Reflection positivity & its application to lattice theories

[see Montvay + Münster for some discussion & Refs.]

As mentioned before, in order for a Euclidean path integral to be "Wick-rotatable" to a physical Minkowski theory (w/ positive norm, bounded hermitian $\hat{H}$, ..) it is necessary that reflection positivity hold.

I want to spend a little time describing & motivating this property & explaining its relevance to lattice field theories.

The set-up is that someone hands us a set of Euclidean corr fcns $G_n(x_1^E, \ldots x_n^E)$ which are symmetric & Euclidean rot^n + trans^n invariant.

These are called "Schwinger fcns" & can either be obtained from a path integral (see 1.6) or by Wick rot^n from a Minkowski theory (for which there are some subtleties involved in obtaining the full correlation fcn for arbitrary arguments — see Montvay & Münster).

Let's follow M+M & call the Schwinger fcns

$$\mathcal{S}(x_1, \ldots x_n).$$

To define refl. pos. we introduce the Euclidean time reflection op -

$$\Theta(x_\mu = (\vec{x}, x_4)) = (\vec{x}, -x_4)$$

We now consider general (complex) fcns $f_j(x_1, \dots x_j)$ which are non-vanishing only if

$$(x_k)_4 \geq 0 \quad \forall k$$

(i.e. all the x_k are at positive times)

& which fall off for large $|x_k|$.

<u>Refl. pos.</u>

$$(\#) \sum_{j,k} \int dx_1 \dots dx_j \; dy_1 \dots dy_k \; f_j^*(x_1, \dots x_j) \, f_k(y_1, \dots y_k) \, J(\theta x_1, \dots \theta x_j, y_1 \cdots y_k) \geq 0$$

for "all" f_j such that the integrals converge.

Simple example:

$$(*) \int dx \, dy \;\; f_1^*(x) \, f_1(y) \; \underbrace{J(\theta x, y)}_{\substack{\text{2 pt-fcn w/ one argument} \\ \text{at positive time, the} \\ \text{other at negative.}}} \geq 0$$

Usually, despite the rather forbidding nature of condition, it is either

- easy to show holds in general
- easy to show that it fails in specific cases

So it really is useful in practice (with some caveats)

Why does it make sense? Where does it come from?

If our Euclidean theory originates w/ a physical Minkowski theory,

then $J(\theta x, y) = J(y, \theta x)$

$$= \langle 0 | \hat{\phi}(\vec{y}, 0) e^{-\hat{H} y_4} \; e^{\hat{H}(-x_4)} \hat{\phi}(\vec{x}, 0) | 0 \rangle$$

MUST be time ordered, since this is the order which is produced by original analytic cont. from Mink. correlators, and so that this expression is well-defined

See notes 1.5

Now let's first assume that $f_1(y) \approx f_1(\vec{y})\, \delta(y_4)$
i.e. support only at $y_4 = 0$ (so that $x_4 = 0$ too).

Then (being sloppy w/ overall factors)

$$(*) \rightarrow \underbrace{\langle 0 | \left[\int d^3\vec{y}\, f_1(\vec{y})\, \hat{\phi}(\vec{y}, 0) \right]}_{\text{then this is the dual } \langle f_1 |} \underbrace{\left[\int d^3\vec{x}\, f_1^*(\vec{x})\, \hat{\phi}(\vec{x}, 0) \right] | 0 \rangle}_{\text{call this } | f_1 \rangle}$$

From previous page

So $(*) \rightarrow \langle f_1 | f_1 \rangle \geq 0$.

Generalizing to the full $\sum_{jk} f_j^* f_k$ structure of (#), but keeping $y_4 = x_4 = 0$, we see that

(#) $\rightarrow$ $\langle \psi | \psi \rangle \geq 0$; $|\psi\rangle$ = most general state created by $\hat{\phi}, \hat{\phi}^2, \ldots$

So this subset of the refl. pos. condition corresponds to existence of a positive norm Hilbert space.

$\widetilde{\#}$

Now consider $f_1(y)$ with support only for $y_4 = \tau$

Then (*) becomes

$$\langle f_1 | \, e^{-\hat{H}\tau} \; e^{-\hat{H}\tau} | f_1 \rangle \geq 0 \quad (\forall \tau \geq 0)$$

which, using (#), generalizes to

$$\langle \psi | \, e^{-2\hat{H}\tau} | \psi \rangle \geq 0 \qquad (\forall \tau \geq 0)$$

For this to be true we must have $\hat{H} = \hat{H}^{+}$ (otherwise have complex eigenvalues & potentially oscillating matrix elements)

Further, since the Schwinger fcns are finite (as long as we keep the x_j different), the spectrum of $\hat{H}$ must be bounded from below.

