Spectrum of GW Dirac operator

- First recall spectrum in continuum of massless Dirac operator

  - $\not{D}^\dagger = -\not{D}$ $\Rightarrow$ pure imaginary e'values & orthonormal e'vectors

  $*$ $\not{D}|v_\lambda\rangle = |v_\lambda\rangle \lambda \qquad \lambda = -\lambda^* \qquad \langle v_{\lambda'}|v_\lambda\rangle = \delta_{\lambda'\lambda}$

  - $\not{D}\gamma_5 = -\gamma_5\not{D}$ $\Rightarrow$ non-zero e'values paired $\{\lambda, -\lambda\}$ & zero modes are e'vectors of γ_5

  $*$ $\not{D}\gamma_5|v_\lambda\rangle = -\gamma_5\not{D}|v_\lambda\rangle = -\underset{\overset{\shortparallel}{\lambda^*}}{\lambda}\,\gamma_5|v_\lambda\rangle$

  $\Rightarrow \gamma_5|v_\lambda\rangle = |v_{-\lambda}\rangle$ (normalization is correct — check!)

  $*$ For zero-modes $|v_0^{(i)}\rangle \quad i=1, n_{zero}$

  $\gamma_5|v_0^{(i)}\rangle \in$ zero mode subspace

  $\not{D}|v_0^{(i)}\rangle = 0$

  $\Rightarrow$ $\not{D}$ & γ_5 are block diagonal with zero-modes & non-zero modes in different blocks

  Furthermore $\not{D}\gamma_5 = \gamma_5\not{D}$ within zero-mode block

  $\Rightarrow$ can simultaneously diagonalize

  $\Rightarrow$ can choose zero-modes to be e'vectors of γ_5.

  $\gamma_5|v_0^{(i)}\rangle = \underset{\text{since } \gamma_5^2 = \mathbb{1}}{\pm}|v_0^{(i)}\rangle \qquad \begin{array}{l} + \Rightarrow \text{RH (positive chirality)} \\ - \Rightarrow \text{LH (negative chirality)} \end{array}$

- How much of this carries over to GW fermions?

To simplify analysis, assume further that

$$D^\dagger = \gamma_5 D \gamma_5 \qquad \text{"}\gamma_5\text{-hermitian"}$$

- holds in continuum
- holds for D_{Wilson} & for standard GW choices

N.B. D is neither hermitian or antihermitian, so e'values are, in general, complex

- For any γ_5-hermitian operator, e'values come in complex conjugate pairs $\{\lambda, \lambda^*\}$ unless they are real

Why? Characteristic eq. determines λ:

$$\begin{aligned} 0 &= \det(D - \lambda \mathbb{1}) \\ &= \det[\gamma_5 (D - \lambda \mathbb{1}) \gamma_5] && \text{since } \det \gamma_5^2 = \det \mathbb{1} = 1 \\ &= \det[D^\dagger - \lambda \mathbb{1}] && \text{using } \gamma_5\text{-hermiticity} \\ &= \left(\det[D - \lambda^* \mathbb{1}]\right)^* \end{aligned}$$

$\Rightarrow \lambda^*$ is an e'value if λ is an e'value

- GW + γ_5-hermiticity $\Rightarrow [D, D^\dagger] = 0$ i.e. D is "normal"

$\Rightarrow$ can diagonalize D by a unitary similarity trans. & e'vectors form orthogonal basis.

GW: $D\gamma_5 + \gamma_5 D = D\gamma_5 D \Rightarrow \gamma_5 D \gamma_5 + D = \gamma_5 D \gamma_5 D$

add in γ_5-hermicity: $\boxed{D^\dagger + D = D^\dagger D}$

Also: $D + \gamma_5 D \gamma_5 = D\gamma_5 D \gamma_5 \Rightarrow \boxed{D + D^\dagger = D D^\dagger}$

Together $\Rightarrow D^\dagger D = D D^\dagger$.

- Finally, GW spectrum.

