Another application of the hopping parameter expansion — propagator in "random walk" approx

- gives an alternative derivation of K_c in mean-field approx which also shows that the particle mass vanishes at K_c

[Based on Smit's discussion, but expanded.]

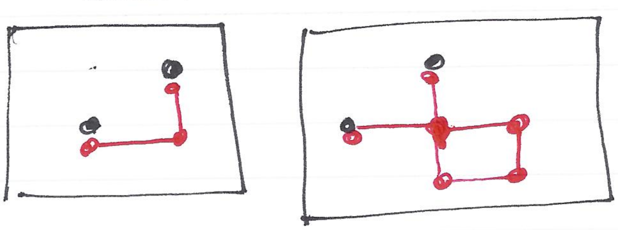
We aim for an approximate calculation of the propagator $G_{n0} = \langle \phi_n \phi_0 \rangle$ keeping some terms to all orders in K.

To set this up, consider examples of contribs to

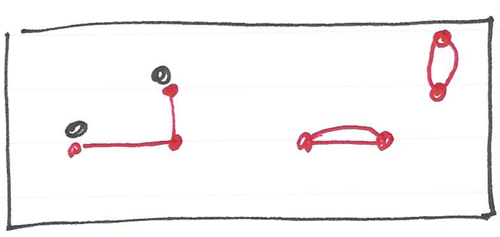
$$G_{(1100)0} = \frac{\prod_n \int_{\phi_n} e^{-S(\phi_n)} \, e^{2K \sum_{<>} \phi_{n_1}\phi_{n_2}} \, \phi_{(1100)} \phi_0}{\prod_n \int_{\phi_n} e^{-S(\phi_n)} \, e^{2K \sum_{<>} \phi_{n_1}\phi_{n_2}}}$$

- we expand numerator & denom. in powers of K as before & the multiplicity factors are unchanged
- Diagrams giving non-zero contributions

Numerator

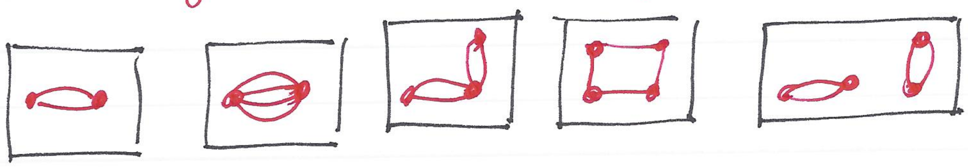


"connected" diagrams (no volume factors)

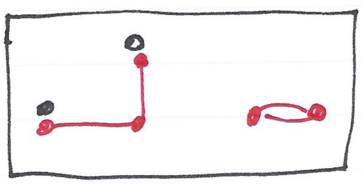
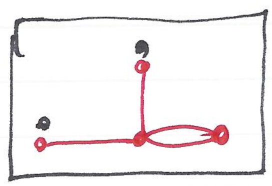


"disconnected" diagrams — with factors of Ω, Ω^2 etc. except for excluded volumes.

- Denominator $= Z$: discussed earlier
  - only "disconnected" diagrams

 etc.

- Such diagrams cancel between numerator & denom, except for excluded volumes, e.g.

in  cannot sum  freely since  etc came with different γ_n factors.

- Now we make an approximation that allows us to, in some sense, completely cancel disconnected contribs
  - we take the free field (Gaussian) approx for γ_n

$$\Rightarrow \gamma_4 = \langle \phi^4 \rangle_1 \approx 3 \langle \phi^2 \rangle_1^2 = 3\,\gamma_2^{\,2}$$

holds exactly if $\lambda = 0$

\# of "contractions"

& $\gamma_6 \approx 5 \cdot 3\ \gamma_2^{\,3}$

etc.

- In this approx, we can recast the hopping param expansion for the propagator as a random walk from, say, n to 0.

To see this, proceed order by order in κ.

