Chiral symmetry on the lattice: Ginsparg Wilson fermions.

Nice discussion in Gattringer & Lang Ch 7.

So far we have discussed naive, staggered & Wilson fermions — all of which mangle chiral symmetry.

e.g. staggered $\Rightarrow N_f = 4$

In the ctm. have $SU(4)_L \times SU(4)_R \times U(1)_V$

on the lattice have $(m=0)$ $U(1)_A \times U(1)_V$

flavor non singlet axial transf. — fermion #.

$$\chi_n \to e^{i\varepsilon(n)\theta}\chi_n \qquad \varepsilon(n) = \begin{cases} +1 & \text{even sites} \\ -1 & \text{odd sites} \end{cases}$$

(No time to explain why it's flavor non-singlet & axial.)

This symmetry implies no tuning is needed to set $m_{phys} = 0$,

But loss of most of chiral symmetry leads to many complications when using staggered fermions.

Wilson fermions $\Rightarrow$ can choose N_f, but break axial symmetries

e.g. $N_f = 2$, degenerate — lattice theory has symmetry

$$SU(2)_V \times U(1)_V$$

Also have to tune m_0 — & do NOT regain full $SU(2)_L \times SU(2)_R$ even when approach $m_\pi = 0$.

$\Rightarrow$ If chiral symmetry matters, then usually Wilson fermions are a poor choice.

Why do we care about Xral symm.?

- One of the characteristic non-perturbative phenomena displayed by QCD is SPONTANEOUS XRAL SYMM. BREAKING. (XSB)

If have N_f massless fermions & N_f not too large$_{\wedge}$ $(N_f < 8\text{-}12)$
then there is overwhelming evidence (from nature if $N_f = 2$ or 3; from simulations otherwise) that

$$\underset{\text{symmetry of quantum theory}}{SU(N_f)_L \times SU(N_f)_R} \xrightarrow[\text{break.}]{\text{spont.}} \underset{\text{symmetry of vacuum.}}{SU(N_f)_V}$$

- Simplest$_{\wedge}$ Order parameter is the condensate

$$C_{ij} = -\langle \bar{\Psi}_j \Psi_i \rangle = -\langle \bar{\Psi}_{Lj} \Psi_{iR} + \bar{\Psi}_{Rj} \Psi_{iL} \rangle$$

flavor $i,j = 1, N_f$

Clearly not invariant if

$$\Psi_R \to R\,\Psi_R \ , \ \bar{\Psi}_L \to \bar{\Psi}_L L^\dagger, \text{ etc.}$$

with $R \in SU(N_f)_R \quad L \in SU(N_f)_L$

Signature of XSB is $\langle \bar{\Psi}_j \Psi_i \rangle \neq 0$.

usually choose vacuum (by global rot^n) s.t.

$$\langle \bar{\Psi}_j \Psi_i \rangle \propto \delta_{ji}$$

- We know this is happening in QCD because of the presence of pions ($N_f^2 - 1$ Goldstone bosons), the lack of parity doubling in the spectrum, etc.

- Approximate Chiral symmetry leads to many phenomenologically important predictions

  e.g. relating processes w/ n pions to those w/ n+1
  e.g. $K \to \pi\pi$ & $K \to \pi\pi\pi$

  e.g. predicting the behavior of matrix elements as $m_q \to 0$ ("chiral behavior")

  $$\langle K^0 | \mathcal{H}_W | \bar{K}^0 \rangle \propto m_K^2$$

  This is particularly useful for chiral extrapolations of lattice "data"

- When calculating matrix elements, chiral symmetry restricts operator mixing

  e.g. $\bar{\psi}_i \psi_i$ cannot mix with $\mathbb{1}$
  (but it can w/ Wilson fermions)
  ($\Rightarrow \langle \bar{\psi}\psi \rangle_{\text{Wilson fermions}} \propto 1/a^3$ - divergent & useless)

Over the last 25 years various clever ideas (twisted mass, Wilson flow ...) have allowed the use of Wilson-like fermions to address many of the phenomena of χSB despite breaking χ symm explicitly.

Nevertheless, not all questions can be asked, and it would clearly be good to have lattice fermions with exact chiral symmetry.

This is <u>essential</u> if we are to discretize chiral gauge theories.

Let $S_{fermion} = -\sum_{np} \bar{\psi}_n D_{np} \psi_p$ so D is lattice Dirac operator (with $M=0$)

e.g. $D_{naive} = \gamma_\mu [\delta_{n,p-\mu} U_{n,\mu} + \delta_{n,p+\mu} U^{\dagger}_{n-\mu,\mu}]$

in ctm limit want $D = a\gamma_\mu D_\mu$ covariant derivative

Chiral symmetry requires $DP_L = P_R D$

$$\text{or } \gamma_5 D + D\gamma_5 = 0 \qquad (*)$$

for then $S = -\bar{\psi}_L D \psi_L - \bar{\psi}_R D \psi_R$.
allowing ψ_L & ψ_R to be rotated independently.

