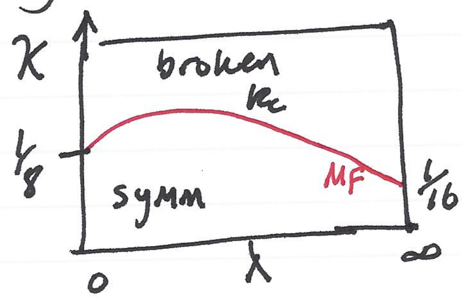
Towards triviality II: Hopping parameter expansion

This step provides an example of a type of expansion – a version of a high-temp expansion in stat. mech – that is available once one discretizes.

The story so far: mean field theory ⇒



Aim: understand properties of theory in the vicinity of κ_c – where we can take continuum limit – on both sides of κ_c.

Possible strategy A: do numerical simulations & determine masses, couplings & $\langle\phi\rangle$ near κ_c

– can by now do very accurate simulations, but still need to satisfy:

$$a \ll \xi_{phys} \ll L_{phys} \quad \text{(box size)}$$

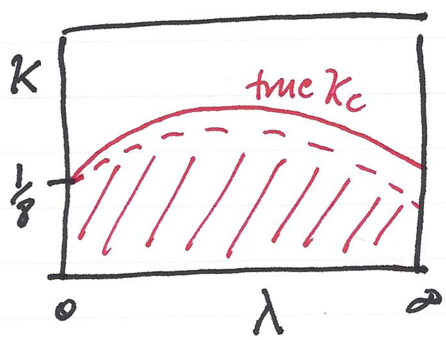
or $$1 \ll \xi = \frac{\xi_{phys}}{a} \ll N_{spatial} = L_{phys}/a$$

requiring very large $N_{spatial}$ (~100?)

Preferred strategy B: do as much as possible analytically & use numerics to check.

e.g. simulations find above phase diagram with κ_c shifted somewhat from MF value

Next analytic step: expand in κ



If work to high enough order (κ^{14}) + use some tricks (changing variables, padé approximants, ...) can obtain accurate results for $\kappa \lesssim 0.95\,\kappa_c$.
[Accurate ⇒ percent-level errors].

The quantities of interest turn out to be:

$$\chi_2 = \sum_n \langle \phi_n \phi_0 \rangle_{conn} = \tilde{G}(p=0) \quad \overset{\sim}{\leftarrow}\text{ F.T.}$$

$$= \sum_n \langle \phi_n \phi_0 \rangle - \langle \sum_n \phi_n \rangle \langle \phi_0 \rangle$$

connected bare propagator at zero 4-momentum

$$\mu_2 = \sum_n n^2 \langle \phi_n \phi_0 \rangle_{conn} \sim \partial_\mu^2 \tilde{G}(p)\big|_{p=0}$$

$$\chi_4 = \sum_{n_1,n_2,n_3} \langle \phi_{n_1} \phi_{n_2} \phi_{n_3} \phi_0 \rangle_{conn}$$

4 pt <u>un</u>amputated vertex with all legs at zero mom.

+ a few others.

As we'll see later in detail, these are related to (particular def$^{\underline{n}}$s) of renormalized mass & coupling.

We will also calculate $\tilde{G}(p)$ for all p in a κ expansion.

term "hops"

Recall: $S = \sum_n s(\phi_n) + 2K \sum_{n,\mu} \phi_n \phi_{n+\hat{\mu}}$

$\phi_n^2 + \lambda(\phi_n^2 - 1)^2$

$2K \sum_{\langle n_1, n_2 \rangle} \phi_{n_1} \phi_{n_2}$

means "nearest neighbors"

Given

$\langle f(\{\phi_n\}) \rangle = \frac{1}{Z} \int_\phi e^{-S} f(\{\phi_n\})$

(e.g. ϕ_n or $\sum_n \phi_n$ or...)

method is to write $e^{-S} = \left[\prod_n e^{s(\phi_n)}\right] e^{2K \sum_{\langle\rangle} \phi_{n_1} \phi_{n_2}}$

and expand second exponential (in both numerator & denominator)

This leads to a sum of terms, each of which factorizes into a product of single site integrals (like those we evaluated in the mean-field approx).

