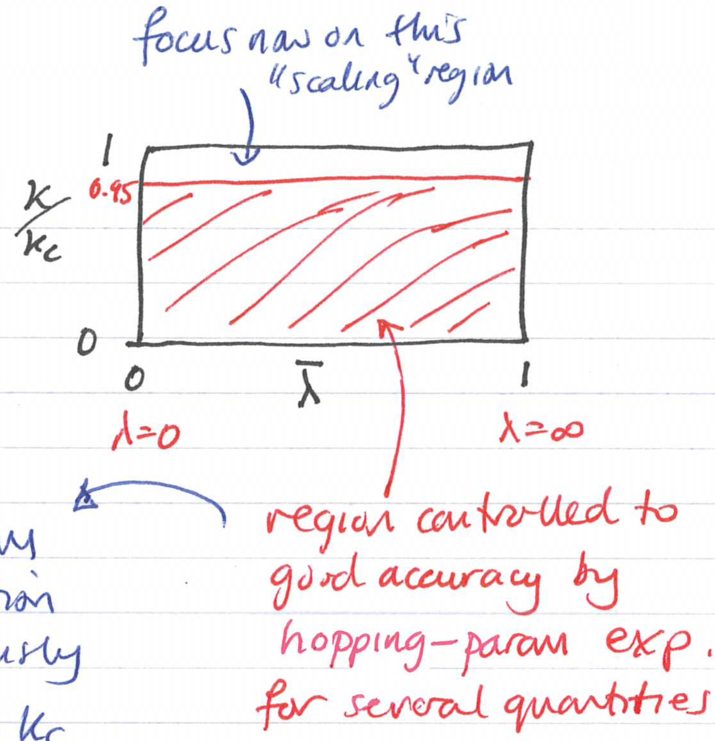
$\bar{\lambda} = 3\lambda + O(\lambda^2)$ $\quad\lambda \ll 1$

$\bar{\lambda} \to 1$ $\quad\lambda \to \infty$ $\quad(\gamma_n \to 1)$

$\bar{\lambda}$ monotonically increases as a fcn of λ.

So LW use $\bar{\lambda}$ instead of, say, $\lambda/1+\lambda$

- So plots become



In fact, hopping param exp. controls calculation of K_c, which obviously requires working up to K_c, which is the radius of convergence.

But it is easier to determine the radius of convergence of a series without being able to calculate other quantities accurately up to that radius.

- Key point is (as we will see) that within the

$$\tau = \frac{K_c - K}{K_c} = 1 - \frac{K}{K_c} \lesssim 0.05 \quad \text{strip},$$

one is close enough to the continuum limit

$$(\Lambda = 1/a \gtrsim 2\, m_{renorm,\,phys})$$

that the theory is described well by continuum pert. thy.

- This leads us to ...

Perturbation theory (PT) w/ a lattice regulator

Bottom line: all methods of P.T. you learned in the ctm, w/ dim reg or Pauli Villars or some other regulator, carry over:

i.e. generating fcns of connected corr. fcns; vertex fcns; renormalization; Callan–Symanzik eqs, beta fcn, anom. dims...

Some details will differ – we will not use counterterms explicitly (although we could), and we will use a mass-dependent renorm. scheme.

Aside – the technical details of the proof of renormalizability of the ϕ^4 theory have been extended to lattice regularization by T. Reisz, Nucl. Phys. B318 (1989) 417 & refs therein

We will go back to the original, more familiar, form of the action

$$S = \sum_{n,\mu} \frac{(\tilde{\phi}_{n+\hat{\mu}} - \tilde{\phi}_n)^2}{2} + \frac{m_0^2}{2}\sum_n \tilde{\phi}_n^2 + \frac{g_0}{4!}\sum_n \tilde{\phi}_n^4$$

For which we know the free propagator

$$\langle \tilde{\phi}_n \tilde{\phi}_p \rangle_{g_0=0} = \int_k \frac{e^{ik\cdot(n-p)}}{m_0^2 + \hat{k}^2}$$

$\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4}$

dim'less bare mass

Means $\sum_\mu \hat{k}_\mu^2$

$\boxed{\hat{k}_\mu = 2\sin\frac{k_\mu}{2}}$

incoming momenta (momentum conserving δ-fcn removed).

