Triviality of ϕ^4 theory – a saga in several parts

Key papers: Martin Lüscher & Peter Weisz,

LW1: "Scaling laws & triviality bound in the lattice ϕ^4 theory (I) One-component model in the symmetric phase," Nucl. Phys. B290 (1987) 25

LW2: "Scaling laws... (II) One-component model in the phase with spontaneous symmetry breaking," Nucl. Phys. B295 (1988) 65

LW3: "Scaling laws... (III) n-component model," Nucl. Phys. B318 (1989) 705

LW4: "Application of the linked cluster expansion to the n-component ϕ^4 theory" Nucl. Phys. B300 (1988) 325

Idea + Relevance to Higgs physics first clearly emphasized in

R. Dashen & H. Neuberger, Phys. Rev. Lett. <u>50</u> (1983) 1897

Both Smit's book & Montvay + Münster have useful discussions

Aim – understand all the tools & results that allow one to conclude that, when we take the continuum limit, the renormalized coupling vanishes – irrespective of the size of the bare coupling (e.g. it can be infinite).

In the S.Model $\Rightarrow M_H/M_W \to 0$!

This result holds also for the n-component model : $\phi_p \rightarrow \vec{\Phi}_p$ n-comp. vector

$$V \rightarrow \frac{m_0}{2} \vec{\Phi}_p \cdot \vec{\Phi}_p + \lambda_0 (\vec{\Phi}_p \cdot \vec{\Phi}_p)^2$$

This theory is more complicated – having goldstone bosons in the broken phase – so we'll stick to the 1-component model.

Tools we will use :

- lattice pert. theory + "renormalized pert. thy"
- mean-field theory (large dimension expansion)
- hopping param. expansion ≡ high temp. exp. ≈ linked cluster exp.
- to fully convince ourselves of the result – need numerical simulations.

– complete analytic proof still lacking

LW notation.

So, here's our theory again:

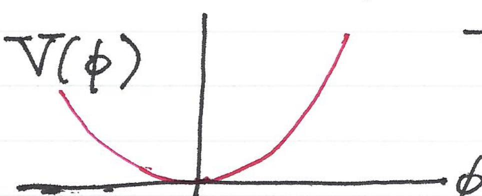
$$S = \sum_n \left[\sum_\mu \frac{(\tilde{\phi}_{n+\mu} - \tilde{\phi}_n)^2}{2} \right] + \frac{m_0^2}{2} \tilde{\phi}_n^2 + \frac{g_0}{4!} \tilde{\phi}_n^4$$

(the reason for the tildes will become clear shortly)

$Z = \int_\phi e^{-S}$ only defined for $g_0 \geq 0$ & we will assume, in fact, $g_0 > 0$ since free theory is not interesting.

Recall that, in the continuum, if we analyze the corresponding theory at weak coupling, there are two possibilities

$m^2 > 0$

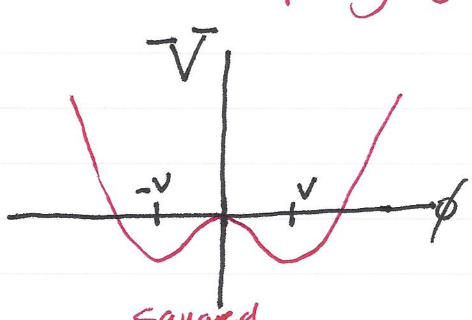


$V = \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$

"Symmetric phase"

$\Rightarrow$ particle of squared mass m^2 with weak four-pt. coupling g + higher-order corrections