– straightforward at low orders; need tricks to reach K^{14}

Key feature: finite radius of convergence (can be shown rigorously – we will only note that it is plausible because can change sign of K and still have well-defined theory).

Unlike expansion in λ which is asymptotic (cannot change sign of λ) (since then path integral diverges)

- In order to get a sense of how the κ-expansion works let's do a simple example, Z itself.

$$Z = \prod_n \int d\phi_n \, e^{-s(\phi_n)} \quad e^{2\kappa \sum_{\langle n_1 n_2 \rangle} \phi_{n_1} \phi_{n_2}}$$

↑ see (6.3)

$\langle n_1 n_2 \rangle$ means nearest neighbors.

$(\kappa)^0$:

$$\boxed{Z^{(0)} = (z_0)^{\Omega}}$$

Ω: "volume" i.e. total number of sites

$$z_0 = \int d\phi_n \, e^{-s(\phi_n)}$$

"single-site partition fcn"

$(\kappa)^{(1)}$

$$e^{2\kappa \sum_{\langle\rangle} \phi_{n_1} \phi_{n_2}} = 1 + 2\kappa \sum_{\langle\rangle} \phi_{n_1} \phi_{n_2} + \ldots$$

4Ω such terms, all equal

$$Z^{(1)} = (2\kappa)(4\Omega)\, z_0^{\Omega} \left(\frac{\int d\phi \, e^{-s(\phi)} \phi}{z_0} \right)^2$$

single-site expectation value – vanishes by Z_2 symmetry

$$\Rightarrow \boxed{Z^{(1)} = 0}$$

- Clearly we will need

$$\gamma_n = \langle \phi^n \rangle_{\text{single-site}} = z_0^{-1} \int d\phi \, e^{-s(\phi)} \phi^n$$

→ generated by

$$z(j) = z_0^{-1} \int d\phi \, e^{-s(\phi) + j\phi} = \sum_{k=0}^{\infty} \frac{1}{k!} j^k \gamma_k .$$

Can calculate numerically for general d

Notes

- Z_2 symmetry $(\phi \to -\phi) \Rightarrow \gamma_{k\,\text{odd}} = 0$
- $\lambda \to \infty \quad \langle \phi^n \rangle_{\text{single site}} \longrightarrow \langle \sigma^n \rangle_{\text{Ising}} = \dfrac{(+1)^n + (-1)^n}{2} = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

Moving on to second order:

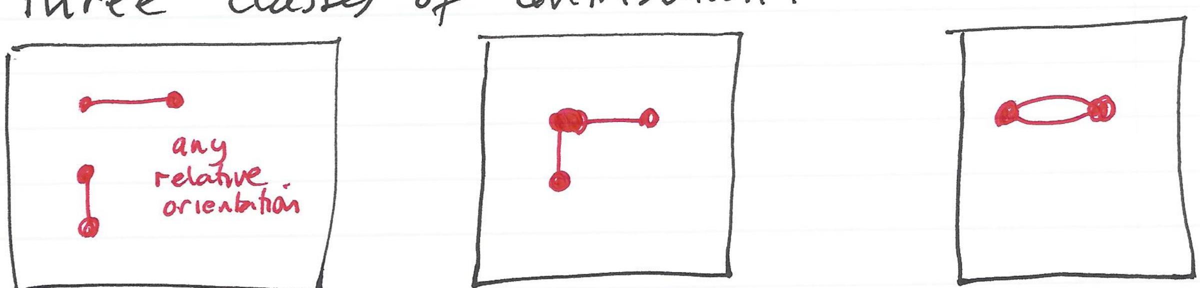
$$Z_1^{(2)} = \prod_n \int d\phi_n \, e^{-S(\phi_n)} \; \frac{1}{2!} (2K)^2 \Big(\sum_{\langle\rangle} \phi_{n_1} \phi_{n_2} \Big)^2$$

Represent diagrammatically: 

dot ⇒ factor of ϕ on a site (one for each link entering site)

link — ⇒ must be nearest neighbors.

