Transfer matrix for scalar field theory

Previously we followed the path

$$\langle 0|T(\phi \ldots \hat{\phi})|0\rangle \longrightarrow \langle 0|T(\hat{\phi} \ldots \hat{\phi})|0\rangle = \frac{1}{Z}\int [D\phi]\, e^{-S}\phi \ldots \phi$$

Mink. Euclid. Euclid path integral

Lattice version.

Basically, constructing the transfer matrix allows us to reverse these steps, crudely speaking.

We can figure out what the Hamiltonian $\hat{H}$ is, what its symmetries are, & reproduce (in discretized form) expressions like

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$$G_2(x^E, 0) = \langle 0|\, \hat{\varphi}(0)\, e^{-\hat{H}x_4} e^{i\vec{p}\cdot\vec{x}}\, \hat{\varphi}(0)\,|0\rangle$$

$$= \langle 0|\, \hat{\varphi}(\vec{x})\, e^{-\hat{H}x_4}\, \hat{\varphi}(0)\,|0\rangle$$

So we can understand better what we are calculating when we do a Euclidean path integral. In stat. mech., transfer matrix methods are widely used.

Warm up: $d=1$ case (QM of anharmonic osc.)

$\phi_n = \{\phi_1, \phi_2, \phi_3, \phi_N\}$ PBC $\phi_{N+1} = \phi_1$

$$Z = \prod_{j=1}^{N} \int_{-\infty}^{\infty} d\phi_j \; e^{-S}$$

Aim to write as $Z = \mathrm{Tr}(\hat{T}^N)$

$\hat{T}$ = transfer-operator
$\equiv$ transfer matrix

$$S = \sum_{j=1}^{N} \left\{ +\frac{1}{2} (\phi_{j+1} - \phi_j)^2 + \frac{m^2}{2} \phi_j^2 \right\}$$

$$= S_A(\phi_1) + S_A(\phi_2) + \dots + S_A(\phi_N)$$

$$+ S_B(\phi_1, \phi_2) + S_B(\phi_2, \phi_3) + \dots + S_B(\phi_N, \phi_1)$$

with
$$S_A(\phi) = \frac{m^2}{2} \phi$$ on-site

&
$$S_B(\phi_1, \phi_2) = \frac{1}{2} (\phi_2 - \phi_1)^2$$ hopping.
$$= S_B(\phi_2, \phi_1)$$

Now introduce Hilbert space of square-integrable fcns in 1-d – i.e. the usual space we consider in 1-d QM.

Use non-normalizable basis akin to $|x\rangle$ (or $|p\rangle$) but call it $|\phi\rangle$; $\langle \phi' | \phi \rangle = \delta(\phi' - \phi)$

& introduce operators $\hat{\phi}$ s.t. $\hat{\phi} |\phi\rangle = |\phi\rangle \phi$
(just like $\hat{x}|x\rangle = |x\rangle x$ in QM) & $\langle \phi | \hat{\phi} = \phi \langle \phi |$

Also introduce canonically conjugate ops $\hat{p}$ s.t.

$$[\hat{p}, \hat{\phi}] = -i \quad \underset{\text{as in QM}}{\Longrightarrow} \quad \underbrace{e^{-i\hat{p}\cdot\Delta}}_{\text{unitary translation op.}} |\phi\rangle = |\phi + \Delta\rangle$$

$p \sim -i\frac{\partial}{\partial\phi}$

Using this technology, we want to find

$\hat{T}$ s.t. $\quad \langle\phi'|\hat{T}|\phi\rangle = e^{-\frac{1}{2}S_A(\phi')} e^{-S_B(\phi',\phi)} e^{-\frac{1}{2}S_A(\phi)}$

For then.

$$Z = \prod_{j=1}^{N} \int d\phi_j \; \langle\phi_N|\hat{T}|\phi_{N-1}\rangle \ldots \langle\phi_2|\hat{T}|\phi_1\rangle \langle\phi_1|\hat{T}|\phi_N\rangle$$

Now $\quad \int d\phi \, |\phi\rangle\langle\phi| = \mathbb{1}$.

& $\quad \int d\phi \, \langle\phi|\hat{O}|\phi\rangle \equiv \operatorname{tr} \hat{O}$.

so we find $\quad Z = \operatorname{tr}(\hat{T}^N)$ as claimed.