$\widetilde{\#}$

This sketch makes plausible that the Refl. Pos. conditions are necessary. O+S showed that they are (combined w/ other properties) sufficient & that one can reconstruct $\hat{H}$ from the Schwinger fcns, in principle.

- These considerations generalize to a lattice theory:
  If a lattice version of refl. pos. holds, then a transfer matrix $\hat{T}$ with physical properties exists.

See e.g. K. Osterwalder & E. Seiler, Ann. Phys. (NY) 110, 440 (1978)
+ Nice, brief discussion in Montvay & Münster.

- To help the discussion, introduce operator Θ:

$$\Theta\, \phi(x) \equiv \phi(\theta x)$$

would include hermitian conjugation if ϕ were complex

$$\Theta \left[f(x)\, \phi(x) \right] \equiv f^*(\ x)\, \phi(\theta x)$$

argument only changed inside operator

antilinear (like time-reversal)

Θ is a "Euclidean version" of hermitian conj.

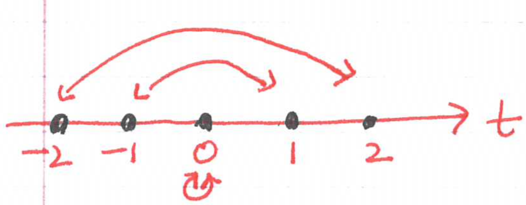
$$\hat{\phi}(\vec{x}, x_4) = e^{\hat{H} x_4}\, \hat{\phi}(\vec{x})\, e^{-\hat{H} x_4} \xrightarrow[\text{conj.}]{\text{herm.}} e^{-\hat{H} x_4}\, \hat{\phi}(\vec{x})\, e^{\hat{H} x_4}$$

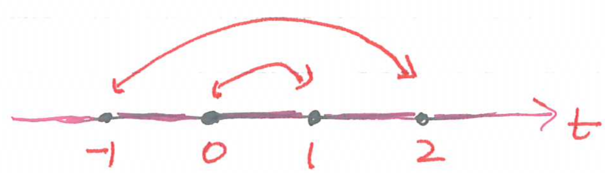
$$= \hat{\phi}(\theta x).$$

- Now, for a lattice can have either

"site reflection" or "link reflection"

e.g. $n_4 \xrightarrow{\theta} -n_4$ e.g. $n_4 \xrightarrow{\theta} 1 - n_4$





- Need refl. pos. under <u>both</u> types to guarantee a physical $\hat{T}$
  (hermitian, positive, bounded, acting on positive norm Hilbert space)

Statement of refl. pos. for lattice th'y defined by a path integral.

Schwinger fcns are $S(n_1, n_2, .. n_j) = Z^{-1} \int_\phi e^{-S} \phi_{n_1} \phi_{n_2} \cdots \phi_{n_j}$

Let $F = \sum_k \sum_{n_1, .. n_k} f_k(n_1, .. n_k)\, \phi_{n_1} \cdots \phi_{n_k}$

with f_k having support only for:

$(n_j)_4 \geq 0$ site-reflection & $(n_j)_4 > 0$ link-refl.

so $\Theta F = \sum_k \sum_{n_1 .. n_k} f_k^*(n_1, \ldots n_k)\, \phi_{\underbrace{\Theta n_1}_{(\vec{n}_1, -n_4)}} \cdots \phi_{\Theta n_k}$

Theory is refl. pos. if

$$(\$) \quad \boxed{Z^{-1} \int_\phi e^{-S}\, F\, \Theta F \geq 0} \quad (\forall \text{ choices of } f_k.)$$

can drop

Can be site-refl. pos or link-refl pos or both.

That this makes sense can be seen by working out this expression if $\hat{T}$ exists.

If f_k have support (for site refl.) only for $n_4 = 0$,

$(\$) \Rightarrow \langle \psi | \psi \rangle \geq 0$ state in general lattice Hilbert space

If f_k has support for site refl. for $n_4 = 1$

$(\$) \Rightarrow \langle \psi | \hat{T}^2 | \psi \rangle \geq 0$

connects $n_4 = -1$ to $n_4 = +1$

(reasoning is same as leads to $e^{-\frac{2}{=}\hat{H}\tau}$ on 4.4 above)

If consider link reflection with support only at $n_4=1$ (which reflects to $n_4=0$) then

$$(\$) \Rightarrow \langle \psi | \hat{T} | \psi \rangle \geq 0 .$$

and w/ support only at $n_4 = 2$ (reflects to $n_4=-1$)

$$(\$) \Rightarrow \langle \psi | \hat{T}^3 | \psi \rangle \geq 0$$

$\Rightarrow$ If have both types of refl. pos. know that $|\psi\rangle$ are positive & $\hat{T}$ positive & real (& bounded)

While if only have site refl. (quite common, e.g. staggered fermions) then know that
$|\psi\rangle$ are positive & $\hat{T}^2$ is pos & real & bounded

This allows $\hat{T}$ to have negative e'values.