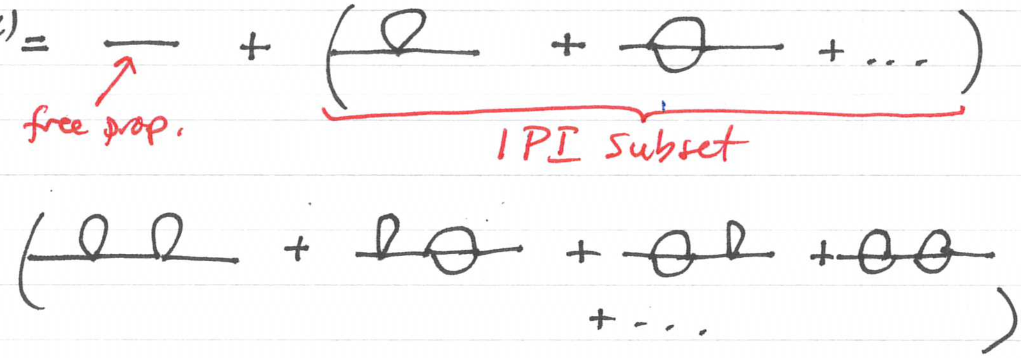
Let $\Gamma^{(n)}(k_1, k_2, \ldots k_{n-1})$ be the n-point amputated vertex function, 1PI (1-particle irreducible)

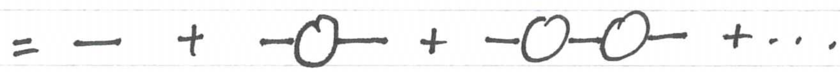
E.g. $\Gamma^{(2)}(k) = -\,G_2(k)^{-1}\; G_2(k)\; G_2(k)^{-1}$

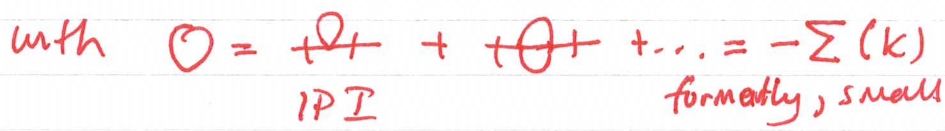
Sign convention. amputation; 2 pt fcn i.e. propagator $\sum_n e^{-ik\cdot n} \langle \tilde{\phi}_n \tilde{\phi}_0 \rangle_{g_0 \neq 0}$; amputation

$$= -\,G_2^{-1}(k).$$

Now (usual Feynman rules) $G_2(k) =$







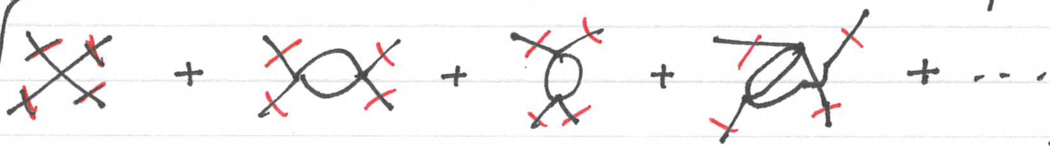
$$= \frac{1}{m_0^2 + \hat{k}^2}\left(1 - \Sigma(k)\frac{1}{m_0^2+\hat{k}^2} + \left[-\Sigma \frac{1}{m_0^2+\hat{k}^2}\right]^2 + \ldots\right)$$

$$= \frac{1}{m_0^2 + \hat{k}^2 + \Sigma(k)}$$

$$\Rightarrow \boxed{\Gamma^{(2)}(k) = -\left(m_0^2 + \hat{k}^2 + \Sigma(k)\right)}$$

- Another important example: 4-pt vertex

$$\Gamma^{(4)}(k_1, k_2, k_3) = -G^{-1}(k_1)\,G^{-1}(k_2)\,G^{(4)}(k_1, k_2, k_3)_{1PI}\,G^{-1}(k_3)\,G^{-1}(-k_1-k_2-k_3)$$

$= -\Big[$  $\Big]$

- Note that, in the symmetric phase (which is where we are working) $\Gamma^{(odd)} = 0$ by Z_2 symm.