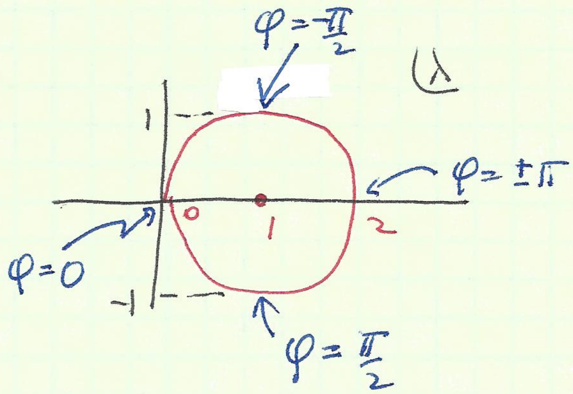
First define $V = 1 - D$

Then $V V^\dagger = (1-D)(1-D^\dagger) = 1 \overbrace{-D - D^\dagger + DD^\dagger}^{\emptyset}$
$= 1 = V^\dagger V$

so V is unitary $\Rightarrow$ e'values are $e^{i\phi}$

$\Rightarrow$ $D = 1 - V$ has e'values $\lambda = 1 - e^{i\phi}$

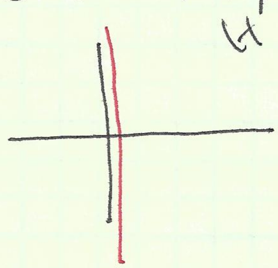
$\Rightarrow$



E'values lie on "GW circle" & come paired $\lambda = 1 - e^{\pm i\phi}$ except for $\lambda = 0$ & 2

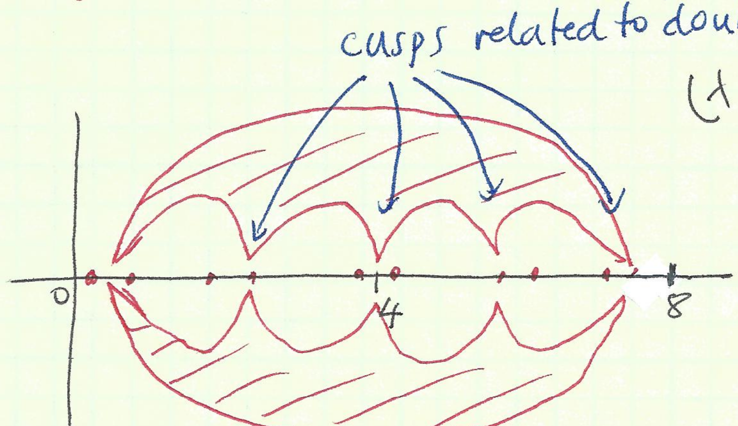
- Contrast this with spectrum of other fermions

Continuum & staggered massless fermions



pure imaginary

Wilson fermions ($m_0 = 0$) [Symmetric about $\mathrm{Im}\,\lambda = 0$ & $\mathrm{Re}\,\lambda = 4$ axes.]



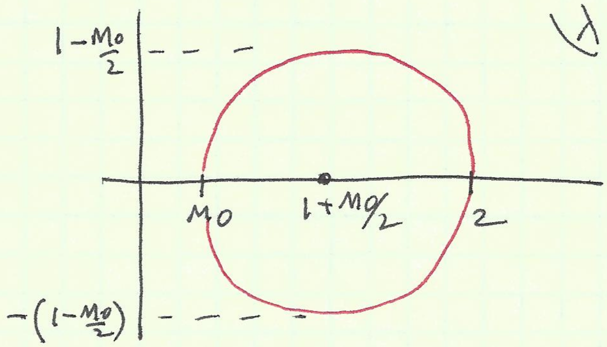
complex w/ isolated real e'values.