$m^2 < 0$



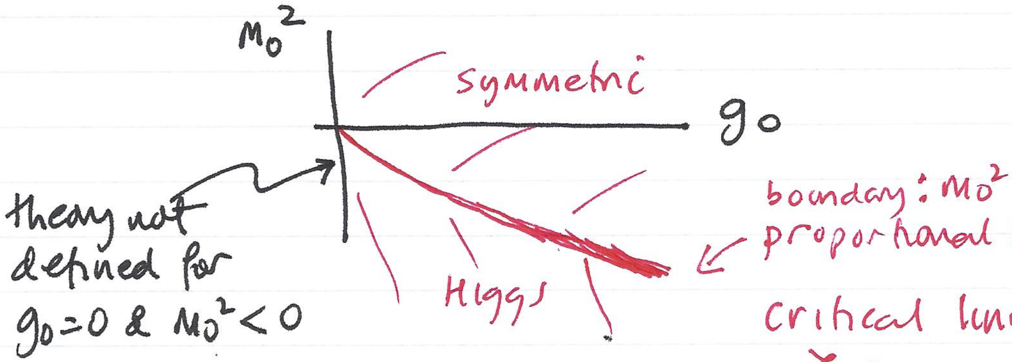
$\langle\phi\rangle$
$\text{V.E.V.} = \pm\left(-\frac{6m^2}{g}\right) = \pm v.$

Spontaneous breaking of Z_2 $\phi \to -\phi$ symm.
$\langle\phi\rangle$ is "order parameter"

$\Rightarrow$ particle of squared mass $-2m^2$ with weak 3-pt & 4-pt interactions: "Higgs phase"

Boundary: $m^2 = 0$ — massless particle (at tree level) w/ interactions.

Atleast for small g_0, we expect to find the same two phases in the lattice theory, and indeed we do for <u>all</u> g_0 (as we'll understand more as we proceed).



critical line with $\xi \to \infty$.
$\Rightarrow$ can take continuum limit there, from either phase.

Phase diagram looks nicer using standard lattice parameters:

$$S = \sum_n \left\{ \left[-2K \sum_\mu \phi_{n+\hat\mu}\phi_n \right] + \phi_n^2 + \lambda(\phi_n^2-1)^2 - \lambda \right\}$$

constant — can be dropped

with $\phi_n = \dfrac{\tilde\phi_n}{\sqrt{2K}}$ (hopping terms match)

$\lambda = \dfrac{K^2}{6} g_0$ (ϕ_n^4 terms match)

& $\dfrac{1-2\lambda}{2K} = \dfrac{m_0^2+8}{2}$ (ϕ_n^2 terms match)

At first sight, this new parametrization seems strange
e.g. it looks like it is forcing one into a symmetry broken phase.

But, in fact, one is not changing the theory at all:

each choice of $\kappa \geq 0$ & $\lambda > 0$
corresponds to a unique m_0^2 & $g_0 > 0$
and vica-versa
i.e. this is a 1-1 "onto" mapping

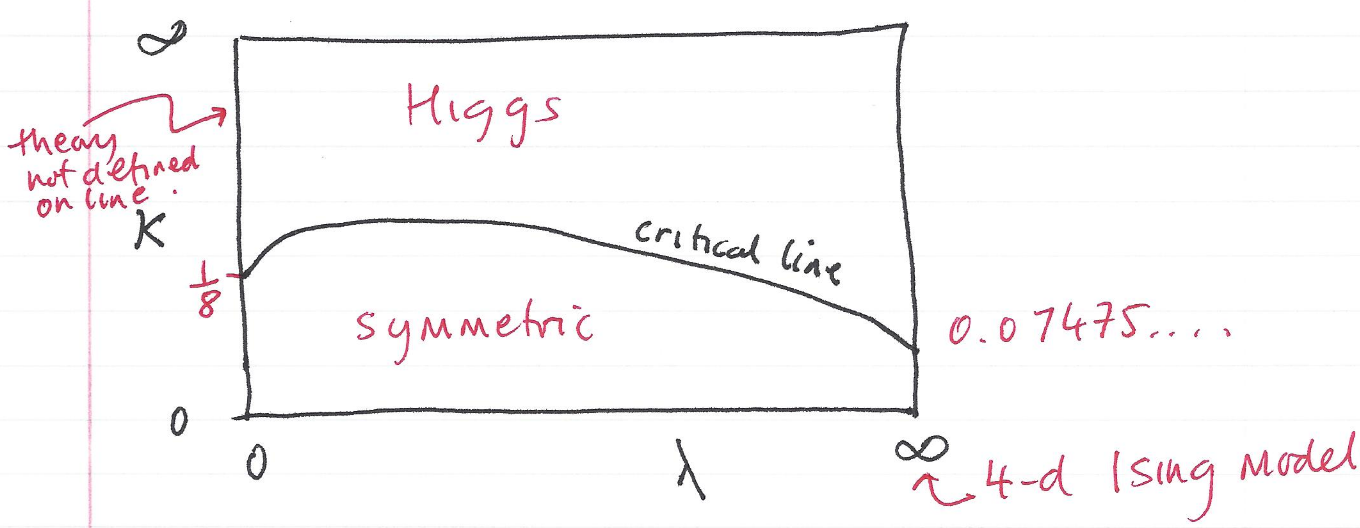
[Aside: same is true for $\kappa \leq 0$ & $\lambda > 0$;