What is $\hat{T}$? The S_A parts are easy

$$\hat{T} = e^{-\frac{1}{2}S_A(\hat{\phi})} \hat{X} e^{-\frac{1}{2}S_A(\hat{\phi})}$$

$$\Rightarrow \langle\phi'|\hat{T}|\phi\rangle = e^{-\frac{1}{2}S_A(\phi')} \underbrace{\langle\phi'|\hat{X}|\phi\rangle}_{\text{must equal } e^{-\frac{1}{2}(\phi'-\phi)^2}} e^{-\frac{1}{2}S_A(\phi)}$$

By inspection: $\hat{X} = \int_{-\infty}^{\infty} d\Delta \; e^{-\frac{\Delta^2}{2}} e^{-i\hat{p}\Delta}$

since $\langle\phi'|\hat{X}|\phi\rangle = \int_{-\infty}^{\infty} d\Delta \; e^{-\frac{\Delta^2}{2}} \underbrace{\langle\phi'|e^{-i\hat{p}\Delta}|\phi\rangle}_{\langle\phi'|\phi+\Delta\rangle = \delta(\phi'-\phi-\Delta)}$

$$= e^{-\frac{(\phi'-\phi)^2}{2}}$$

Simplify $\hat{X}$ by doing Δ integral:

$$\hat{X} = \int_{-\infty}^{\infty} d\Delta \quad e^{-\frac{1}{2}(\Delta + i\hat{p})^2} \; e^{-\hat{p}^2/2}$$

In momentum basis, $\hat{X}$ is a diagonal operator, will $\hat{p} \to p$. Then get do Gaussian integral w/ $\Delta \to \Delta + ip$.

$$\Rightarrow \quad \hat{X} = \sqrt{2\pi} \; e^{-\hat{p}^2/2}$$

Putting the above together

$$\hat{T} = \sqrt{2\pi} \quad e^{-\frac{1}{2} S_A(\hat{\phi})} \; e^{-\frac{\hat{p}^2}{2}} \; e^{-\frac{1}{2} S_A(\hat{\phi})}$$

do not commute,

so cannot combine exponents

Properties of $\hat{T}$ (hold also in field theory)

- $\hat{T} = A^{+} A$ w/ $A = e^{-\frac{\hat{p}^2}{4}} e^{-\frac{1}{2} S_A(\hat{\phi})}$

$\Rightarrow \hat{T}^{+} = \hat{T}$ hermitian, w/ orthog e'states $|n\rangle$ & ~~r~~real e'values λ_n

$\Rightarrow \lambda_n = \langle n | A^{+} A | n \rangle = \langle An | An \rangle \geq 0$

so we say $\hat{T}$ is positive.

- Order e'values $\lambda_0 > \lambda_1 > \lambda_2 \cdots$ etc.

(assuming gaps – no degeneracies – for simplicity)

Then $tr(\hat{T}^N) = Z = \sum_{j=0}^{\infty} \lambda_j^N$

$= \lambda_0^N \left(1 + \left(\frac{\lambda_1}{\lambda_0}\right)^N + \ldots \right)$

<u>Largest</u> e'value is picked out as $N \to \infty$

We will shortly see that the correspond e'state $|0\rangle$ should be thought of as the vacuum state.

- Here we are using $\hat{T}$ in its eigen basis

$$\hat{T} = \sum_n |n\rangle \lambda_n \langle n|$$

$$\hat{T}^P = \sum_n |n\rangle \lambda_n^P \langle n|$$

Transfer matrix for field theory.

Follow same method of dividing action:

$$S = \sum_n \left\{ \left[\sum_\mu \tfrac{1}{2} (\phi_{n+\hat\mu} - \phi_n)^2 \right] + V(\phi_n) \right\}$$

