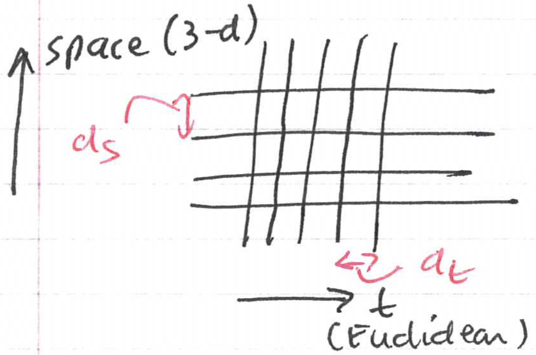
Lattice field theory: 1st page intro



Non perturbative regularization of QFT provided by discretizing space-time (Euclidean)

a = lattice spacing

Mostly consider hypercubic lattices, but sometimes asymmetric w/ $a_s \neq a_t$

- Key point : $a \Rightarrow$ UV cutoff $\Lambda \sim 1/a$

Combined with path integral formulation of QM, lattice provides regularization INDEPENDENT of pert. thy (unlike, say, dim. reg.).

$\Rightarrow$ Can address foundational issues
e.g. are chiral gauge theories well-defined?

Continuum limit is $a \to 0$.

- Numerical simulations allow non-pert. study of QFTs in strong-coupling domain
  - QCD at low energies (now a practical tool)
  - triviality in scalar theories
  - light dilaton in nearly conformal theories?

- Allow different series expansions & use of methods borrowed from stat. mech
  - strong coupling expansion
  - mean-field methods
  - critical phenom & universality

Challenges for LFT?

- breaks continuous (Euclidean) rot^n symm
- calculates Euclidean correlators
  - need to analytically continue to obtain physical scattering ampls.
- Numerically, forced to use finite vol, $a > 0$ & other approxs

We will see (in part) how to deal w/ these challenges.

Basic results from QFT

(Discuss in context of a real scalar field for notational simplicity)

We can determine everything about a QFT (at $T=0$, at least) from time-ordered corr. fcns

$$\langle 0 | T\{ \hat{\varphi}(x_1)\, \hat{\varphi}(x_2) \ldots \hat{\varphi}(x_n) \} | 0 \rangle = G_n(x_1, \ldots x_n)$$

Heisenberg op.

Minkowski positions

vacuum

symmetric fcn of arguments (if $\hat{\varphi}$ bosonic)

$$\hat{\varphi}(x) = e^{i\hat{H}t - i\hat{\vec{P}}\cdot\vec{x}}\, \hat{\varphi}(0)\, e^{-i\hat{H}t + i\hat{\vec{P}}\cdot\vec{x}}$$

e.g. $G_2(x_1, x_2)$ is propagator

Fourier transform $G(k) \sim \dfrac{iZ}{k^2 + m^2 - i\varepsilon}$ + regular

Mostly + metric

$\Rightarrow$ particle mass

e.g. $G_4(x_1, x_2, x_3, x_4)$ gives, via LSZ reduction formula, scattering amplitude

Path integral rep. (formal):

fcns, not operators $\Rightarrow$ manifestly symmetric

$$G_n(x_1, \ldots x_n) = \frac{\int [D\phi]\, e^{iS_M} \phi(x_1) \ldots \phi(x_n)}{\int [D\phi]\, e^{iS_M}}$$

$$S_M = \int d^4x \left[-\frac{(\partial_\mu \phi)(\partial^\mu \phi)}{2} - V(\phi) \right]$$