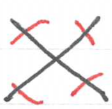
- Feynman rules are like in the continuum, except for $|k_\mu| \le \pi$ and that Dirac delta-fcns are periodic

$$(2\pi)^4\,\delta^4_{per}(k-q) = \sum_n e^{i(k-q)\cdot n} \qquad \left(\to \Omega\,\delta_{k,q} \text{ in finite volume}\right)$$

sets $k_\mu = q_\mu$ mod 2π

Also, because we have e^{-S} $\qquad S = \ldots + \sum_n g_0 \frac{\tilde{\phi}_n^4}{4!}$

the vertex comes with a minus sign

 $= -\frac{g_0}{\cancel{4!}}\,\cancel{4!}$ ← contractions

Aside: generating fcnals in Euclidean space

(Mainly to get signs clear)

- $W[j] = \ln Z[j]$; $Z[j] = \int [D\phi]\, e^{-S + \sum_n j_n \tilde{\phi}_n}$

W generates connected correlation fcns, e.g.

ordinary derivs here → $\left.\dfrac{\partial^2 W[j]}{\partial j_p \partial j_n}\right|_{j=0} = \langle \tilde{\phi}_p \tilde{\phi}_n \rangle_{conn.} = \langle \tilde{\phi}_p \tilde{\phi}_n \rangle - \langle \hat{\phi}_p \rangle \langle \tilde{\phi}_n \rangle$
$\equiv G^{(2)}_{pn}$

- (*) $\langle \tilde{\phi}_n \rangle_j \equiv \bar{\varphi}_n = \dfrac{\partial}{\partial j_n} W[j]$ expectation value in presence of source j_n (which can depend on n)

Invert: $j_n = j_n[\bar{\varphi}]$ ← j_n depends, in general, on all $\bar{\varphi}_p$.

[N.B. $\bar{\varphi} = 0 \Leftrightarrow j = 0$ if no symmetry breaking]

- Generator of amputated 1PI vertex fcns

$$\Gamma[\bar{\varphi}] = \left[W - \sum_n j_n \bar{\varphi}_n \right]\Big|_{j = j[\bar{\varphi}]}$$

$$\Rightarrow \quad \frac{\partial \Gamma}{\partial \bar{\varphi}_n} = -j_n$$

$$\Rightarrow \quad \left.\frac{\partial^2 \Gamma}{\partial \bar{\varphi}_n \partial \bar{\varphi}_m}\right|_{\bar{\varphi}=0} \equiv \Gamma^{(2)}_{nm} = -\left.\frac{\partial j_m}{\partial \bar{\varphi}_n}\right|_{\bar{\varphi}=0}$$

$$\underbrace{\Gamma^{(2)}_{nm} G^{(2)}_{mp}}_{\text{implicit sum}} = -\left.\frac{\partial j_m}{\partial \bar{\varphi}_n}\right|_{\bar{\varphi}=0} \left.\frac{\partial^2 W}{\partial j_m \partial j_p}\right|_{j=0} = \left.\frac{-\partial^2 W}{\partial \bar{\varphi}_n \partial j_p}\right|_{\substack{\bar{\varphi}=0 \\ j=0}} \overset{(*)}{=} -\delta_{nm}$$

so $\Gamma^{(2)} = -G^{(2)^{-1}}$ as claimed.

1-loop result for $\Gamma^{(2)}$

$$\Sigma^{(1)}(k) = -\,\text{-}\!\!\bigcirc\!\!\text{-} = -\underbrace{\frac{(-g_0)}{2}}_{\text{symmetry factor}} \underbrace{\int_k \frac{1}{M_0^2+\hat{k}^2}}_{\text{dim'less integral } I(M_0)}$$

$$\Rightarrow -\Gamma^{(2)} = M_0^2 + \hat{k}^2 + \underbrace{\frac{g_0}{2} I(M_0)}_{\text{mass renorm: } \delta M_0^2 \text{ since indep of } \hat{k}} + O(g_0^2)$$

- $I(M_0)$ is a finite function of M_0
  e.g. $I(0) = 0.154933\ldots$

- Corresponding continuum integral is quadratically UV divergent:

  $M_0 = \overline{m}_0 a$; $k = \overline{k} a$ — $a \to 0$ w/ $\overline{m}_0, \overline{k}$ fixed

  LW notation for "phys"

  cut off at $\overline{k} \sim 1/a \equiv \Lambda$

$$\Rightarrow I \to a^2 \underbrace{\int \frac{d^4\overline{k}}{(2\pi)^4} \frac{1}{\overline{m}_0^2 + \overline{k}^2 + O(\overline{k}^4 a^2)}}_{\sim \Lambda^2 \sim 1/a^2} \quad \Rightarrow I \to \text{const.}$$

N.B. On lattice we are not afraid of quadratic divergences!

