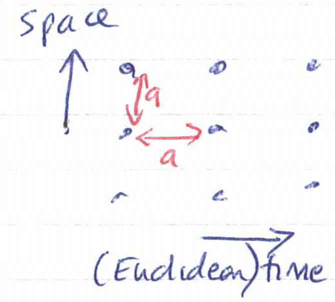
Lattice scalar field theory

Recall why: UV regularization.

Pick simplest lattice: hypercubic
(used for most simulations
since naturally parallelizes)



Since will almost always be in Euclidean space, henceforth drop subscript E: $x_E \to x$, $k_E \to k$.

Want to discretize

$$\int [D\phi]\, e^{-S}\ \phi(x_1) \dots \phi(x_n).$$

$$S = \int d^4x \left[\tfrac{1}{2}\partial_\mu\phi\,\partial_\mu\phi + V(\phi)\right]$$

In continuum, how you regularize UV doesn't matter (dim reg vs Pauli Villars, for example) once you include appropriate counterterms & take $n \to 4$ (dim reg) or $M_{PV} \to \infty$ (Pauli Villars).

Similarly, expect choice of discretization to be irrelevant when $a \to 0$.
(For numerical calcs. choice is important — can minimize discretization errors — but we are concerned w/ matters of principle for now.)

So we will pick simplest choice.

Fields live only on lattice sites; choose to be dim'less

$$\underbrace{a\phi(x)}_{\text{dim'less}} \longrightarrow \phi_n \qquad n = (n_1, n_2, n_3, n_4)$$

4-vector of integers labeling sites

$$\int d^4x \longrightarrow a^4 \sum_n$$

$$\Rightarrow \int d^4x\, V(\phi) = \int d^4x \left(\frac{m^2\phi^2}{2} + \lambda \phi^4 \right)$$

$$\longrightarrow \sum_n \left\{ \frac{(ma)^2}{2} \phi_n^2 + \lambda \phi_n^4 \right\}$$

Derivatives:

$$a^2 \partial_\mu \phi \longrightarrow \Delta_\mu^+ \phi_n = \phi_{n+\hat{\mu}} - \phi_n$$

OR

$$\Delta_\mu^- \phi_n = \phi_n - \phi_{n-\hat{\mu}}$$

OR

$$\Delta_\mu \phi_n = \frac{1}{2} (\phi_{n+\hat{\mu}} - \phi_{n-\hat{\mu}})$$

OR ... ?

Simplest most local choice for kinetic term is:

$$\int d^4x\, (\partial_\mu \phi)(\partial_\mu \phi) \rightarrow \sum_n \sum_\mu (\Delta_\mu^+ \phi_n)^2 = \sum_n \sum_\mu (\Delta_\mu^- \phi_n)^2$$

$$= \sum_{n,\mu} -\phi_n (\phi_{n+\mu} - 2\phi_n + \phi_{n-\mu})$$

- analog of $\int (\partial_\mu \phi)^2 = -\int \phi \Box \phi$
- assumes infinite lattice or finite lattice with PBC.

Finally, $\int [D\phi] = \prod_n \int_{-\infty}^{\infty} d\phi_n$ $\Rightarrow$ now well-defined!

So, in summary,

$$G_2(x_1, x_2) \longrightarrow \frac{1}{Z} \prod_n \left(\int_{-\infty}^{\infty} d\phi_n \right) e^{-S_{lat}} \; \phi_{n_1} \; \phi_{n_2}$$

$x_1 \approx a n_1$ $\quad$ $x_2 \approx a n_2$

$$S_{lat} = \sum_{n,\mu} \frac{(\phi_n - \phi_{n-\hat{\mu}})^2}{2} + \sum_n \frac{(ma)^2}{2} \phi_n^2 + \lambda \sum_n \phi_n^4$$

like classical stat. mech system

$$Z = \prod_n \left(\int_{-\infty}^{\infty} d\phi_n \right) e^{-S_{lat}}$$ $\equiv$ "partition fcn" (ubiquitous denominator)

What have we gained? If we also consider finite volume: $n_\mu = 0, 1, 2, \ldots, N_\mu - 1$

with, say, PBC $\phi_{(N_1, n_2, n_3, n_4)} = \phi_{(0, n_2, n_3, n_4)}$ etc.

then have finite number of simple integrals, with exponential damping.