in fact, theories with $\kappa < 0$ can be obtained from those with $\kappa > 0$ by changing variables:

$$\phi_n \Rightarrow \phi_n (-1)^{n_1+n_2+n_3+n_4}$$

(which leaves the measure unchanged).
So we lose no info. by restricting to $\kappa \geq 0$.]

In terms of new variables, phase diagram is (qualitatively)



Mantra: physical meaning of lattice parameters is often obscure — our aim is always to tune (fine tune in particle parlance) to critical points.

- Still, what is κ, crudely?

$$\kappa \to 0 \Rightarrow m_0^2 \to \pm\infty$$ sign depends on λ

This makes intuitive sense since
(a) when $\kappa \to 0$ the sites are decoupled
(b) an infinitely massive particle does not move since that would require infinite kinetic energy

$\kappa = 0$:
- all sites (in space & time) decoupled
- correlation length = 0
- no symmetry breaking since have simple integral on each site, so

$$\langle\phi\rangle = \frac{\int d\phi \; e^{-\phi^2+\lambda(\phi^2-1)^2} \phi}{\int d\phi \; e^{-\phi^2+\lambda(\phi^2-1)^2}} = 0$$

- Another important limit : $\lambda \to \infty$

single-site measure becomes Ising-like

$$\int d\phi \; e^{-\phi^2-\lambda(\phi^2-1)^2} = \int d\mu(\phi)$$

$$d\mu(\phi) \propto \delta(\phi-1) + \delta(\phi+1)$$

i.e. $\phi_n \to \pm 1$

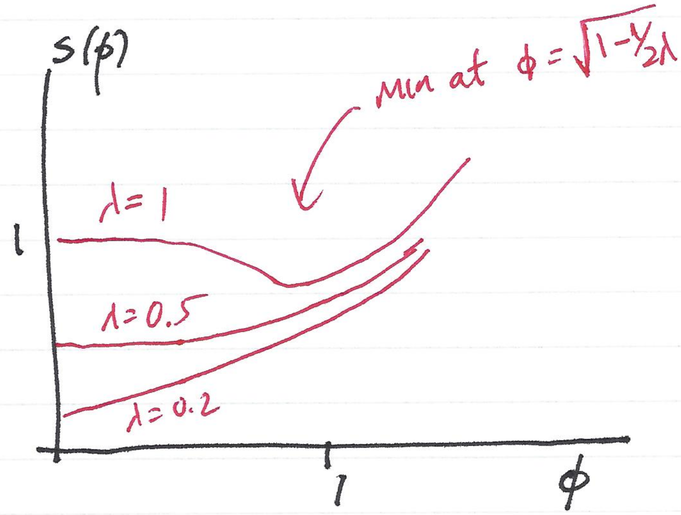
κ term is spin-spin interaction which wants to align spins