Minkowski

$$V(\phi) = \tfrac{1}{2} m^2 \phi^2 + \lambda \phi^4 \text{, say.}$$

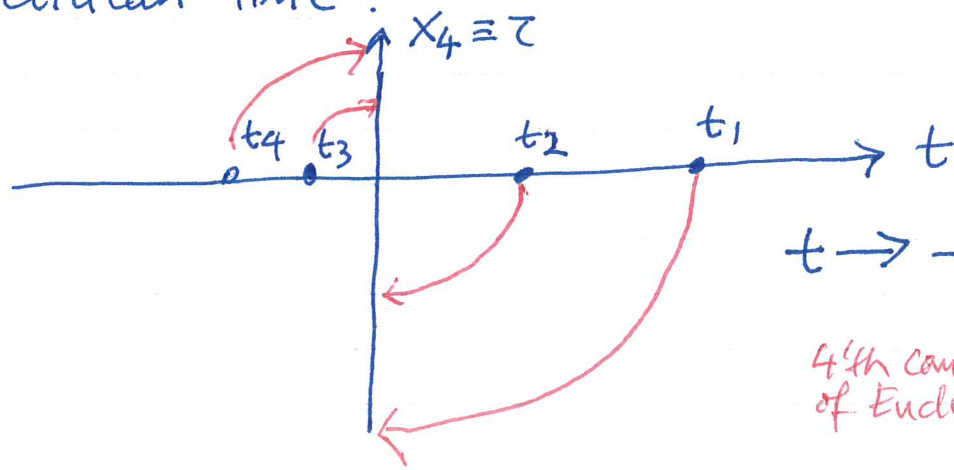
(really $(m^2 - i\varepsilon)$ to project onto $|0\rangle$)

Note: Srednicki likes to package these corr. fcns using sources, $S_M \to S_M + \int J(x)\phi(x)\, d^4x$, with fields in the functional integrand produced by $\frac{\delta}{\delta J(x)} Z[J]$.

This is nice formalism, and useful for developing pert. thy, but is less useful when considering numerical simulations.

The path integral is formal because of e^{iS_M} factor, which is an oscillating complex phase that does not obviously lead to a convergent integral.

To obtain a well defined quantity, we go to Euclidean time:



$$t \to -i x_4$$

real.

4'th component of Euclidean vector

- Since we will spend almost all our time in Euclidean space in this course, & since this "Wick rotation" may not be too familiar to you, let's spend a little time on it.

- This is an analytic continuation of G_n. In the operator form,

$$\hat{\varphi}(x) \longrightarrow \underbrace{e^{+\hat{H}x_4 - i\vec{\hat{P}}\cdot\vec{x}}\,\hat{\varphi}(0)\,e^{-\hat{H}x_4 + i\vec{\hat{P}}\cdot\vec{x}}}_{\equiv\ \hat{\varphi}(x^E)}$$

(x_1, x_2, x_3, x_4)

Assume $\hat{H}$ is bounded below & add const. so that $\hat{H}|0\rangle = \phi$ (zero) ($\& \ \vec{\hat{P}}|0\rangle = \phi$)

Consider for example $G_2(x, 0)$ with $t > 0$

Wick

$$\longrightarrow G_2(x^E, 0) = \langle 0|\, \underbrace{e^{\hat{H}x_4 - i\vec{\hat{P}}\cdot\vec{x}}}_{1}\, \hat{\varphi}(0)\, e^{-\hat{H}x_4}\, e^{i\vec{\hat{P}}\cdot\vec{x}}\, \hat{\varphi}(0)\,|0\rangle$$

key point: by Wick rotating clockwise, end up with e^{-Hx_4} instead of e^{+Hx_4}.

$$= \sum\!\!\!\!\!\!\int_n |\langle 0|\hat{\varphi}(0)|n\rangle|^2\, e^{-E_n x_4}\, e^{i\vec{p}_n\cdot\vec{x}}$$

spectral representation ($x_4 > 0$)

sum over states n has chance of converging (usually spectral density grows like a power of E so exponential wins).

N.B. Exercise: check same holds when $t < 0$ due to time ordering.

In a similar way, rotation of all G_n are well defined.
What happens to the functional integral?