$\Rightarrow$ rigorous def$^{\underline{n}}$ of correlators, calculable numerically.

Equivalent to a set of coupled anharmonic oscillators, one per site.
(coupling due to $\mu = 1-3$ kinetic term)

Meaning of "$\longrightarrow$" in discretization above:

Not equality. Meaning is "replaced by", with original object being formal & it's replacement well-defined.

For smooth $\phi(x)$ the continuum & lattice forms became equal when $a \to 0$.

But we know that most paths in a path integral are jagged & do NOT have a smooth ctm. limit.

So once we have regularized, it becomes a priori unclear what really happens as $a \to 0$

- for an asymptotically free theory like QCD, we expect jagged = high momentum modes to be treatable using pert. thy & thus to be able to analytically study $a \to 0$.

- for a non-asymptotically free theory like $\lambda \phi^4$ we don't know what to expect.

The answer, as we'll discuss, is that the renormalized coupling vanishes as $a \to 0$, whatever λ we use in the action (even $\lambda = \infty$).

Message: quantum continuum limit need not be close to/related to the classical cont. limit.

Bare vs renormalized parameters

- In QFT texts, it is standard to write the Lagrangian in terms of renormalized fields & params, e.g.

Srednicki (9.1) $\mathcal{L}_M = -\frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_m m^2 \varphi^2 + \frac{1}{6} Z_g g \varphi^3 + \dots$

wave fcn renorm

field normalized so that $\langle k | \varphi(x) | 0 \rangle = e^{-ik\cdot x}$

physical mass

mass renorm

This makes sense when developing pert. th^{y}. (PT)

- However, when we are studying a theory non-perturbatively, e.g. using lattice simulations, we often cannot usefully use standard PT. Also we want to express results in terms of the parameters we put in the computer codes. So we mostly use BARE parameters

$$\text{e.g. } \mathcal{L}_E = \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4$$

bare field

bare mass

bare coupling

$$\& \quad \mathcal{L}_{latt} = \frac{1}{2} (\Delta_\mu^- \phi_n)^2 + \frac{1}{2} (m a^2) \phi_n^2 + \frac{\lambda}{4} \phi_n^4$$

dim'less bare mass

dim'less bare field

bare coupling

- So our m has a <u>different meaning</u> than Srednicki's
  – perhaps we should use m_0, λ_0, ϕ_0, but we want to avoid too much notation.
- Please keep this in mind!

Continuum limits & general philosophy

- When we study an interacting lattice QFT in a non-pert. domain, in order to take the continuum limit in such a way that results do not depend on the lattice regularization, we need to find CRITICAL POINTS where ξ = corr. length $\to \infty$

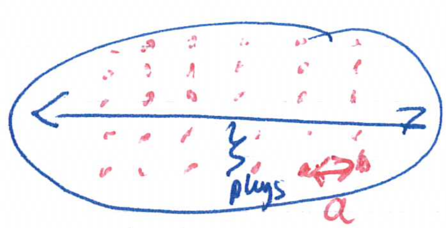
* Here ξ is defined by $G_2(n,0) \underset{x\to\infty}{\propto} e^{-|n|/\xi}$ (up to powers of $|n|$)

* Since $G_2 \sim e^{-M|n|}$ we have $\xi = \dfrac{1}{M_{min}}$
MINIMUM.

so $\xi \to \infty \iff M_{min} \to 0$ (Both in "lattice units")

* since $\xi = \xi_{phys}/a$

$\xi \to \infty$ means



Lattice details become unimportant

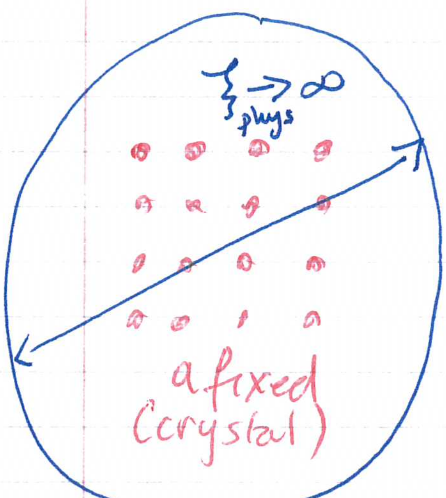
* This is the regime in studies of 3-d materials in stat. mech. where critical phenomena & universality set in. (More on this later.)