- M_0^2 dependence? $\quad \dfrac{dI}{dM_0^2} = \displaystyle\int_k \frac{-1}{(M_0^2+\hat{k}^2)^2}$

  UV & IR log. divergent $\Rightarrow \ln(\overline{m}_0/(1/a)) = \ln \overline{m}_0 a = \ln M_0$

Can show (see Smit) that, for $\overset{\text{dim'less}}{\downarrow} M_0 \ll 1$

$$I(M_0) = \underset{\text{quad. div.}}{\underset{\uparrow}{I(0)}} + \underset{\substack{\text{log div.} \\ \ln(\bar{M}_0/\Lambda)}}{\underset{\uparrow}{\frac{M_0^2 \ln M_0^2}{16\pi^2}}} + \underset{\substack{-0.0303\ldots \\ \text{another constant.}}}{\underset{\uparrow}{C_2}} M_0^2 + \underbrace{\underset{\text{up to logs}}{\underset{\uparrow}{O(M_0^4)}}}_{\substack{\text{ignore since} \\ \text{suppressed by } a^2 \\ \text{in cont. limit.}}}$$

So, at 1-loop, bare lattice mass replaced by

$$M_0^2 + \delta M_0^2 = M_0^2 + \frac{g_0}{2}\left[I(0) + \frac{M_0^2}{16\pi^2}\ln M_0^2 + C_2 M_0^2\right] + O(a^4; g_0^2)$$

or

$$\bar{m}_0^2 + \delta\bar{m}_0^2 = \bar{M}_0^2 + \frac{g_0}{2}\left[\frac{I(0)}{a^2} + \frac{\bar{M}_0^2}{16\pi^2}\ln(\bar{M}_0 a)^2 + C_2 \bar{M}_0^2\right] + O(a^2; g_0^2)$$

shows quad. div. explicitly.

N.B. No wave fcn renorm. at 1-loop – needs k dependence, comes in first at 2 loop 

N.N.B. In continuum dim. reg. analysis usually work at $M_0 = 0$ in which case "tadpole"

 vanishes – no quad. divs.

Renormalized mass

(but mass dependent)

One fairly standard def^{n} of renormalized mass is the pole mass: Wick rotate $k_E^2 \to -k_M^2$ and go to the pole in $G^{(2)}$ or zero in $\Gamma^{(2)}$.

More convenient choice here is

$$\boxed{-\Gamma^{(2)} \underset{k\to 0}{=} \frac{m_R^2 + k^2 + O(k^4)}{Z}}$$

i.e. based on small k behavior of $\Gamma^{(2)} = G^{(2)^{-1}}$

N.B. We continue to use dim'less quantities, although we are imagining working close to the ctm limit by having $m_R, |k| \ll 1$

So, at 1-loop.

$$m_R^2 = m_0^2 + \frac{g_0}{2}\left[I(0) + \frac{m_0^2 \ln m_0^2}{16\pi^2} + C_2 m_0^2\right] + \dots$$

$$Z = 1 + O(a^2, g_0^2)$$

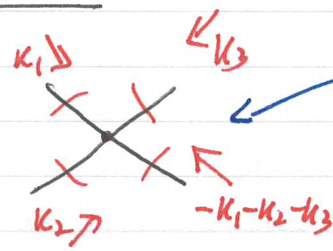
$\Rightarrow$ For $m_R \to 0$, need to FINE TUNE

$$m_0^2 \to \approx -\frac{g_0}{2} I(0)$$

(as in phase diagram sketch on 5.4)

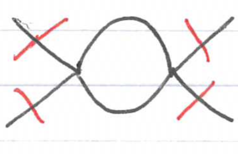
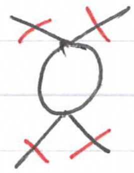
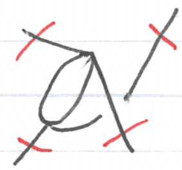
but that is NOT a PROBLEM – just do it by hand.