Three classes of contribution:

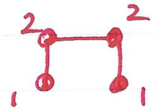


disconnected	central dot means ϕ^2	$\propto \gamma_2^{\,2}$
⇒ 0 since proportional to $\gamma_1^{\,4}$	⇒ proportional to $\gamma_1^{\,2}\gamma_2 = 0$	NON-VANISHING.

→ combinatorics: 4Ω such terms (# of links)

$$\Rightarrow \boxed{Z_1^{(2)} = z_0^{\,\Omega} \, \frac{(2K)^2}{2} (4\Omega) \, \gamma_2^{\,2}}$$

- By now it is clear that $Z^{(odd)} = 0$, because one always ends up with an odd # of ϕ's on some sites, e.g.

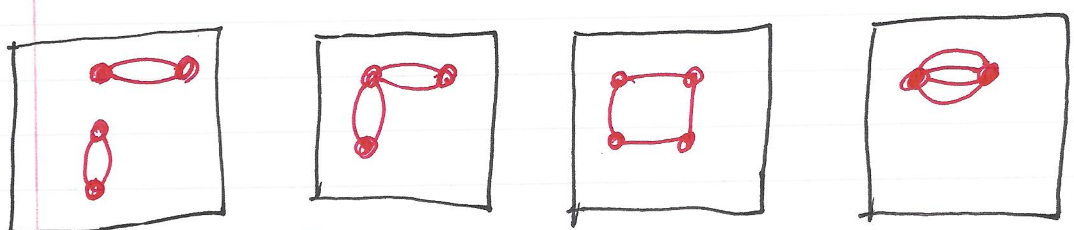


(would be different on a triangular lattice)

$\Rightarrow$ expansion in κ^2, showing, again, that changing $\kappa \to -\kappa$ does not change the theory

- So next term is $Z^{(4)}$

Diagrams with non-vanishing contributions:



disconnected — "connected"

To evaluate these, use nice result (taken from (Montvay + Münster))

$$e^{2\kappa\sum_{\langle\rangle}\phi_{n_1}\phi_{n_2}} = \sum_{\text{diagrams}} \underbrace{(2\kappa)^{\#\text{links}}}_{\text{obvious}} \left(\prod_{\substack{\text{links}\\ \text{in diagram}}} \frac{1}{(\text{mult})!}\right) \underbrace{\prod_{\substack{\text{links}\\ \text{in diagram}}} \phi_{\text{link end } 1}\,\phi_{\text{link end } 2}}_{\text{obvious.}}$$

"Multiplicity" i.e. # of hopping terms on link

↑ Only non-trivial part of this result.

This combinatoric factor comes from expanding out

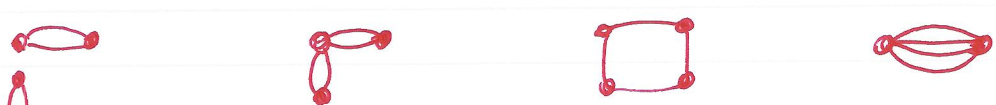
$$\frac{\left(\sum_{\langle\rangle} \phi_{n_1}\phi_{n_2}\right)^n}{n!} = \frac{\overbrace{(\Sigma\phi\phi)(\Sigma\phi\phi)\dots(\Sigma\phi\phi)}^{n \text{ times}}}{n!}$$

- If all links differ, get $n!$ choices in numerator, cancelling $n!$ in denom $\Rightarrow$ mult factor $= 1$
  This agrees w/ the general formula because each link (being different) has a multiplicity of 1.

- If two links are the same (the rest differing), one loses the "last" choice, so there are $n!/2$ choices
  $\Rightarrow \frac{n!/2}{n!} = \frac{1}{2} = \frac{1}{2!}$, agreeing w/ general formula

- etc. – enter a dark room & think about it to convince yourself that the formula holds in general.

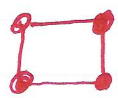
So, multiplicity factors & single-site integrals are



$\frac{\gamma_2^4}{2!\,2!}$ $\qquad$ $\frac{\gamma_2^2\gamma_4}{2!\,2!}$ $\qquad$ γ_2^4 $\qquad$ $\frac{\gamma_4^2}{4!}$

This leaves the counting factors – how many orientations & translations of each diagram are there?