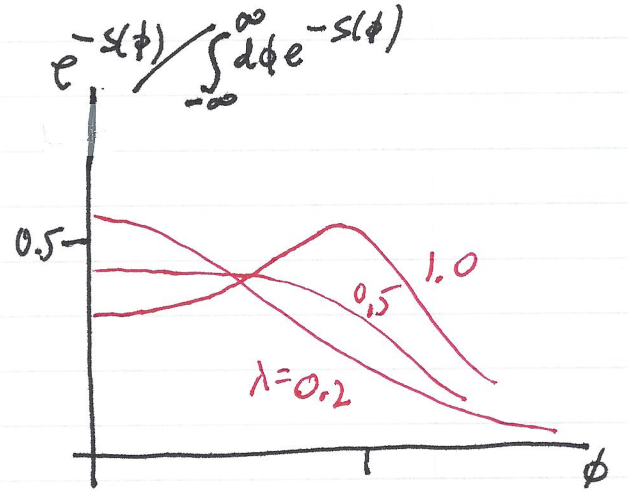
$\Rightarrow$ 4-d ISING MODEL.

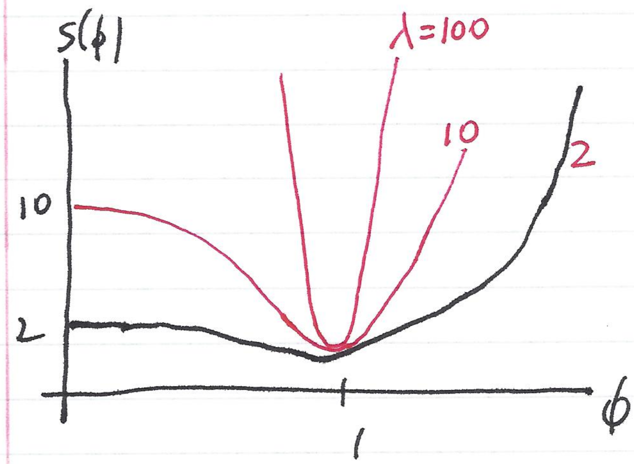
Let's make Ising limit clearer.

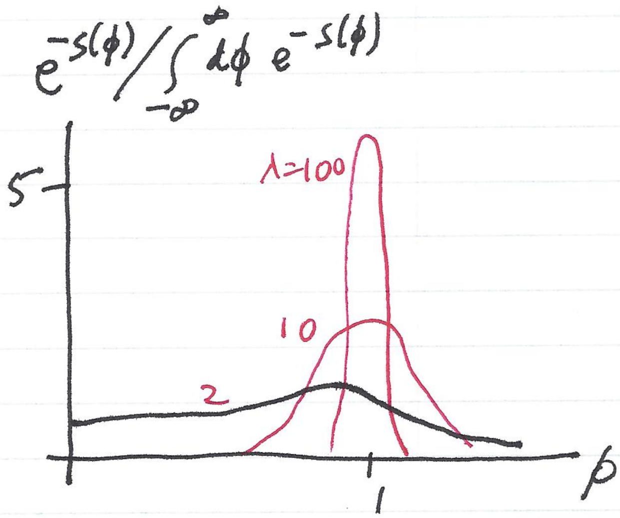
$$Z = \prod_n \left(\int_{\phi_n} d\mu(\phi_n) \right) \; e^{2k \sum_\mu \phi_{n+\hat\mu}\, \phi_n}$$

$$\int d\mu(\phi) = \int_{-\infty}^{\infty} d\phi \; e^{-s(\phi)} \; ; \qquad s(\phi) = \phi^2 + \lambda(\phi^2 - 1)^2$$









Note that these are symmetrical under $\phi \to -\phi$

Mean field analysis of symmetry breaking

When first studying a lattice theory (or stat. mech system) it is useful to get an approximate sense of the phase diagram using analytic tools.

Here we begin with mean field theory, which is (as we will see) a large dimension expansion.