1-loop result for $\Gamma^{(4)}$

$-\Gamma^{(4)}(k_1,k_2,k_3) =$ 

$-\frac{g_0}{\cancel{4!}}\cancel{4!}$

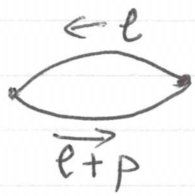
contractions

 $+$  $+$  $+\ O(g_0^3)$

s-channel t-channel u-channel

$$\Rightarrow \Gamma^{(4)} = g_0 - \frac{g_0^2}{2}\Big[\underset{s}{J(M_0,k_1+k_2)} + \underset{t}{J(M_0,k_1+k_3)} + \underset{u}{J(M_0,k_2+k_3)}\Big] + \dots$$

symm factor

$J(M_0,p) =$  $= \displaystyle\int_\ell \frac{1}{(M_0^2+\hat{\ell}^2)(M_0^2+\widehat{\ell+p}^{\,2})}$

log divergent UV & IR

$$\overset{\text{Smit}}{=} -\frac{1}{16\pi^2}\int_0^1 dx\,\ln\left[a^2\left(\bar{M}_0^2 + x(1-x)\,\bar{p}^2\right)\right] - \left(C_2 + \frac{1}{16\pi^2}\right) + O(a^2)$$

Feynman param.

treat $p = \bar{p}a$ as small (unlike ℓ which is integrated over) $-\pi < \ell_\mu \le \pi$

same as above

N.B. If send $\bar{p}^2 < -4\bar{M}_0^2$ by inverse Wick rot^n, pick up imaginary part from unitary cut.

N.N.B. Very similar to dim. reg. result which has $a^2 \to 1/\mu^2 \times$ factor

Renormalization condition for $\Gamma^{(4)}(k_1, k_2, k_3)$:

Recall $-\Gamma^{(2)} \xrightarrow[M_R,\, k \to 0]{} \frac{M_R^2 + k^2}{Z} = G^{(2)\,-1}$

so $G^{(2)} \xrightarrow[M_R,\, k \to 0]{} \frac{Z}{M_R^2 + k^2}$ = Fourier transform $\langle \tilde{\varphi}_n \, \tilde{\varphi}_0 \rangle$

$\Rightarrow \frac{\tilde{\varphi}_n}{\sqrt{Z}} = \tilde{\varphi}_{n,R}$. yeilds a renormalized propagator F.T $\langle \tilde{\varphi}_{n,R} \, \tilde{\varphi}_{0,R} \rangle \to \frac{1}{M_R^2 + k^2}$

$\Rightarrow$ 4-pt fcn $\sim \langle \hat{\varphi} \, \hat{\varphi} \, \hat{\varphi} \, \hat{\varphi} \rangle$ must be multiplied by $\frac{1}{(\sqrt{Z})^4}$ to be renormalized.

Since $\Gamma^{(4)} \sim (G^{-1})^4 \langle \hat{\varphi} \, \hat{\varphi} \, \hat{\varphi} \, \hat{\varphi} \rangle$

must multiply by $(\sqrt{Z})^4$ to renormalize:

$$(\sqrt{Z})^4 \, \Gamma^{(4)} = \Gamma_R^{(4)} = \underbrace{(Z \, G^{-1})^4}_{-(M_R^2 + k^2)} \underbrace{\frac{1}{(\sqrt{Z})^4} \langle \hat{\varphi} \, \hat{\varphi} \, \hat{\varphi} \, \hat{\varphi} \rangle}_{\text{renormalized}}$$

Renorm. condition (def^n of renormalized coupling)

$$-\Gamma_R^{(4)}(0,0,0) \equiv g_R$$

zero external momentum

$$\text{Thus} \quad g_R = g_0 - \overset{s+t+u}{3}\!\!\Big/2\; g_0^2 \left[\frac{-1}{16\pi^2} \ln \underset{\overline{M_0}a}{M_0^2} - \left(C_2 + \frac{1}{16\pi^2}\right) \right]$$

$$+ O(a^2, g_0^3).$$

Key result – basic reason for triviality

- For fixed g_0, as $M_0 = \overline{M_0}\, a \to 0$, $g_R \downarrow$
- Equivalently, to hold g_R fixed as $M_0 \to 0$, $g_0 \uparrow$

(Opposite to gauge theory: $g_0(a) \to 0$ as $a \to 0$ in order to hold g_R fixed).

- When g_0 get's too large, can't obviously trust PT so need more work.