Claim:

defined as described above

$$G_n(x_1^E, x_2^E, \ldots x_n^E)$$

$$= \frac{\int [D\phi]\, e^{-S_E[\phi]} \phi(x_1^E)\,\phi(x_2^E) \ldots \phi(x_n^E)}{Z}$$

$$Z = \int [D\phi]\, e^{-S_E[\phi]}$$

$S_E[\phi]$ is Euclidean action obtained as follows

$$i\, S_M = i \int d^4x \left[-\frac{\partial_\mu \phi\, \partial^\mu \phi}{2} - V(\phi) \right]$$

$$\xrightarrow{x^0 = t \to -i x_4} (i)(-i) \int \underbrace{d^4x^E}_{dx_1 dx_2 dx_3 dx_4} \left[-\frac{(\partial_\mu \phi)(\partial_\mu \phi)}{2} - V(\phi) \right]$$

both are $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right)$ contracted with Euclidean inner product.

$$\equiv -S_E[\phi]$$

$$S_E[\phi] = \int d^4x^E \left[\underbrace{\frac{(\partial_\mu \phi)(\partial_\mu \phi)}{2}}_{\geq 0} + V(\phi) \right]$$

- One can "derive" this claim following the usual steps for obtaining the path integral, except that the time evolution op. is now $e^{-\hat{H}x_4}$

- Note that since, in most theories, $S_E[\phi]$ is real & bounded below, $e^{-S_E[\phi]}$ damps highly oscillating fields (which have large S_E)

Thus the Euclidean path integral is much nicer than the Minkowski one. Indeed, when we discretize space-time to avoid UV problems, we get a completely well-defined object.

- Since we can, in principle, analytically continue our correlation fcns "back" to Minkowski time, we have lost nothing.

In practice, however, we have "gained" something (ground states are easier in Euclidean space) & effectively "lost" a lot (excited states & scattering amps are harder).

In particular, when calculating G_n numerically, we cannot do the requisite analytic continuation in a controlled way:

$$\sum_n e^{-E_n x_4} \xrightarrow{???} \sum_n e^{-iE_n t}$$

Euclidean: Dominated by ground state

Minkowski: all states contribute.

Aside on when a "Euclidean field theory",
e.g. correlators defined by a Euclidean path integral,
can be analytically continued to obtain
<u>physical</u> Minkowski correlation fcns.

Physical here means described by a Hilbert space w/ positive norm (i.e. no ghosts), with hermitian Hamiltonian $\hat{H}$ bounded below (so there is a ground state = vacuum). and on which Poincaré invariance is implemented by unitary ops.

The answer was provided in
K. Osterwalder & R. Schrader, Comm. Math. Phys 42 ('75) 281

If S_E is Euclidean invariant [SO(4) symmetry + translations],

$\Rightarrow$ Euclidean cor. fcns are invariant (for scalar fields)

& corr fcns are symmetric, e.g. $G_2(x_1^E, x_2^E) = G_2(x_2^E, x_1^E)$.

(automatically true in path integral formulation)

+ REFLECTION POSITIVITY holds
(discussed in Montvay & Münster)

then guaranteed that can analytically continue Euclidean cor. fcns & obtain physical Minkowski ones.

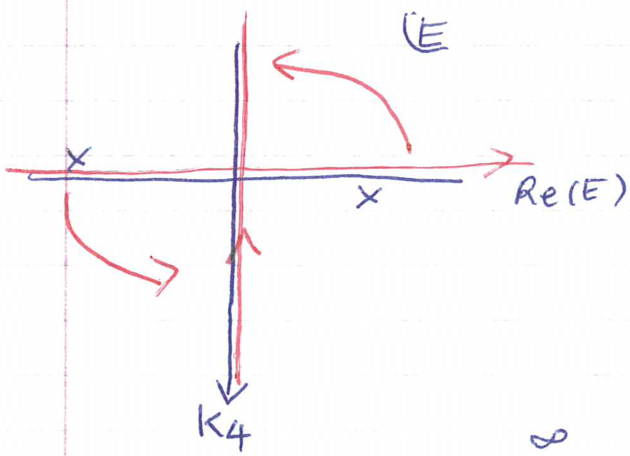
N.B. This doesn't always hold e.g. quenched QCD

Aside on Wick rotation

We have described this as $t \to -i x_4$.
However, you have probably seen this previously as a mathematical trick to evaluate Feynman diagrams.
These are the <u>same</u> rotation, but there is one small subtlety which I want to mention.