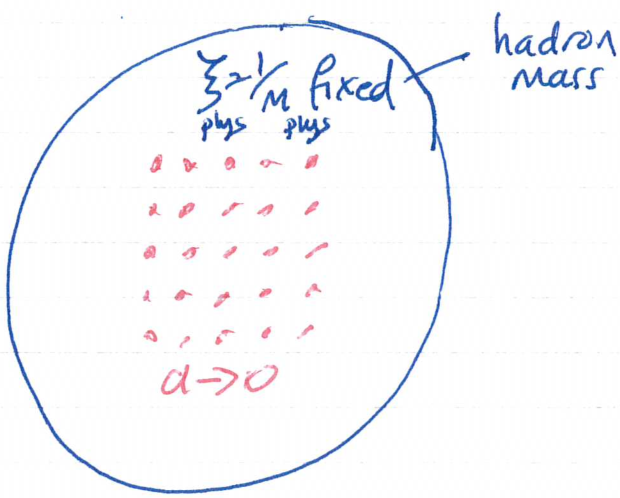
- We usually don't a priori know where, in parameter space, critical points or critical manifolds are, so we have to search for them, and then study the interactions of long-distance modes to determine what sort of continuum theory we have.

  E.g. triviality: $\lambda_{bare} = \infty$ but $\lambda_{phys} \to 0$

- For QCD (discussed later) we do know where the ctm limit is due to asymp. freedom
  $g \to 0$, $m \to 0$ (in appropriate manner)

- stat mech vs part. thy view of crit. pts.





same description: $\xi = \xi_{phys}/a \to \infty$

different interpretation

A basic calculation - free lattice propagator.

Set $a \to 1$; can recover by dimensional analysis at the end. (Here $aM \Leftrightarrow M$).

$$G_2(n,p) = Z^{-1} \int_\phi e^{-S} \phi_n \phi_p$$

$$S = \sum_{n,\mu} -\frac{1}{2}\phi_n(\phi_{n+\mu} - 2\phi_n + \phi_{n-\mu}) + \sum_n \frac{M^2}{2}\phi_n^2$$

$$= \frac{1}{2}\sum_{n,p} \phi_n M_{np} \phi_p \qquad (M = M^T = M^\dagger = M^*)$$

with kernel $M_{np} = \underbrace{\delta_{np}(M^2+8) - \sum_\mu (\delta_{n,p+\hat\mu} + \delta_{n,p-\hat\mu})}_{\text{tridiagonal} \Rightarrow \text{v. sparse}}$

$$\int_\phi \equiv \prod_n \left(\int_{-\infty}^{\infty} d\phi_n \right)$$

Multidimensional Gaussian integral.

$$Z = \int_\phi e^{-\phi_n \frac{M_{np}}{2} \phi_p} = (\det M)^{-1/2} (\sqrt{2\pi})^{N_{site}}$$

recall: show by diagonalizing M

$$\int_\phi e^{-S} \phi_n \phi_p = (\det M)^{-1/2} (\sqrt{2\pi})^{N_{site}} (M^{-1})_{np}.$$

can show similarly, or using sources.

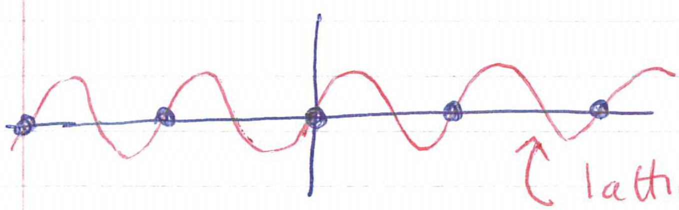
$$\Rightarrow G_2(n,p) = (M^{-1})_{np}$$

- Fourier transform to evaluate inverse

$$\phi_n = \underbrace{\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4}}_{\equiv \int_k} e^{ik\cdot n}\, \phi(k) \qquad (*)$$

(assuming infinite volume)

Note that range of k is 2π, since $e^{i(k+2\pi\hat{\mu})\cdot n} = e^{ik\cdot n}$



lattice cannot "see" this high wavelength.