Neilsen-Ninomiya (Karsten-Smit) thm says that we cannot separate ψ_L & ψ_R — they always come in pairs as part of the same lattice fermion field. So, even if (*) holds (as it does, say, for minimally doubled fermions — HW4) we still cannot rotate ψ_L & ψ_R independently since they are part of the same field.

After a tortuous history, beginning w/ a 1982 paper by Ginsparg & Wilson, subsequently forgotten, key papers by David Kaplan ("domain-wall" fermions) & Narayan & Neuberger ("overlap" fermions), and a rediscovery of Ginsparg & Wilson's work by P. Hasenfratz & others, a simple formulation of lattice chiral symmetry was presented.

- The trick turned out to be to back off from (*) to

$$(\#) \quad \boxed{\gamma_5 D + D\gamma_5 = D\gamma_5 D}$$

(can include extra numerical constants in here, but we won't)

Ginsparg–Wilson relation

- If write $D = a\bar{D}$

dim'less dim'full $\Rightarrow \not{D}$ in cotm.

then have $[\gamma_5, \bar{D}]_+ = a\bar{D}\gamma_5\bar{D}$

so the violations of XSymm. are formally suppressed as $a \to 0$, as well as being local.

- Another way of writing (#) is

$$\bar{D}^{-1}\gamma_5 + \gamma_5\bar{D}^{-1} = \gamma_5 \quad \text{or} \quad [\bar{D}^{-1}, \gamma_5]_+ = a\gamma_5$$

$\Rightarrow$ the violation of XS for the propagator $\bar{D}^{-1}$ is local & vanishes as $a \to 0$.

(this is a useful form when studying Ward identities)

- It turns out that this is a weak enough violation of XS (locality is key) that essentially all of the desired consequences of XS are retained for non-zero lattice spacing.

We will try & understand this in the following.

- Of course, to be useful, we need choices of D that satisfy (#) & are local – & there are (overlap, domain-wall & fixed-pt fermions)

– more on this later

Explicit construction of chiral transformation (Lüscher '98)

- As already noted, we want to be able to rotate L & R handed fields independently.
- This is equivalent to being able to both do a vector transformation and an axial trans$^{\underline{n}}$.

(for notational convenience)

To see this consider $N_f = 1$ & forget about the axial anomaly:

$$\text{vector: } \psi \to e^{i\theta_V}\psi \; ; \; \bar\psi \to \bar\psi e^{-i\theta_V}$$

$$\text{axial: } \psi \to e^{i\gamma_5\theta_A}\psi \; ; \; \bar\psi \to \bar\psi e^{i\gamma_5\theta_A}$$

$$\Rightarrow \psi_L \to e^{-i\theta_A}\psi_L \; ; \; \bar\psi_L \to \bar\psi_L e^{+i\theta_A}$$

$$\psi_R \to e^{+i\theta_A}\psi_R \; ; \; \bar\psi_R \to \bar\psi_R e^{-i\theta_A}$$

recall $L = \left(\frac{1-\gamma_5}{2}\right)$

recall $\bar\psi_R \doteq \bar\psi_R P_L$

Thus if $\theta_V = \theta_A$ $\quad \psi_L \to \psi_L \; ; \; \bar\psi_L \to \bar\psi_L$

$$\psi_R \to e^{2i\theta_V}\psi_R \; ; \; \bar\psi_R \to \bar\psi_R e^{-2i\theta_V}$$

i.e. only RH fields transform

Similarly if $\theta_V = -\theta_A$ only LH fields transform.

- Now, we always have vector symmetry (fermion #), so the key issue is whether there is an axial transformation.

[Aside on minimally doubled fermions — which have one doubler (of opposite chirality), and satisfy $[\gamma_5, D]_+ = 0$. Here can do an apparently axial rot^n: $\psi \Rightarrow e^{i\gamma_5 \theta_A}\psi$ $\quad \bar\psi \Rightarrow \bar\psi e^{i\gamma_5\theta_A}$ & it is a <u>non-anomalous</u> symmetry. [No anomaly since lattice measure is invariant.]

But, since chirality of two fermion poles is opposite, this is, infact, a flavor <u>non</u>-singlet axial transf.

I.E. in ctm limit corresponds to $\tilde\psi \Rightarrow e^{i\gamma_5 \cdot \tau_3 \theta_A}\tilde\psi$ where $\tilde\psi$ is a 2-component flavor vector.