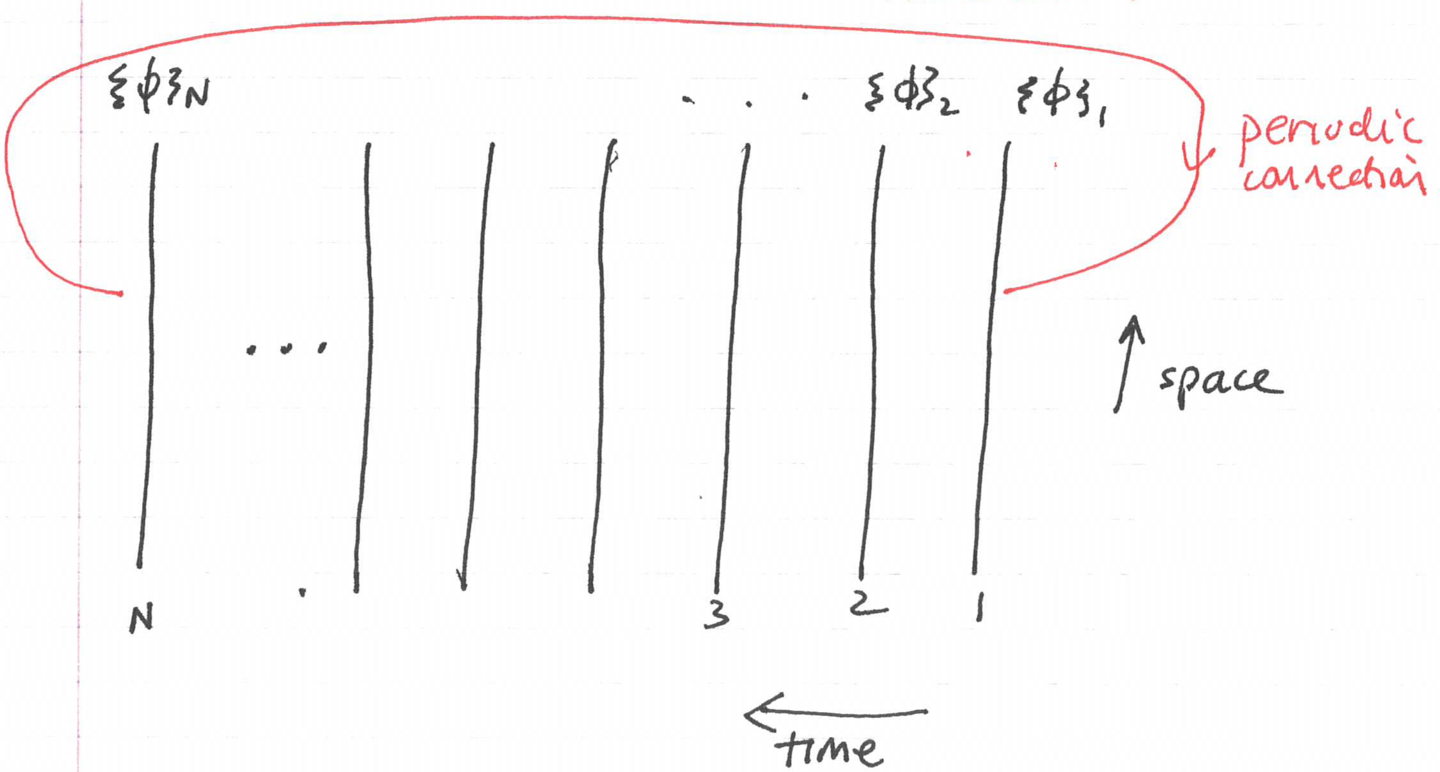
$$= S_A(\{\phi\}_N) + S_A(\{\phi\}_{N-1}) + \dots \quad \dots + S_A(\{\phi\}_1)$$

$$+ S_B(\{\phi\}_N, \{\phi\}_{N-1}) + \dots \quad \dots + S_B(\{\phi\}_1, \{\phi\}_N)$$

where $\{\phi\}_1$ is all fields on first "timeslice" $(n_4 = 1)$

so
$$S_A(\{\phi\}_j) = \sum_{\vec n} \left\{ \left[\sum_{i=1}^{3} \underbrace{\tfrac{1}{2} \left(\phi_{(\vec n, j)+\hat\imath} - \phi_{(\vec n, j)} \right)^2}_{\text{includes spatial part of kinetic term}} \right] + V(\phi_{\vec n, j}) \right\}$$

&
$$S_B(\{\phi\}_i, \{\phi\}_j) = \frac{1}{2} \sum_{\vec n} \underbrace{\left(\phi_{\vec n, i} - \phi_{\vec n, j} \right)}_{\text{time derv.}}{}^2$$



with $[\hat{\phi}_{\vec{n}}, \hat{\phi}_{\vec{n}'}] = 0$ & $\hat{\phi}_{\vec{n}} |\phi\rangle = |\phi\rangle \phi_{\vec{n}}$