- Including mass term $\bar{\Psi} M_0 (1 - \frac{D}{2}) \Psi$,

$$D \to D + M_0\left(1 - \frac{D}{2}\right) = D\left(1 - \frac{M_0}{2}\right) + M_0$$

$\Rightarrow$ circle's radius is $1 - \frac{M_0}{2}$

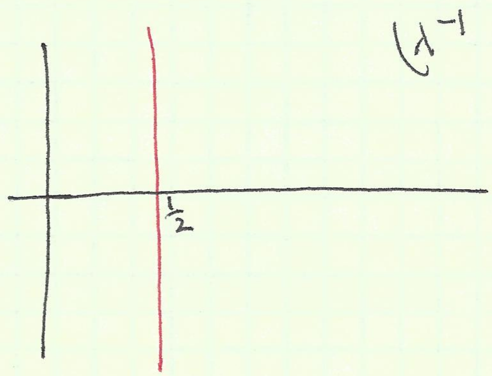
real e'values $\left.\begin{array}{l} \lambda = 0 \to \lambda = M_0 \\ \lambda = 2 \to \lambda = 2 \end{array}\right\} \Rightarrow$ center moves to $1 + \frac{M_0}{2}$



[- Aside: spectrum of D^{-1} (massless propagator)

$$\lambda^{-1} = \frac{1}{1 - e^{i\varphi}} = \frac{1 - e^{-i\varphi}}{(1 - e^{i\varphi})(1 - e^{-i\varphi})} = \frac{1 - c\phi + i s\phi}{2(1 - c\phi)}$$

$$= \frac{1}{2} + i\,\frac{s\phi}{1 - c\phi}$$



Like continuum, but offset horizontally.]

Chirality of GW eigemodes

(really: can be chosen to be)

- Recall for continuum $\not{D}$, zero modes are $_\wedge$ chiral while non-zero modes are connected by action of γ_5
- For GW, real modes ($\lambda = 0, 2$) are chiral while λ & λ^* are connected by γ_5.

i.e. very similar situation

$$GW: \quad D\gamma_5 \overbrace{(1-D)}^{V} = -\gamma_5 D$$

$$\Rightarrow \quad D\gamma_5 V|v_\lambda\rangle = -\gamma_5 D|v_\lambda\rangle$$

$$\Rightarrow \quad D\gamma_5|v_\lambda\rangle e^{i\phi} = -\gamma_5|v_\lambda\rangle \lambda$$

$$\Rightarrow \quad D(\gamma_5|v_\lambda\rangle) = (\gamma_5|v_\lambda\rangle)\underbrace{\left[-(1-e^{i\phi})e^{-i\phi}\right]}_{1-e^{-i\phi}=\lambda^*}$$

- Note that $(\langle v_\lambda|\gamma_5)(\gamma_5|v_\lambda\rangle) = \langle v_\lambda|v_\lambda\rangle = 1$

so normalization is maintained & $\boxed{\gamma_5|v_\lambda\rangle = |v_{\lambda^*}\rangle}$

(just saying that γ_5 is unitary)

- For $\lambda=0$ subspace $\quad \gamma_5 D|v_0\rangle = 0 = D\gamma_5|v_0\rangle$

so, as in ctm $\quad [D,\gamma_5]=0$ in subspace & both are block-diag

$\Rightarrow$ can choose zero modes to have definite chirality ± 1

- For $\lambda=2$ subspace $\quad \gamma_5 D|v_2\rangle = 2\gamma_5|v_2\rangle = D\gamma_5|v_2\rangle$

so again $[D,\gamma_5]=0$ & again can choose modes to have definite chirality ± 1.

- Back to change in measure when do $U(1)_A$ transf.$^{\underline{n}}$

$$\text{Jacobian} = 1 - i\alpha \, \mathrm{tr}(\gamma_5 D)$$

- Use spectral decomposition to evaluate $\mathrm{tr}(\gamma_5 D)$

$$D = \sum_\lambda |v_\lambda\rangle \lambda \langle v_\lambda|$$

$|v_\lambda\rangle$ is just convenient & familiar not^n for vector v_λ & $\langle v_\lambda|$ is not^n for $v_\lambda^\dagger$

means "sum over all eigenstates" (even if have degeneracies)

$$\text{So } \mathrm{tr}(\gamma_5 D) = \sum_\lambda \lambda \langle v_\lambda | \gamma_5 | v_\lambda \rangle$$

But we've seen this vanishes except for real λ – here $\lambda = 0$ & 2 (Only $\lambda=2$ contributes here.)

$$= 2\left[n_+(\lambda=2) - n_-(\lambda=2) \right]$$

$n_+(\lambda=2)$: # RH @ $\lambda=2$; $n_-(\lambda=2)$: # LH @ $\lambda=2$

OK, interesting, starts to look like an index th$^{\underline{m}}$.