- To simplify calculation, rescale numerator & denominator of G_{n0} by z_0^{Ω}, i.e. choose measure s.t. $\int_\phi e^{-s(\phi)} = 1$

At $O(\kappa^0)$ only contribution is from $n=0$ and is

$n=0$ $\quad \delta_{n0}\, \gamma_2$ $\qquad$ (no hops)

At $O(\kappa^1)$ only contrib is from $n = \pm\hat{\mu}$:

n, 0 $\quad 2\kappa\, \gamma_2^{\,2} \left(\sum_\mu \delta_{n,\hat{\mu}} + \delta_{n,-\hat{\mu}} \right)$

- arrow is for later convenience (counting paths from n to 0)
- does not impact evaluation of γ_n' factors or link multiplicities

- Useful to introduce hopping matrix

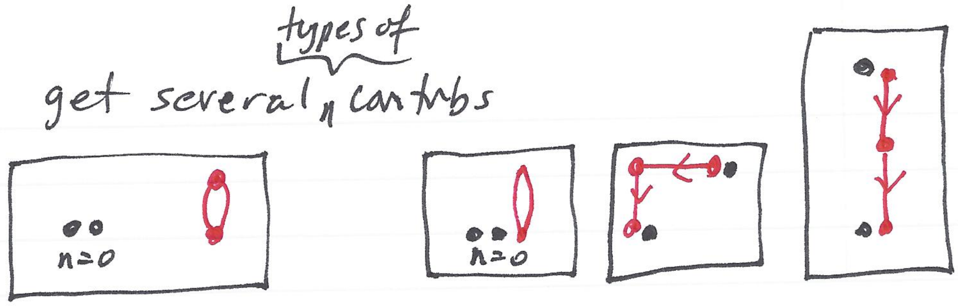
$$H_{np} = \sum_\mu \left(\delta_{n,p+\hat{\mu}} + \delta_{n,p-\hat{\mu}} \right)$$

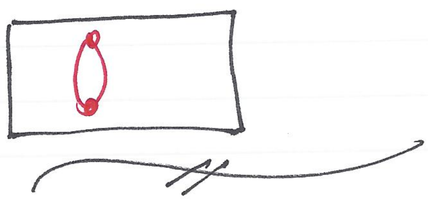
hops to nearest neighbor w/ unit step & unit weight

Then $O(\kappa^1)$ contrib to prop. is

$$2\kappa\, \gamma_2^{\,2}\, H_{n0}$$

At $O(k^2)$ get several types of contribs

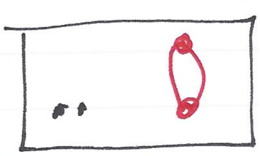
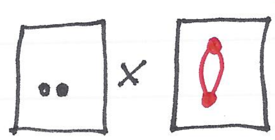
Numerator: 

Denom: 

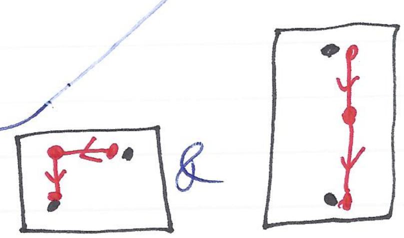
Focus on  $\Rightarrow (2k)^2 \delta_{n0} \frac{\gamma_2 \gamma_4}{2!}$ ← link multiplicity

add to 3

Make Gaussian approx: $\approx (2k)^2 \delta_{n0} \gamma_2^3 \left(\frac{1}{2!} + \frac{2}{2!}\right)$