Simplest example is a "tadpole diagram" 

$$\Rightarrow I = \int d^4k \, \frac{1}{k^2+m^2-i\varepsilon} = \int d^3k \int_{-\infty}^{\infty} dE \, \frac{-1}{E^2-\omega_k^2+i\varepsilon} \qquad \omega_k^2 = \vec{k}^2+m^2$$



Given $i\varepsilon$ pole prescription can rotate contour as shown (counterclockwise)

$$E \to i(-k_4) \qquad (*)$$

this sign is subtlety.

$$\text{So } I = i\int d^3k \int_{-\infty}^{\infty} d(-k_4) \frac{1}{k_4^2+\omega_k^2+i\varepsilon} = i\int d^4k_E \, \frac{1}{k_E^2+m^2}$$

can drop.

Similar rot$^{\text{n}}$s work for all diagrams, as long as also rotate <u>external</u> momenta in same way
$\Rightarrow$ can rotate "entire theory"

Choice of sign in (*) (which is mathematically trivial) implies that $-tE + \vec{x}\cdot\vec{k} \longrightarrow +x_4k_4 + \vec{x}\cdot\vec{k} = x_E \cdot k_E$

so that Fourier transforms have Euclidean inner products in their phase factors

Let's end this discussion of basic cfM (= continuum) results by returning to the Euclidean two-point fcn & seeing what we can learn from it.

We assume our theory satisfies the O-S axioms so that $\hat{H}$ exists, as does $|0\rangle$, & choose $\hat{H}|0\rangle = \emptyset$ as above.

On (1.5) we found.

$$G_2(x^E, 0) = \sum\!\!\!\!\!\!\int_n |\langle 0|\hat{\varphi}(0)|n\rangle|^2 e^{-E_n x_4} e^{i\vec{p}_n\cdot\vec{x}} \qquad (x_4 > 0)$$

Similarly one finds

$$G_2(x^E, 0) = \sum\!\!\!\!\!\!\int_n |\langle 0|\hat{\varphi}(0)|n\rangle|^2 e^{+E_n x_4} e^{-i\vec{p}_n\cdot\vec{x}} \qquad (x_4 < 0)$$

$\Rightarrow$ as $|x_4| \to \infty$ G_2 always falls off, asymptotically $\propto e^{-E_n^{min}|x_4|}$

(an example of cluster decomposition)

E_n^{min} is the lowest energy state coupled to the vacuum by $\hat{\varphi}$, i.e. having the same quantum numbers as $\hat{\varphi}$.

E.g. in $\hat{\varphi} = \pi_0$ then $|n\rangle$ could be $|\pi_0\rangle$ or (in QCD) $|\pi_0 \pi_+ \pi_-\rangle, \ldots$ each with any momentum.

To simplify our expression, assume parity invariance

$\Rightarrow$ if there is a state $|n\rangle$ w/ $(E_n, \vec{p}_n)$
then " " another, degenerate, state $|n_\pi\rangle$
w/ $(E_n, -\vec{p}_n)$

Alternatively $G_2(x^E, 0)$ invariant under $\vec{x} \Rightarrow -\vec{x}$.