Hilbert space is now $|\phi\rangle = \prod_{\vec{n}} \otimes |\phi_{\vec{n}}\rangle$

direct product of 1-d QM spaces, one for each spatial site

We thus want $\hat{T}$ to satisfy

$$\langle \phi' | \hat{T} | \phi \rangle = e^{-\frac{1}{2} S_A(\{\phi'\})} e^{-S_B(\{\phi'\}, \{\phi\})} e^{-\frac{1}{2} S_A(\{\phi\})}$$

For then, exactly as for the 1-d case,

$$Z = \int_\phi e^{-S}$$

Hilbert space for all fields on timeslice N

$$= \prod_{j=1}^{N} \left(\int_{\{\phi\}_j} \right) \langle \phi_N | \hat{T} | \phi_{N-1} \rangle \cdots \langle \phi_2 | \hat{T} | \phi_1 \rangle \langle \phi_1 | \hat{T} | \phi_N \rangle$$

integral over all fields on timeslice j

$$= \mathrm{tr}(\hat{T}^N)$$

as before — though space is bigger now

To construct $\hat{T}$, need conjugate momenta $\hat{p}_{\vec{n}}$ for each spatial site, with

$$[\hat{p}_{\vec{n}}, \hat{\phi}_{\vec{n}'}] = -i \delta_{\vec{n}\vec{n}'} \qquad [\hat{p}_{\vec{n}}, \hat{p}_{\vec{n}'}] = 0$$

Construction goes through as in 1-d case, since hopping term on each site can be treated separately.

spatial volume

Result:

$$\hat{T} = (2\pi)^{\frac{L^3}{2}}\; e^{-\frac{1}{2} S_A(\{\hat{\phi}\})}\; e^{-\frac{1}{2}\sum_n \hat{\vec{\pi}}_n^{\,2}}\; e^{-\frac{1}{2} S_A(\{\hat{\phi}\})}$$

set of all $\vec{\phi}_n$

which has (as claimed) same properties as for 1-d:

$\hat{T} = \hat{T}^{+}$; $\hat{T} > 0$; $\hat{T}|n\rangle = |n\rangle \lambda_n$ $\quad \lambda_n > 0$

λ_n real.

$\langle n' | n \rangle = \delta_{n'n}$

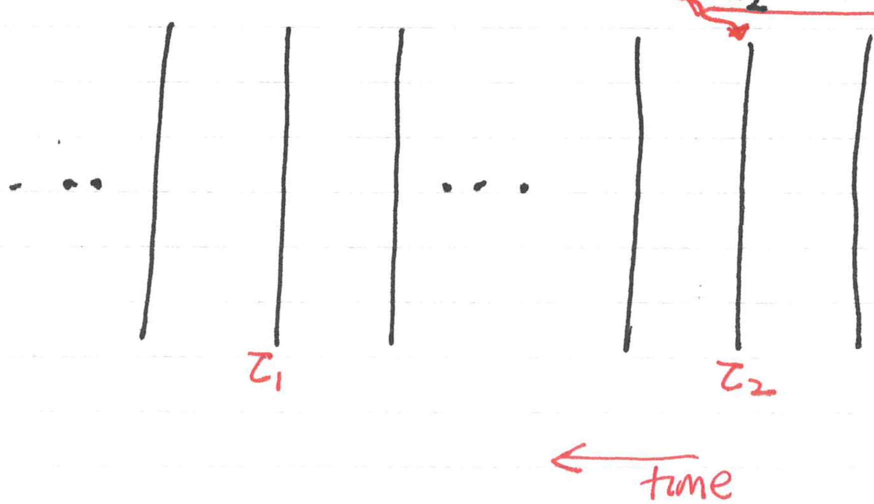
To fully understand the meaning of $\hat{T}$, consider a correlation fcn.

$$G_2(n_1, n_2) = \frac{1}{Z} \int [\mathcal{D}\phi]\, e^{-S}\, \phi_{n_1}\, \phi_{n_2}$$

with $n_1 = (\vec{n}_1, \tau_1)$ & $n_2 = (\vec{n}_2, \tau_2)$ with, say, $\tau_1 > \tau_2$.

Break up action as before

$$\ldots e^{-S_B(\{\phi\}_{\tau_2+1}, \{\phi\}_{\tau_2})}\; e^{-\frac{1}{2} S_A(\{\phi\}_{\tau_2})}\; \phi_{n_2}\; e^{-\frac{1}{2} S_A(\{\phi_{\tau_2}\})}\; e^{-S_B(\{\phi\}_{\tau_2}, \{\phi\}_{\tau_2-1})} \ldots$$



Same as before except two extra ϕ's.