Start in Ising limit for illustration:

$$Z = \sum_{\{\sigma_n\}} e^{2K \sum_{\mu} \sigma_{n+\hat{\mu}} \sigma_n}$$

up to overall constant, which cancels in expectation values

where $\phi_n \to \sigma_n = \pm 1$

We are interested in symmetry breaking, so consider $\langle \phi_n \rangle \equiv v = Z^{-1} \sum_{\{\sigma_n\}} e^{2K \sum_{\mu} \sigma_{n+\hat{\mu}} \sigma_n} \sigma_n \equiv \langle \sigma_n \rangle$

which should be indep. of n by translation invariance.

Now, for any <u>finite</u> volume we know what this is:

$\langle \phi_n \rangle = 0$ since $S(\phi) = S(-\phi)$.

Z_2 symmetry.

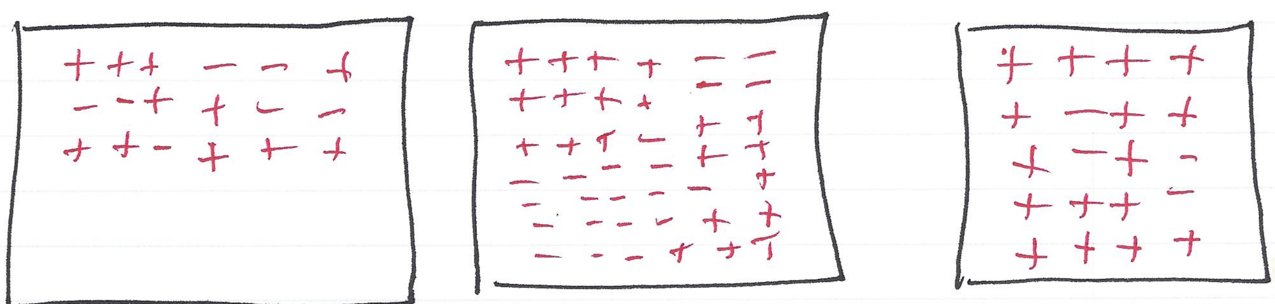
How to treat S.S.B (spont. symm. breaking)?

For $K > 0$, $e^{2K\sum_{\mu}\sigma_{n+\hat{\mu}}\sigma_n}$ favors alignment of neighboring spins, so expect "ferromagnetism" at large enough K.

Countering this is "entropy" – $\sum_{\sigma_n}$ leads, by itself, to a random dist^n of σ_n.

So, if simulate a finite system (by Monte Carlo methods, which we may discuss later); expect

small K | medium K | larger K



small scale clustering; mostly random; many sign flips | larger scale clustering, but regions still flip sign | almost complete "magnetization" with small fluctuations

Key point is that, although the Z_2 symmetric configuration ($+ \leftrightarrow -$) is equally likely in principle, most algorithms will not make the jump, since it requires changing almost all spins at once.

Still, this is an algorithmic argument & thus not fully satisfying – there are "cluster algs." which flip large regions all at once.

To be rigorous we need to add to the action a source which slightly biases the dist^n.

$$S \to S - j\sum_n \phi_n \Rightarrow s \to s - j\phi$$

infinitessimal source

In Ising case leads to

$$\frac{\text{prob}(\sigma=+1)}{\text{prob}(\sigma=-1)} = \frac{e^j}{e^{-j}} = e^{2j}$$

\# of $\sigma=+$ sites

Thus a state w/ $\left\langle \frac{\sum_n \sigma_n}{\Omega} \right\rangle \equiv \langle\sigma\rangle = V = \frac{N_+ - N_-}{\Omega} = \frac{2N_+}{\Omega} - 1$

Ω = Number of sites

will be favored over the Z_2 "symmetric" state with $\langle\sigma\rangle = -V$

by a relative prob. of

$$e^{2j\cdot(N_+ - N_-)} = e^{2jV\Omega}$$

$$\xrightarrow[\Omega\to\infty]{} \infty$$

(however small j is as long as fixed)