So theory has an exact $U(1)_A \times U(1)_V$ symmetry at finite lattice spacing — a subgroup of the $SU(2)_L \times SU(2)_R \times U(1)_V$ of the ctm limit.

[This is similar to staggered fermions.]

This is nice, but comes at the cost of rotation invariance.

In any case, we want <u>MORE</u> — we want to have the complete

$$SU(2)_L \times SU(2)_R$$

at finite lattice spacing!]

Axial transformation for GW fermions:

$$\psi \to \psi' = e^{i\alpha\gamma_5(1-\frac{D}{2})}\psi$$

$$\bar\psi \to \bar\psi' = \bar\psi\, e^{i\alpha(1-\frac{D}{2})\gamma_5}$$

- α = parameter
- axial-like with + sign in both rotations

N.B. Depends on $\mathcal{U}$ through D — $O(a)$ correction which is slightly non-local

Check:

$$\bar\psi' D \psi' = \bar\psi\, e^{i\alpha(1-\frac{D}{2})\gamma_5} D\, e^{i\alpha\gamma_5(1-\frac{D}{2})}\psi$$

Now use $D\gamma_5(1-\frac{D}{2}) = -(1-\frac{D}{2})\gamma_5 D$

which is identical to GW rel^n.

$$\Rightarrow \quad D\, e^{i\alpha\gamma_5(1-D/2)} = e^{-i\alpha(1-\frac{D}{2})\gamma_5} D$$

$$\Rightarrow \quad \bar\psi' D \psi' = \bar\psi D \psi \quad \checkmark$$

Note that vector transformation remains unchanged:

$$\psi \to e^{i\alpha_V}\psi \qquad \bar\psi \to \bar\psi\, e^{-i\alpha_V}.$$

So we have both vector & (slightly funny) axial transformations — can we define LH & RH projectors? Yes.

- We need the usual $P_{\substack{L\\R}} = \frac{1 \mp \gamma_5}{2}$

and also $\hat{P}_{\substack{L\\R}} = \frac{1 \mp \hat{\gamma}_5}{2}$

where $\hat{\gamma}_5 = \gamma_5 (1-D)$

GW ensures these are projectors, e.g.

$$\hat{P}_L^{\,2} = \frac{1 + \hat{\gamma}_5^{\,2} - 2\hat{\gamma}_5}{4} = \hat{P}_L$$

since
$$\begin{aligned}\hat{\gamma}_5^{\,2} &= \gamma_5 (1-D) \gamma_5 (1-D) \\ &= 1 - \gamma_5 D \gamma_5 - D + \gamma_5 \underbrace{D \gamma_5 D}_{GW} \\ &= 1 - \gamma_5 D \gamma_5 - D + \gamma_5 (\gamma_5 D + D\gamma_5) \\ &= 1\end{aligned}$$

Also, clearly have $\hat{P}_L + \hat{P}_R = 1$; $\hat{P}_R^{\,2} = \hat{P}_R$ & $\hat{P}_{\substack{R\\L}} \hat{P}_{\substack{L\\R}} = 0$

- Final key property is

$$\underbrace{P_L D = D \hat{P}_R}_{\text{true since}} \quad \& \quad \underbrace{P_R D = D \hat{P}_L}_{\text{similarly here}}$$

true since

$$\gamma_5 D = - D \hat{\gamma}_5$$

(which is just GW)

• Thus $\bar{\psi} D \psi$

$$= (\bar{\psi} P_L + \bar{\psi} P_R)\, D\, (\hat{P}_L \psi + \hat{P}_R \psi)$$

$$= \underbrace{\bar{\psi} P_L}\, D\, \underbrace{\hat{P}_R \psi} + \underbrace{\bar{\psi} P_R}\, D\, \underbrace{\hat{P}_L \psi} \quad \text{(others vanish)}$$

$$\equiv \overline{\psi_R}\, D\, \psi_R + \overline{\psi_L}\, D\, \psi_L$$

Same decomposition as in ctm. (although have funny "latticey" projectors on ψ.)

• So we can rotate L & R-handed components separately, e.g.

$$\psi_L \to e^{i\alpha_L} \psi_L \; ; \; \overline{\psi_L} \to \overline{\psi_L}\, e^{-i\alpha_L} \; ; \; \psi_R, \overline{\psi_R} \text{ unchanged.}$$

• If we have a single fermion, then the axial (or LH or RH) transformation is anomalous (see below how this happens on the lattice).