So $$G_2(n_1, n_2) = \frac{1}{Z} \prod \int_{\{\phi\}_j} \langle \phi_N | \hat{T} | \phi_{N-1} \rangle \ldots \langle \phi_{\tau_1} | \hat{\phi}_{\vec{n}_1} \hat{T} | \phi_{\tau_1 - 1} \rangle \ldots$$

$$\ldots \langle \phi_{\tau_2} | \hat{\phi}_{\vec{n}_2} \hat{T} | \phi_{\tau_2 - 1} \rangle \ldots \langle \phi_1 | \hat{T} | \phi_N \rangle$$

$$= \frac{\mathrm{tr}\left(\hat{\phi}_{\vec{n}_1} (\hat{T})^{\tau_1 - \tau_2} \hat{\phi}_{\vec{n}_2} (\hat{T})^{N - \tau_1 + \tau_2} \right)}{\mathrm{tr}\left(\hat{T}^N \right)}$$

Now send $N \to \infty$ & assume $\lambda_0 > \lambda_1$ (gapped spectrum)

$$G_2(n_1, n_2) = \frac{\sum_{m, m'} \lambda_m^{N - \tau_1 + \tau_2} \langle m | \hat{\phi}_{\vec{n}_1} | m' \rangle \lambda_{m'}^{\tau_1 - \tau_2} \langle m' | \hat{\phi}_{\vec{n}_2} | m \rangle}{\sum_m \lambda_m^N}$$

Euclid. discrete lattice times

$$\xrightarrow{N \to \infty} \sum_{m'} \langle 0 | \hat{\phi}_{\vec{n}_1} \underbrace{| m' \rangle \left(\frac{\lambda_{m'}}{\lambda_0} \right)^{\tau_1 - \tau_2} \langle m' |}_{\hat{T}^{(\tau_1 - \tau_2)} / (\lambda_0)^{(\tau_1 - \tau_2)}} \hat{\phi}_{\vec{n}_2} | 0 \rangle$$

cf. continuum field theory expression: cts. Euclid. times

$$G_2(x_1^E, x_2^E) = \langle 0 | \hat{\varphi}(\vec{x}_1) e^{-\hat{H}(\tau_1 - \tau_2) a} \hat{\varphi}(\vec{x}_2) | 0 \rangle$$

$(\vec{x}_1, \tau_1 a)$ $(\vec{x}_2, \tau_2 a)$

$$= \sum_m \langle 0 | \hat{\varphi}(\vec{x}_1) | m \rangle e^{-E_m (\tau_1 - \tau_2) a} \langle m | \hat{\varphi}(\vec{x}_2) | 0 \rangle$$

Forms match! With $\boxed{\frac{\hat{T}}{\lambda_0} \Longleftrightarrow e^{-\hat{H} a}}$

- Since in correlation fcns one always ends up with the same number of $\hat{T}$'s in the numerator & denominator, we can rescale $\hat{T}$ to absorb the λ_0 factor

$$\hat{\hat{T}} \equiv \frac{\hat{T}}{\lambda_0} \longleftrightarrow e^{-\hat{H}a} \quad ; \quad \bar{\lambda}_m = \frac{\lambda_m}{\lambda_0}$$

Largest eigenvalue is now 1.

Eigenvector $|0\rangle$ of $\hat{\hat{T}} \longleftrightarrow$ vacuum of QFT.

- The name "transfer matrix" or "transfer op." now makes sense: $\hat{\hat{T}}$ translates distance a in Euclidean time.

- Spectrum of $\hat{\hat{T}}$ goes over, in the ctm limit, to that of the field theory. At finite a identify:

$$\bar{\lambda}_m = e^{-E_m a}.$$

- Can define a Hamiltonian at finite a

$$\hat{H} = -\frac{\ln \hat{T}}{a} + \frac{\ln \lambda_0}{a}$$

Hermitian, lowest e'value is $\emptyset$, but (using Baker-Campbell-Hausdorff) is non-local & not useful.

a^2 needed to get correct form below

- If we define $\hat{\phi}_{\vec{n}} = a\,\hat{\phi}^{phys}(\vec{n}a)$

  & $\hat{p}_{\vec{n}} = a^2\,\hat{p}^{phys}(\vec{n}a)$

recall, this is NOT the spatial momentum!