$\Rightarrow$ SSB

Recipe: (1) Introduce source j
(2) Send $\Omega \to \infty$
(3) Send $j \to 0$

Mean-field approx

Assume $\langle\sigma_n\rangle \neq 0$ (positive, say) due to implicit addition of a source
$\equiv v$
& check for self-consistency (given some assumptions)

A useful theoretical quantity is the single-site prob. dist.

$$P_1(\sigma_n') = Z^{-1} \sum_{\{\sigma_n\}} e^{-S_{lat}} \, \delta_{\sigma_n, \sigma_n'}$$

i.e. do full sum except constraining σ_n to equal σ_n'

Clearly $\sum_{\sigma_n' = \pm 1} P_1(\sigma_n') = Z^{-1} Z = 1$

Now, approximate P_1 assuming that the sum over all other spins allows the replacement of $\sum_\mu (\sigma_{n+\hat\mu} + \sigma_{n-\hat\mu})$ with $8v$

$$P_1(\sigma_n') = Z^{-1} \sum_{\{\sigma_n\}} \delta_{\sigma_n,\sigma_n'} \, e^{-S_{lat}} = Z^{-1} \sum_{\{\sigma_n\}} \delta_{\sigma_n,\sigma_n'} \, e^{\{2k\sigma_n \sum_\mu (\sigma_{n+\hat\mu} + \sigma_{n-\hat\mu}) + \dots\}}$$

interactions involving other sites but not σ_n

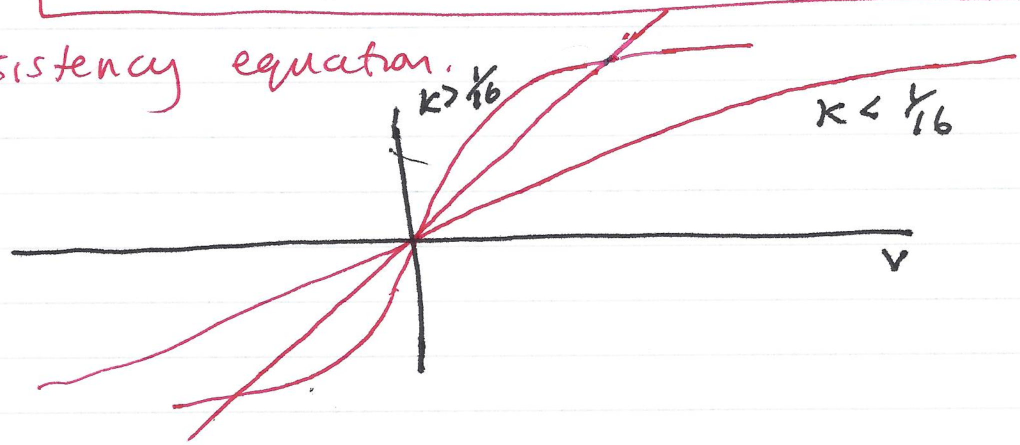
$$\approx \frac{e^{2k\sigma_n' 8v}}{\sum_{\sigma_n' = \pm 1} e^{2k\sigma_n' 8v}}$$

Idea: for d=dim large, there are so many neighbors that their average value does not fluctuate, and is not influenced by σ_n.

Then $\langle \sigma_n \rangle = \sum_{\sigma_n' = \pm 1} P_1(\sigma_n')\, \sigma_n'$