Using either result, we can write

$$G_2(x^E, 0) = \sum_n |\langle 0|\hat{\varphi}(0)|n\rangle|^2 e^{-E_n|x_4|} e^{i\vec{p}_n\cdot\vec{x}}$$

Now lets Fourier transform in space

$$\int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}} G_2(x^E, 0) =$$

$$\sum_n |\langle 0|\hat{\varphi}(0)|n\rangle|^2 e^{-E_n|x_4|} (2\pi)^3 \delta^3(\vec{p} - \vec{p}_n)$$

$$\propto \sum_{\substack{n \\ \vec{p}_n = \vec{p}}} |\langle 0|\hat{\varphi}(0)|n\rangle|^2 e^{-E_n|x_4|}$$

$\Rightarrow$ G_2 at fixed $\vec{p}$ is a sum/integral over all states in spectrum with $\vec{p}_n = \vec{p}$, weighted by $e^{-E_n|x_4|}$. As $|x_4| \to \infty$, lightest such state dominates (assuming a gap) & can read off E_n^{min}.

$\Rightarrow$ Can get E_n^{min} + a few excited states w/out analytic continuation.

Let's pursue this calculation to the bitter end in the case of a <u>free</u> scalar field with mass M.

Spectrum: $|\vec{p}\rangle \qquad E = \sqrt{\vec{p}^{\,2}+M^2} \equiv E_{\vec{p}}$

Up to now we've assumed non-rel norm $\langle n|n\rangle = 1$, but now let's use rel. norm.

$$\langle \vec{q}\,|\vec{p}\,\rangle = 2E\,(2\pi)^3\,\delta^3(\vec{p}-\vec{q}\,) \xrightarrow[\text{vol. } V = L^3]{\text{large finite}} 2E_{\vec{p}}V\,\delta_{\vec{p},\vec{q}}$$

so momenta $\vec{p} = \frac{2\pi}{L}\vec{n}$ (easier to count states)

$$\Rightarrow \quad G_2(x^E,0) = \sum_{\vec{p}} \frac{|\langle 0|\hat{\varphi}(0)|\vec{p}\,\rangle|^2}{2E_{\vec{p}}V} e^{-E_{\vec{p}}|x_4|} e^{i\vec{p}\cdot\vec{x}}$$

$$\int_V d^3x\, e^{-i\vec{q}\cdot\vec{x}}\, G_2(x^E,0) = \cancel{V}\frac{|\langle 0|\hat{\varphi}(0)|\vec{q}\,\rangle|^2}{2E_{\vec{q}}\cancel{V}} e^{-E_{\vec{q}}|x_4|}$$

one of discrete finite-vol momenta

Full Fourier transform:

$$\int dx_4\, e^{-iq_4x_4}\int_V d^3x\, e^{-i\vec{q}\cdot\vec{x}}\, G_2(x^E,0) = G_2(q_E)$$

$(q_4, \vec{q}\,)$

$$= \frac{|\langle 0|\hat{\varphi}(0)|\vec{q}\,\rangle|^2}{2E_{\vec{q}}}\left[\int_0^\infty dx_4\, e^{-E_{\vec{q}}x_4 - iq_4x_4} + \int_{-\infty}^0 dx_4\, e^{+E_{\vec{q}}x_4 - iq_4x_4}\right]$$

so $$G_2(q_E) = \frac{|\langle 0|\hat{\varphi}(0)|\vec{q}\rangle|^2}{2E_{\vec{q}}} \underbrace{\left(\frac{1}{E_{\vec{q}}+iq_4} + \frac{1}{E_{\vec{q}}-iq_4}\right)}_{\frac{2E_{\vec{q}}}{E_{\vec{q}}^2+q_4^2}}$$

$$= \underbrace{|\langle 0|\hat{\varphi}(0)|\vec{q}\rangle|^2}_{\text{This is unity for a free scalar field.}} \underbrace{\frac{1}{q_E^2+M^2}}_{\text{Recover expected form of Euclidean propagator}}$$

If analytically continue back to Mink. momenta

$q_4 \rightarrow i(-E)$ (see 1.9)

then $$G_2 \rightarrow \frac{1}{-E^2+\vec{q}^2+M^2} = \frac{-1}{(E^2-\omega_q^2)}$$

$\Rightarrow$ usual poles are recovered.