But if we have N_f fermions, then flavor non-singlet axial transformations are non-anomalous

$$\left.\begin{aligned} \psi &\to e^{i\alpha_a T^a \gamma_5 (1 - \frac{D}{2})} \psi \\ \bar{\psi} &\to \bar{\psi}\, e^{i\alpha_a T^a (1 - \frac{D}{2}) \gamma_5} \end{aligned}\right\} \begin{array}{l} \mathrm{Tr}\, T^a = 0 \\ T^a \text{ are generators} \\ \text{of flavor rot}^n \end{array}$$

Full

$\Rightarrow \; SU(N_f)_L \times SU(N_f)_R \times U(1)_V$

is respected by lattice theory (in massless limit)

Form of mass term?

In ctm,

$$m\bar\psi\psi = m(\overline{\psi_L}\,\psi_R + \overline{\psi_R}\,\psi_L)$$

Breaks $SU(N_f)_L \times SU(N_f)_R$ in a "maximal" way.

Natural choice on lattice matches this breaking by having an identical form

$$m(\overline{\psi_L}\,\psi_R + \overline{\psi_R}\,\psi_L)$$

$$= m(\bar\psi P_R \hat{P}_R \psi + \bar\psi P_L \hat{P}_L \psi)$$

$$= m\Big(\bar\psi \underbrace{\underbrace{\Big[\frac{1+\gamma_5}{2}\,\frac{1+\gamma_5(1-D)}{2} + \frac{1-\gamma_5}{2}\,\frac{1-\gamma_5(1-D)}{2}\Big]}_{\frac{1}{2}(1+\gamma_5^2(1-D))}}_{1-\frac{D}{2}} \psi\Big)$$

$$= m\bar\psi\left(1-\frac{D}{2}\right)\psi$$

additional, gauge field dependent, $O(a)$ term

- Similarly $\bar\psi\gamma_5\psi\big|_{cont} \to \bar\psi(P_R\hat{P}_R - P_L\hat{P}_L)\psi$

$$= \bar\psi\gamma_5\left(1-\frac{D}{2}\right)\psi$$

The flavored versions of these transform under $SU(N_f)_L \times SU(N_f)_R$ in the same way as in the ctm. More generally, reproduce ctm. "current algebra"

Axial anomaly (discussed in context of single flavor)

- We know from ctm. regularizations that $U(1)_A$ is anomalous — it is a symmetry of the Lagrangian but not the measure.
- On the lattice, as we've discussed above, any "standard" (gauge field independent) global transformation is not anomalous

How is this puzzle resolved? By the presence of "D" in the transformation.

$$Z_{GW} = \int [D\psi][D\bar\psi][DU]\, e^{\underbrace{-S_{gauge} + \bar\psi D \psi}_{\text{invariant under Lüscher transformation}}}$$

lattice action w/ GW fermions

$$\psi \to \psi' = [1 + i\alpha\gamma_5(1-D/2)]\psi$$

$$\bar\psi \to \bar\psi' = \bar\psi\,[1 + i\alpha(1-D/2)\gamma_5]$$

(with $|\alpha| \ll 1$)

Need Jacobian for Grasmann variables — just inverse of Jacobian for commuting variables.

$$[D\psi][D\bar\psi] = [D\psi'][D\bar\psi']\, \det\left(\mathbb{1} + i\alpha\gamma_5(1-D/2)\right),\ \det\left(\mathbb{1} + i\alpha(1-D/2)\gamma_5\right)$$

determinant over space-time (lattice) indices, color indices & Dirac indices.

Key point is dependence on U inside of D

Now.

$$\det\left(1+i\alpha\gamma_5(1-\tfrac{D}{2})\right) = \exp\left[\mathrm{tr}\ln\left(1+i\alpha\gamma_5(1-\tfrac{D}{2})\right)\right]$$

$$= \exp\left\{\mathrm{tr}\, i\alpha\gamma_5(1-D/2) + O(\alpha^2)\right\}$$

$$= 1 + i\alpha\,\mathrm{tr}\left[\gamma_5(1-\tfrac{D}{2})\right] + O(\alpha^2)$$

$$= 1 - \frac{i\alpha}{2}\,\mathrm{tr}\,\gamma_5 D + O(\alpha^2)$$ since $\mathrm{tr}\,\gamma_5 = 0$

$$= \det\left(1+i\alpha(1-\tfrac{D}{2})\gamma_5\right)$$

Thus Jacobian becomes

$$\left(1 - \frac{i\alpha}{2}\,\mathrm{tr}\,\gamma_5 D + O(\alpha^2)\right)^2 = 1 - i\alpha\,\underbrace{\mathrm{tr}(\gamma_5 D)} + O(\alpha^2)$$

as we will see, this is non-vanishing in general (for $U \neq 1$)

Indeed $\mathrm{tr}(\gamma_5 D) \propto$ topological charge $\propto \int_x F_{\mu\nu}\tilde{F}_{\mu\nu}$ in cont.

Bottom line — axial transformation is NOT a symmetry of Z.