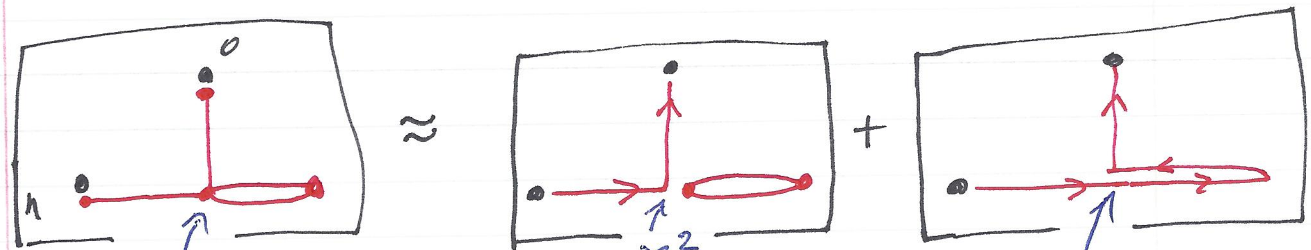
This combines with  to give 

i.e. factorizes out denom. contrib with no excluded volume

This combines with types 

to give $(2k)^2 \gamma_2^3 \, (H^2)_{n0}$

Another example at $O(\kappa^4)$



$\gamma_4 \approx 3\gamma_2^2$

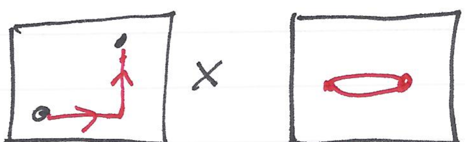
γ_2^2

one contraction

$\gamma_2^2 \cdot 2$

cancels $\frac{1}{2!}$ from multiplicity on link

Contribution to



cancels term from Z

Contribution to four hop term

$(2\kappa)^4 \gamma_2^5 (H^4)_{n0}$

Plausible that all orders result is (in Gaussian approx.)

$$\boxed{\langle \phi_n \phi_0 \rangle = \sum_{\ell=0}^{\infty} (2\kappa)^\ell \gamma_2^{\ell+1} (H^\ell)_{n0}}$$

random walk with each step weighted by $2\kappa\gamma_2$

Now H is an hermitian matrix which can be diagonalized

$$H = \mathcal{U}^{-1} H_D \, \mathcal{U} \qquad H^{\ell} = \mathcal{U}^{-1} H_D^{\ell} \, \mathcal{U} \qquad \mathcal{U}^{-1}\mathcal{U} = 1$$

unitary diagonal
$\mathcal{U}^{-1} = \mathcal{U}^{+}$

$$(H_D)_{p_1 p_2} = h_{p_1} \delta_{p_1 p_2}$$

(diagonalized by Fourier transforming)

$$\Rightarrow \mathcal{U} G \mathcal{U}^{-1} = \gamma_2 \sum_{\ell=0}^{\infty} (2k\gamma_2 H_D)^{\ell}$$

$$= \frac{\gamma_2}{1 - 2k\gamma_2 H_D}$$

geometric series converges for $2k\gamma_2 h_p^{max} < 1$

$$\Rightarrow \mathcal{U} G^{-1} \mathcal{U}^{-1} = \frac{1 - 2k\gamma_2 H_D}{\gamma_2}$$

$$\Rightarrow G^{-1} = \frac{1 - 2k\gamma_2 \mathcal{U}^{-1} H_D \mathcal{U}}{\gamma_2} = \frac{1 - 2k\gamma_2 H}{\gamma_2}.$$

(which is to say that we didn't need to do the diagonalization to sum the series).

- Note that, in this approximation, the finite radius of convergence of the K expansion is manifest (assuming h_p^{max} is finite, as we'll shortly see)

Form of G? We can piggy back on earlier work.

Recall a free lattice scalar FT has action (2.6)

$$S = \frac{1}{2} \phi_n \hat{M}_{np} \phi_p$$

$$M_{np} = \mathbb{1}_{np}(m_0^2 + 8) - H_{np} = (G^{-1})_{np}.$$

We have exactly this form up to rescalings:

$$G^{-1}_{\substack{\text{random}\\ \text{walk}}} = 2K\left(\frac{1}{2K\gamma_2} - H\right)$$

$$\Rightarrow \quad G^{r.w.}_{n0} = \frac{1}{2K} \int_k \frac{e^{ik\cdot n}}{m_0^2 + \sum_\mu \hat{k}_\mu^2}$$

with $m_0^2 + 8 = \frac{1}{2K\gamma_2} \Rightarrow \frac{1}{2K_c\gamma_2} = 8$ so $M_0 = 0$

or $\boxed{K_c = \frac{1}{16\gamma_2}}$

This is exactly the mean-field result.
- mean-field is valid when dim >> 1
- random walk valid when Gaussian approx holds for "crossings".
- In large dim, chance of crossings decreases, vanishing when dim → ∞
- Thus expect agreement on K_c.

Can extend to calculate other properties → HW