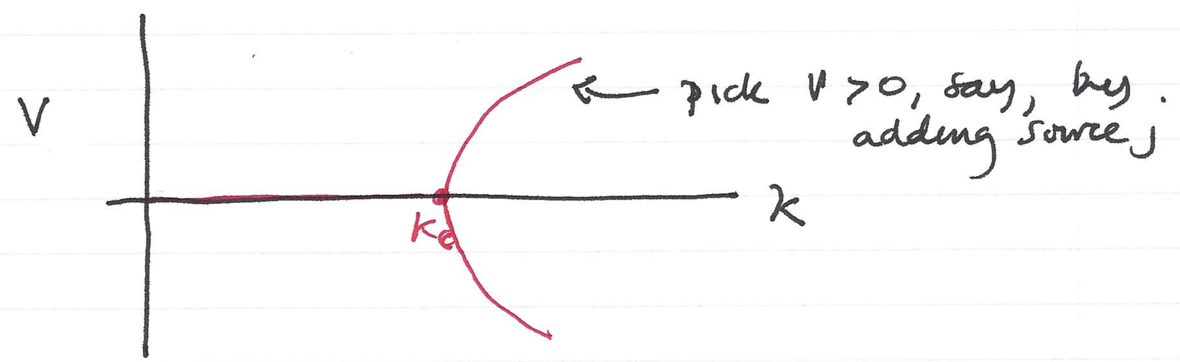
or $$\boxed{V = \frac{e^{16kV} - e^{-16kV}}{e^{16kV} + e^{16kV}} = \tanh(16kV)}$$

consistency equation.



- Clearly, if slope of tanh at $V=0 < 1$, get only $V=0$ solution, while if slope > 1 get $V=0$ and nontrivial sol^{n}s $V = \pm V_0$ too.

$$\frac{d}{dV} \tanh(16kV)\Big|_{V=0} = 16k \Rightarrow \boxed{k_c = 1/16}$$



- Shape of v for $K \gtrsim K_c$?

$$v = \tanh(16Kv) \approx 16Kv\left(1 - \frac{(16Kv)^2}{3} + \dots\right)$$

for small v_0

$$\approx (16(K-K_c)+1)\,v - \cancel{(16K_c)^3}^{\,1}\,\frac{v^3}{3}$$

$$\Rightarrow\ 16(K-K_c) \approx \frac{v^2}{3}$$

$$\Rightarrow\ v = \pm\sqrt{48(K-K_c)}$$

- This non-analytic behavior, $v \propto (K-K_c)^\beta$, is characteristic of critical phenomena (2nd order transitions)

- Here $\beta = \frac{1}{2}$ (mean-field value).

  In $d=3$ (real world classical stat. mech. systems), these exponents are changed from mean-field values.

  In $d=4$ (the "upper critical dimension" above which mean-field is exact) there are logarithmic corrections $v \propto (K-K_c)^{\frac{1}{2}}\,|\ln(K-K_c)|^{1/3}$

  [See e.g. Sec 26.2 of J. Zinn-Justin, "QFT & Critical phenomena".]

- How well does MF work numerically?

  $(K_c)_{MF}\Big|_{\lambda=\infty} = \frac{1}{16} = 0.0625$

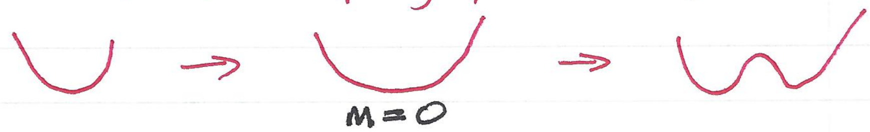
  whereas in fact (from simulations) result is $0.07475\ldots$

  reasonably accurate (& can do better – see below)

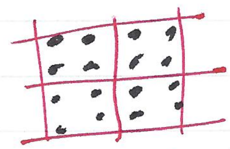
Expect massless particle at K_c $(\xi \to \infty)$.

- will see this explicitly in another (approximate) calculation below.

- Note that in the Ising limit, we do not have the weak coupling potential picture:

V: 

- However we can define an effectively continuous ϕ_{block} by averaging σ_n over a block (e.g. hypercube) of lattice sites



and integrating out in-block degrees of freedom

- We can, in principle (in practice, numerically — Monte Carlo R.G.) determine the "effective action" for ϕ_{block}, and, in particular, for space-time indep ϕ_{block} fields, the blocked ("coarse-grained") potential.

- Z_2 symmetry is retained by blocking, so the blocked potential will change with K as in the pictures above, leading to a critical K_c with a massless particle

- In HW2 — extend MF analysis to all λ.