$n_+ - n_-$ is generically called the index of an operator

Alternative evaluation:

$$\text{Use } \mathrm{tr}(\gamma_5 \otimes \mathbb{1}) = 0 = \sum_\lambda \langle v_\lambda | \gamma_5 | v_\lambda \rangle = n_+(\lambda=0) - n_-(\lambda=0) + n_+(\lambda=2) - n_-(\lambda=2)$$

$\mathbb{1}$: color & space-time

$\Rightarrow$ total index of D vanishes

$$\text{Thus } \mathrm{tr}(\gamma_5 D) = -2 \underbrace{\left[n_+(\lambda=0) - n_-(\lambda=0) \right]}_{\text{index of } D \text{ @ } \lambda=0}$$

Key point is that this index does not vanish on a general configuration.

Indeed, in the continuum, for smooth A_μ, one has the Atiyah-Singer index thm. for the zero e'values of $\not{D}$:

$$\underbrace{n_- - n_+}_{} = \int_x q(x) \equiv Q_{top}$$

topological charge (integer w/ certain BC) (does not change when make local change in A_μ)

$$q(x) = \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \, tr_{color}[F_{\mu\nu}(x) F_{\rho\sigma}(x)]$$

top. charge density.

– index of $\not{D}$: $n_\pm$ = # of RH/LH zero modes

Matching to the lattice result – assuming

$n_- - n_+ |_{cont}$ matches to $n_-(\lambda=0) - n_+(\lambda=0)|_{latt}$

we have

$$\boxed{Q_{top,\,latt} = \frac{1}{2} tr\, \gamma_5 D} \quad (*)$$

& a possible choice of $q(x) \to q_n = \frac{1}{2} tr_{\substack{color\\Dirac}} (\gamma_5 D_{nn})$
(up to total derivatives) not space-time

The result (*) [which had been found somewhat earlier by alternative arguments] "solves" the problem of defining a topologically invariant quantity on a discrete lattice. The quotes around "solve" indicate this is not a unique sol^n, since the value of $tr\,\gamma_5 D$ on a given configuration depends on the details of D – although these differences vanish for smooth configs as $a \to 0$.

[Aside: a little more background on the discussion of the previous page.

In continuum (Euclidean) gauge theories on S^4 (R^4 w/ suitable fall off properties for fields) gauge configs divide into topological sectors w/ different, INTEGER, values of Q_{top}.

In the $Q_{top} = \pm 1$ sectors the lowest action configs are 't Hooft–Polyakov instantons, which are known explicitly.

Smooth deformations change the action but not Q_{top}.

Semi classical expansion of Z including instanton contribs yeilds additional $e^{-\#/g^2}$ contributions, which are non-perturbative.

//

When discretize, there are no topological sectors of gauge fields – one can continuously deform a discretized instanton to $A_\mu^{(n)} \simeq 0$ ($U_{n,\mu} = 1$ in an appropriate gauge).

The construction above replaces a gauge-field based def^n of Q_{top} w/ an index-based one.

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In the last few years, alternative formulations have been given, the most far reaching being that based on "Wilson flow" – a controlled smoothing of lattice gauge fields that allows a continuum-like def^n of top. charge using the gauge field to be used.]

- Can also show for slowly varying "classical" fields that

$$\mathrm{tr}_{\substack{\text{Color}\\ \text{Dirac}}}(\gamma_5 D_{nn}) = 2\,\frac{1}{32\pi^2}\,\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}_{\text{Color}}\left[F_{n,\mu\nu}F_{n,\rho\sigma}\right]$$

where $F_{n,\mu\nu} = \left\{\left(\Delta^+_\mu A_{n,\nu} - \Delta^+_\nu A_{n,\mu}\right) + ig\left[A_{n,\mu}, A_{n,\nu}\right]\right\}\left(1 + O(a)\right)$

& $U_{n,\mu} = e^{-igA_{n,\mu}}$

- This requires an explicit form for D_{GW} (e.g. overlap – see below).

Explicit solutions to GW relation

For all this discussion to be useful, we need an explicit form for D with which we can compute numerically (at least to good approximation).