and take $a \to 0$ formally

(which makes no rigorous sense since e'values of $\hat{\phi}_{\vec{n}}$ need not be small)

then can ignore commutator terms in $\hat{T}$

$$\hat{T} \to (2\pi)^{L^3/2}\, e^{-S_A(\{\hat{\phi}_{\vec{n}}\}) - \frac{1}{2}\sum_{\vec{n}} \hat{p}_{\vec{n}}^2}$$

Using $\sum_{\vec{n}} \to \frac{1}{a^3}\int d^3x$ & $m = a\, m_{phys}$.

$$\hat{T} \to \exp\Big[-a\Big\{\underbrace{\int d^3x\, \frac{\hat{p}^{phys}(\vec{x})^2}{2} + \frac{m_{phys}^2\,\hat{\phi}^{phys}(\vec{x})^2}{2} + \lambda\,\hat{\phi}^{phys}(\vec{x})^4}\Big\} + \text{const.}\Big]$$

This is, indeed, the Hamiltonian of the scalar field when canonically quantized.

Further comments on transfer matrix

- $$Z = \mathrm{tr}\,\hat{T}^N = \lambda_0^N\, \mathrm{tr}\,\hat{\hat{T}}^N .$$
$$= \lambda_0^N\, \mathrm{tr}\left[e^{-aN\hat{H}}\right]$$
$$= \mathrm{tr}\left[e^{-aN(\hat{H} + \ln\lambda_0)}\right]$$

Up to a constant this is $\mathrm{tr}\, e^{-\beta\hat{H}}$ ← thermal partition fcn.
with $\beta \equiv \frac{1}{kT} = aN$.

- Thus doing the path integral with a finite number, N, of sites in the Euclidean time direction means one is studying the field theory in thermal equilibrium at $kT = 1/aN$.

Thus $N \to \infty \Rightarrow T \to 0$.

- All numerical simulations are thus necessarily done at $T > 0$, although usually at $T \ll \Lambda_{QCD}$ when studying the spectrum of QCD.
- To study QCD at non-zero T, however, one must use finite N

- For the correspondence with a finite temp partition fcn to hold we MUST HAVE PBC for bosons (& APBC for fermions)

• A major use of $\hat{T}$ is in understanding the physical content of corr. fcns.

e.g. $\frac{1}{Z}\int[D\phi]\, e^{-S}\left(\phi_{\vec{n}_1,\tau_1}\ \phi_0^2\ \phi_{\vec{n}_2,\tau_2}\right)$

$$\underset{\substack{N\to\infty \\ \tau_1>0 \\ \tau_2<0}}{=} \langle 0|\hat{\phi}_{\vec{n}_1}\ \hat{T}^{\tau_1}\,\hat{\phi}_{\vec{0}}^{\,2}\ \hat{T}^{-\tau_2}\ \hat{\phi}_{\vec{n}_2}|0\rangle$$

$$= \sum_{m,m'} \langle 0|\hat{\phi}_{\vec{n}_1}|m\rangle\, e^{-E_m\tau_1 a}\,\langle m|\hat{\phi}_{\vec{0}}^{\,2}|m'\rangle\, \underbrace{e^{-E_{m'}|\tau_2|a}\ \langle m'|\hat{\phi}_{\vec{n}_2}|0\rangle}$$

where $\lambda_m \equiv e^{-E_m a}$

so as $\tau_1\to\infty$, $|\tau_2|\to\infty$ one picks out the lightest states $|m\rangle$ & $|m'\rangle$ that couple to $\hat{\phi}$, & obtain the corresponding matrix element $\langle m|\hat{\phi}_{\vec{0}}^{\,2}|m'\rangle$

• In fact, dividing by the 2-pt fcn

$$\frac{1}{Z}\int[D\phi]\, e^{-S}\ \phi_{\vec{n}_1,\tau_1}\ \phi_{\vec{n}_2,\tau_2}$$

$$\xrightarrow[\substack{N\to\infty \\ \tau_1\to\infty \\ \tau_2\to-\infty}]{} \langle 0|\hat{\phi}_{\vec{n}_1}|m\rangle_{min}\ {}_{min}\langle m|\hat{\phi}_{\vec{n}_2}|0\rangle\, e^{-E_m(\tau_1-\tau_2)}$$

one cancels everything except
$\langle m|\hat{\phi}_{\vec{0}}^{\,2}|m\rangle$