Putting in factors of a: $\quad k_\mu = k_\mu^{phys}\, a$

$$\Rightarrow k_\mu^{phys}\Big|_{max} = \pi/a \qquad \text{UV cut-off}$$

- Substitute $(*)$ into action $\frac{1}{2}\phi_n M_{np} \phi_p = S$

$$\Rightarrow S = \frac{1}{2}\sum_{n,p}\int_{k,q} e^{ik\cdot n} e^{iq\cdot p}\, \phi(k)\,\phi(q)\, M_{np}.$$

$$= \frac{1}{2}\sum_n \int_{k,q} e^{i(k+q)\cdot n}\, \phi(k)\,\phi(q) \Big[m^2 + \sum_\mu \underbrace{(2 - e^{iq_\mu} - e^{-iq_\mu})}_{2(1-\cos q_\mu) \,=\, 4\sin^2 q_\mu/2}\Big]$$

$$= \frac{1}{2}\int_k \phi(k)\, \overset{\text{"}\phi(k)^*\text{"}}{\phi(-k)} \Big[m^2 + \sum_\mu \hat{k}_\mu^2\Big]$$

with $\hat{k}_\mu = 2\sin k_\mu/2$

Since S is now essentially diagonal (and looks like the free cfm form) we know that the Fourier transform of the propagator is

$$\frac{1}{m^2 + \sum_\mu \hat{k}_\mu^2}$$

so that

$$G_2(n,p) = \int_k \frac{e^{ik\cdot(n-p)}}{m^2 + \sum_\mu \hat{k}_\mu^2}$$

depends only on $n-p$ (trans^n inv)

or

$$\sum_{n,p} e^{-ik\cdot n} G_2(n,p)\, e^{iq\cdot p} = G_2(k,-q)$$

$$= (2\pi)^4 \delta^4(k-q) \frac{1}{m^2 + \sum_\mu \hat{k}_\mu^2} \longrightarrow = G_2(k).$$

from trans$^\text{n}$ inv.

really this is a periodic δ-fcn: $2\pi\, \delta_{per}(k) = \sum_n e^{ik\cdot n}$.

[Can also obtain simply by Fourier transforming M^{-1}_{np}]

Notes:

- $\hat{k}_\mu$ is lattice version of k_μ – FOR THIS DISCRETIZATION!
- $k_\mu = k_\mu^{phys} a$. If k_μ^{phys} fixed, $a \to 0$

$$\hat{k}_\mu = 2 \sin \frac{k_\mu}{2} \longrightarrow k_\mu^{phys} a \to 0.$$

& $G_2 \to \dfrac{1}{a^2(m_{phys}^2 + \sum_\mu k_\mu^{phys\,2})}$ i.e. continuum prop. "classical cont. limit"

$m = (m_{phys}\, a)$

Time dependence of propagator at fixed spatial momentum.

- Recall, in continuum, for free particle (1.12)

$$\int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}\, G_2(x^E, 0) \propto e^{-E_{\vec{q}}|x_4|}$$

$E_{\vec{q}} = \sqrt{\vec{q}_{phys}^{\,2} + M_{phys}^2}$

exponential fall off gives energy.

So look at $\sum_{\vec{n}} e^{-i\vec{k}\cdot\vec{n}}\, G_2(n, 0)$

$$= \int_{-\pi}^{\pi} \frac{dk_4}{2\pi} \frac{e^{ik_4 n_4}}{m^2 + \hat{\vec{k}}^2 + \hat{k}_4^2}$$

$\hat{\vec{k}}^2 = \sum_{i=1}^{3} \hat{k}_i^2$

$$\overset{\text{HW1}}{=} \frac{e^{-E|n_4|}}{\sinh E}$$

with $2\sinh\frac{E}{2} = \sqrt{m^2 + \hat{\vec{k}}^2}$

- We find: exact exponential fall off, even though time is discretized (Why? see below)

For $m \to M_{phys}\, a$ $\quad k_\mu \to k_\mu^{phys}\, a$ & $a \to 0$

fixed

$$E \to a \underbrace{\sqrt{M_{phys}^2 + \vec{k}_{phys}^2}}_{E_{phys}}$$

So $\xi \sim \frac{1}{E} \to \infty$

Recover free scalar field in cont. limit.