But can still define $\hat{H}$ through.

$$\hat{T}^2 = e^{-2\hat{H}a} .$$

Checking refl. pos. for scalar field theory.
(which must work since we have constructed $\hat{T}$).

Site-reflection

Decompose action $S = S_+ + S_0 + S_-$

S_+: depends only on fields at $n_4 \geq 0$ (includes $\phi_{\vec{n},1}$ $\phi_{\vec{n},0}$)

S_0: depends only on fields at $n_4 = 0$ $\Rightarrow$ potential term & diag. part of kinetic term.

S_-: depends only on fields at $n_4 \leq 0$

$S_- = \Theta S_+$

Recall: $S = \sum_n \left(\frac{m^2+8}{2} \phi_n^2 + \frac{\lambda}{4} \phi_n^4 \right) - \sum_{n,\mu} \phi_n \phi_{n-\hat{\mu}}$

$$\int_\phi e^{-S} F \,\Theta F = \int_{\phi_0} e^{-S_0} \underbrace{\left(\int_{\phi_+} e^{-S_+} F\right)\left(\int_{\phi_-} e^{-S_-} \Theta F\right)}_{\text{depend also on } \phi_0}$$

ϕ_0: fields w/ $n_4 = 0$; ϕ_+: fields w/ $n_4 > 0$; ϕ_-: $n_4 < 0$

$$= \int_{\phi_0} e^{-S_0} \underbrace{\left(\int_{\phi_+} e^{-S_+} F\right)}_{\mathcal{F}(\phi_0)} \Theta \underbrace{\left(\int_{\phi_+} e^{-S_+} F\right)}_{\mathcal{F}(\phi_0)^*}$$

≥ 0 ✓

<u>Link-reflection</u> : slightly more tricky

$$S = S_{conn} + S_{1+} + S_{1-}$$

$\left(-\sum_{\vec{n}} \phi_{\vec{n},1} \phi_{\vec{n},0}\right)$

depends only on fields at $n_4 \geq 1$

depends only on fields at $n_4 \leq 0$

$S_{1-} = \Theta S_{1+}$

So:

$$I = \int_\phi e^{-S} F \,\Theta F$$

$$= \int_{\phi_{1+}} \int_{\phi_{1-}} \underbrace{e^{-S_{conn}}}_{\text{does not factorize!}} \left(F e^{-S_{1+}}\right) \Theta \left(F e^{-S_{1-}}\right)$$

$n_4 \geq 1$ $\quad$ $n_4 \leq 0$

so, expand

$$e^{+\sum_{\vec{n}} \phi_{\vec{n},1} \phi_{\vec{n}0}} = 1 + \sum_{\vec{n}} \phi_{\vec{n},1} \phi_{\vec{n},0} + \left(\sum_{\vec{n}} \phi_{\vec{n},1} \phi_{\vec{n},0}\right)^2 + \dots$$

→ when expand out, each term factorizes into a ϕ_{1+} & ϕ_{1-} piece, the latter related by Θ e.g. $\underbrace{\phi_{\vec{n}_1 1} \phi_{\vec{n}_2 1}}\ \underbrace{\phi_{\vec{n}_1 0} \phi_{\vec{n}_2 0}}$

Crucially, each term has a positive coefficient

Thus I = positive sum of terms of the form

$$\left| \left[\int_{\phi_{1+}} \left(\phi's\, F e^{-S_{1+}} \right) \right] \right|^2$$

& thus $I \geq 1$.

When does refl. pos fail?

One example (on HW1) is when use higher order derivatives including 2-steps

Then, you will show, site reflection pos. <u>and</u> link refl. pos fail.

In a free theory of this form, can trace this to an oscillation (sign change) in the propagator for large $\vec{k}$.

Such an oscillation $\Rightarrow$ $\hat{T} = \hat{T}^{+} > 0$ cannot exist – indeed $\hat{T}$ has complex e'values (which come in complex conjugate pairs)
$\Rightarrow$ $\hat{H} \neq \hat{H}^{+}$ & there are complex energies.

In fact, such higher-order actions (which can reduce discretization errors) are very common in numerical simulations today ! o

The lore is that, because the complex energies have large real parts ($E_{phys} \sim 1/a$), they do not impact the low energy physics.

Indeed, oscillations (or non-monotonic behavior) in correlators dies out very quickly in practice.