Easy:  4Ω

 $\Omega \cdot \frac{4 \cdot 3}{2}$

(position of lower-left corner)

number of forward directed "plaquettes"

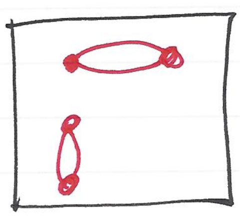
 $4\Omega \cdot 7 \cdot \frac{2}{2}$

1st link; # of orientations of second link

to which end does 2nd link attach

double counting (choice of "1st link")

↓ harder

 $4\Omega \, \frac{(4\Omega - 7\cdot 2 - 1)}{2}$

1st link

"excluded volume"

double counting.

$$= \frac{(4\Omega)^2}{2} - 4\Omega \underbrace{\left(7 + \tfrac{1}{2}\right)}_{15/2}$$

Collecting all contributions $\propto \kappa^4$:

$$Z_1^{(4)} = z_0^{\Omega} (2k)^4 \left\{ \gamma_2^{\,4} \left[\frac{(4\Omega)^2}{8} - 4\Omega \, \frac{15}{8} \right] \right.$$ 

$$+ \gamma_2^{\,4} (4\Omega) \, \frac{3}{2}$$ 

$$+ \gamma_2^{\,2} \gamma_4 \, \frac{7}{4} (4\Omega)$$ 

$$\left. + \gamma_4^{\,2} \, \frac{(4\Omega)}{4!} \right\}$$ 

$\Rightarrow$ All contributions to Z through $O(\kappa^4)$

$$Z_1 = z_0^{\Omega} \left\{ 1 + (2k)^2 \gamma_2^{\,2} \frac{4\Omega}{2} + \frac{(2k)^4 \gamma_2^{\,4}}{2} \left(\frac{4\Omega}{2}\right)^2 \right.$$

$$+ (2k)^4 \, 4\Omega \left[\gamma_2^{\,4} \left(-\frac{3}{8}\right) + \gamma_2^{\,2} \gamma_4 \frac{7}{4} + \gamma_4^{\,2} \frac{1}{4!} \right]$$

$$\left. + O(\kappa^6) \right\}$$

First line is building up an exponential

$$e^{(2k\gamma_2)^2 2\Omega}$$

due to independent summation of  factors – disconnected diagrams.

Can show that this continues to all orders ("extensivity")

$$Z = e^{-\Omega f}$$ ← "free energy density" volume indep.

To the order we are working

$$f = -\ln z_0 - (2K)^2 \gamma_2^2 z$$
$$- (2K)^4\, 4 \left[-\tfrac{3}{8}\gamma_2^4 + \tfrac{7}{4}\gamma_2^2\gamma_4 + \tfrac{1}{24}\gamma_4^2 \right]$$

One can work out diagramatic rules directly for f: this is called the "linked-cluster expansion".

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OK, this was a warm up — we won't directly use Z_1.

What we do want are hopping-parameter expansions for

$$\chi_2 = \langle \sum_n \phi_n \phi_0 \rangle_{conn}$$

(irrelevant since $\langle\phi\rangle = 0$)

$$= Z^{-1} \prod_m \left(\int d\phi_m e^{-s(\phi_m)} \right) e^{2K \sum_{\langle\rangle} \phi_{n_1}\phi_{n_2}} \sum_n \phi_n \phi_0$$

and related quantities.

The only new feature here is the presence of the fields from the expectation value.

So the leading order contribution to χ_2 is at $(K)^0$ from $n=0$ term.

$$\chi_2^{(0)} = \frac{z_0^\Omega \gamma_2}{z_0^\Omega} = \gamma_2$$

Diagramatically: •• where black dots indicate "external" fields & they are supposed to be on same site.

At $(K)^1$ have  i.e. when $n = \pm\hat{\mu}$.

This gives $\chi_2^{(1)} = 2K\,\gamma_2^{\,2} \cdot 8$ ← # of choices of $\pm\hat{\mu}$

You get the idea by now & HW has you extend such calculations to higher order.

- Note: multiplicity factors discussed above still apply (since they came from expanding out hopping terms)

Lüscher & Weisz have extended these series out to 14th order — and in special cases (Ising) others have gone to much higher order.