The most conceptually easy to understand (at least qualitatively) is the domain-wall fermion approach of David Kaplan. Here one adds an additional discretized "fifth" dimension, which one should think of as a flavor space, which has ends (i.e. is finite).

By a clever choice of the action in the fifth dimension, combined w/ the 4-d Wilson action, one finds LH $\lambda \approx 0$ modes on one end & RH on the other. (This is actually Shamir's variant where the domain-"wall" is the end of a slab.)

If they are separated by an infinitely large slab then they decouple from each other & can be rotated indep.
$\Rightarrow$ chiral symmetry.

A practical implementation must involve a finite slab & thus a small effective mass term.

In the infinite slab limit these fermions satisfy a GW-like relation.

The details are complicated so I will describe instead the "overlap" Dirac operator of Neuberger (based on earlier work with Narayanan).

Overlap operator (simplest version – there are several variants)

$$D = 1 + \gamma_5 \,\text{sign}\, H \qquad H = \gamma_5 (\underbrace{D_W}_{\text{massless Wilson fermion op.}} \underbrace{-1}_{\text{large negative mass term.}})$$

- What does this mean? Use spectral decomposition.

$$H\,|v_{\lambda_H}\rangle = |v_{\lambda_H}\rangle\, \lambda_H \quad \leftarrow \text{real.}$$

$$\text{sign}\, H = \sum_{\lambda_H} |v_{\lambda_H}\rangle \,\text{sign}(\lambda_H)\, \langle v_{\lambda_H}|$$

- Well defined as long as $\lambda_H \neq 0$.

Note that $H = H^\dagger$ since

$$H^\dagger = (D_W^\dagger - 1)\gamma_5$$
$$= \gamma_5\, \gamma_5 (D_W^\dagger - 1)\gamma_5$$
$$= \gamma_5 (D_W - 1)$$

H is called the "hermitian Wilson-Dirac Hamiltonian"

Before proceeding to gain understanding of the properties of D, let's check that it works.

(1) $\gamma_5 D \gamma_5 = 1 + \text{sign}(H)\gamma_5 = D^\dagger$ ✓ γ_5-hermitian

(2) $D = 1 - V$, $\underbrace{V = -\gamma_5 \,\text{sign}\, H}_{\text{must be unitary}}$

$$V V^\dagger = \gamma_5 \,\text{sign}H\, \text{sign}H\, \gamma_5 = 1 \checkmark$$
$$V^\dagger V = \text{sign}H\, \gamma_5^2\, \text{sign}H = 1 \checkmark$$

So it satisfies GW rel$^{\underline{n}}$ – but strange looking object.

Satisfying GW is not enough. We also need

(a) It's Fourier transform $\tilde{D}(p) = i\cancel{p}$ for $|p| \ll 1$, $U_{n,\mu} = 1$

i.e. it describes a (single) continuum fermion in classical continuum limit.

(b) It is local: $|D_{nm}| \leq C e^{-\gamma|n-m|}$

with C, γ indep of gauge field & $\gamma \gg m_{min}$

(c) $\tilde{D}(p) \neq 0$ except for $p = 0$ when $U_{n,\mu} = 1$

(no doubles)

(b) can be shown to be satisfied except on configurations of measure zero in path integral

Note that D is not "ultra-local" (meaning that only connects a finite # of close sites), but instead couples all sites together.

To study (a) & (c) need to rewrite D in more useful form these two commute

$$\text{sign}\, H = \frac{H}{\sqrt{H^2}} = \sum_{\lambda_H} |v_{\lambda_H}\rangle \frac{\lambda_H}{|\lambda_H|} \langle v_{\lambda_H}|$$

$$\Rightarrow \gamma_5\, \text{sign} H = \frac{(D_W - 1)}{\sqrt{(D_W^\dagger - 1)(D_W - 1)}}$$

since $H^2 = H^\dagger H$
$= (D_W^\dagger - 1)\cancel{\gamma_5 \gamma_5}(D_W - 1)$

$$\Rightarrow \boxed{D = 1 + \frac{(D_W - 1)}{\sqrt{(D_W^\dagger - 1)(D_W - 1)}}}$$

Meaning: "e'vectors w/ real e'values"

Can show that only real e'vectors of D_W lead to real e'vectors of D. (General e'vectors of D_W are NOT e'vectors of D; diagonalizing leads to complex e'values of D.)