1-loop renormalized 4-pt vertex fcn.

$$-\Gamma_R^{(4)}(k_1,k_2,k_3) = g_R - \frac{g_R^2}{2} \left\{ \frac{-1}{16\pi^2} \int_0^1 dx \ln\left[\frac{m_R^2 + x(1-x)(k_1+k_2)^2}{m_R^2} \right] \right.$$

$$\left. + \; t \;\&\; u \text{ permutations} \right\}$$

$$+ O(g_R^3, a^2)$$

where I have used $m_0^2 = m_R^2 + O(g)$; $g_0 = g_R(1+O(g))$ to rewrite $\Gamma_R^{(4)}$ in terms of g_R & m_R.

N.B. Finite fcn independent of cut-off.
(since in argument of $\ln[\;]$ can change $m_R \to \overrightarrow{m_R}$, $k_j \to \overrightarrow{k_j}$)

Renormalizability

$$\Gamma_R^{(n)} = (Z)^{n/2}\,\Gamma^{(n)}$$ are finite functions of g_R & m_R only, to all orders in PT, up to corrections $\propto$ powers of a^2

This is true when expand $m_R = \overline{m}_R a$ & $k_j = \overline{k}_j a$ and take $a \to 0$

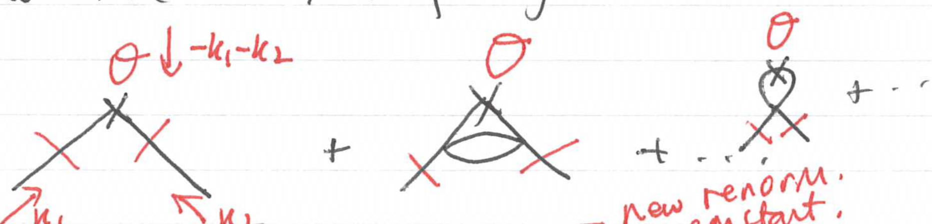
Renormalizability is a highly non-trivial result!

LW extend discussion by adding in corr^{n} fcns involving "hopping" operator $\mathcal{O}_n = \sum_\mu \phi_n(\phi_{n+\hat\mu} + \phi_{n-\hat\mu})$

in terms of which $S = \sum_n \left(s(\phi_n) - K\,\mathcal{O}_n \right)$.

$\Gamma^{(n,e)}$ is 1PI vertex fcn w/ n φ's & e $\mathcal{O}$'s amputated ONLY on φ legs.

e.g. $\Gamma^{(2,1)}_{(k_1,k_2)} =$



new renorm. constant.

Renormalization: $\Gamma_R^{(n,e)} = Z^{n/2}\, Z_{\mathcal{O}}^{e}\, \Gamma^{(n,e)}$

finite fcn of g_R & m_R only

action in "κ, λ" form

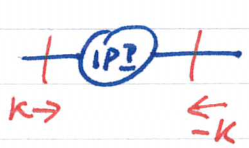
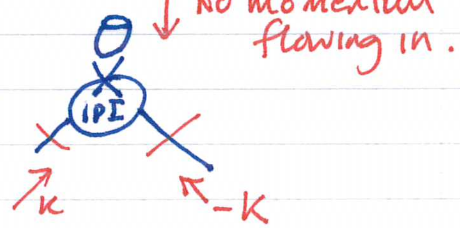
<u>Useful result:</u>

Since $\displaystyle e^{-S} = \underbrace{e^{-\sum_n s(\phi_n)}\, e^{\kappa \sum_n \theta_n}}$

$$\Rightarrow \frac{\partial}{\partial \kappa} \Gamma^{(n)}_{(k_1, \ldots k_{n-1})}\Big|_{\lambda} = \Gamma^{(n,1)}_{(k_1, \ldots, k_{n-1}, -k_1 - k_2 \ldots - k_{n-1})}$$

fixed λ

insertion of θ at zero momentum

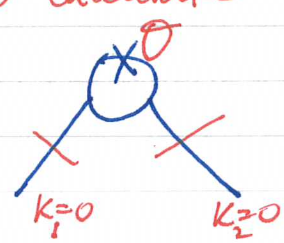
e.g. $\frac{\partial}{\partial \kappa}$  $\Big|_{\lambda}$ = 

By calculating the RHS learn about κ dependence of vertices on LHS

Renormalization condition used to determine Z_θ:

$$\Gamma_R^{(2,1)}{}_{(0,0)} = Z\, Z_\theta\, \Gamma^{(2,1)}_{(0,0)} = 1$$

(1) calculate



(2) already know

(3) Determine $Z_\theta = \frac{1}{8}\left(1 + O(g_0)\right)$

log divergent