If $D_W |v_{\lambda_W}\rangle = |v_{\lambda_W}\rangle \lambda_W$ $\quad \lambda_W$ real

then can show from γ_5-Hermiticity that

$D_W^\dagger |v_{\lambda_W}\rangle = |v_{\lambda_W}\rangle \lambda_W$ also.

$\Rightarrow |v_{\lambda_W}\rangle$ is an e'vector of D, with e'value

$$1 + \frac{\lambda_W - 1}{|\lambda_W - 1|} = \begin{cases} 0 & \lambda_W < 1 \\ 2 & \lambda_W > 1 \end{cases}$$

i.e. real evalues $< 1 = |$magnitude of Wilson fermion mass$|$ are "projected" onto zero-modes of D, while the rest give $\lambda = 2$ modes.

Further properties of the free case.

Can block diagonalize D by Fourier transforming

Recall $\tilde{D}_W(p) = i\not{s} + \frac{\hat{p}^2}{2}$ $\quad s_\mu = \sin p_\mu \quad \hat{p}_\mu = 2\sin p_\mu/2$

$$\tilde{D} = 1 + \frac{i\not{s} + x}{\sqrt{(-i\not{s}+x)(i\not{s}+x)}} \qquad x = \frac{\hat{p}^2}{2} - 1$$

$$= 1 + \frac{i\not{s} + x}{\sqrt{s^2 + x^2}}$$

$\sum_\mu s_\mu^2$

Now send $p \to 0 \Rightarrow x \to -1$

$\Rightarrow D \to i\not{s}$ desired form – single ctm fermion.

For general p can diagonalize.

- $\frac{\not{s}}{\sqrt{s^2}}$ is unitary & hermitian $\Rightarrow$ e'values ± 1
- If $\frac{\not{s}}{\sqrt{s^2}}|1\rangle = |1\rangle$ then $\frac{\not{s}}{\sqrt{s^2}}(\gamma_5|1\rangle) = -(\gamma_5|1\rangle)$

  $\Rightarrow$ e'values come in pairs

Thus, since $\not{s}$ has four e'values, they are

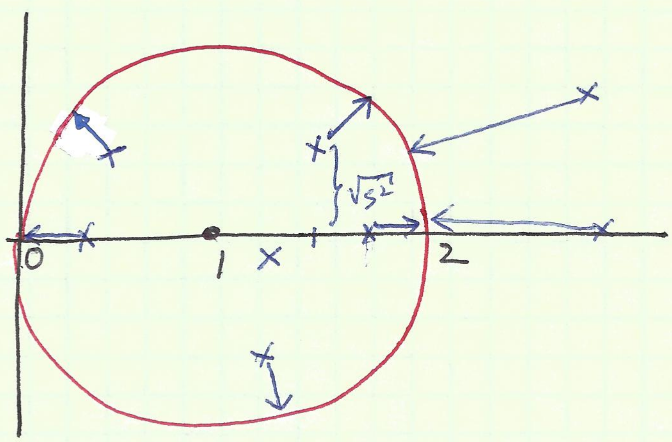
$\sqrt{s^2} \cdot \{1, 1, -1, -1\}$

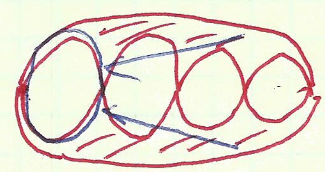
So e'values of $\tilde{D}$ are

$$1 + \frac{\pm i\sqrt{s^2} + x}{\sqrt{s^2 + x^2}}$$

where D_W e'values are $\pm i\sqrt{s^2} + x + 1$

This is a projection onto the GW circle (from center)





Wilson spectrum "collapses" onto GW circle

$\Rightarrow$ only real e'values project to $\lambda = 0, 2$.

For free propagator, only e'values projected to $\lambda = 0$ are those at $p = 0$

All would-be doublers collapse to $\lambda = 2$ — far from continuum.