Lecture 15: dealing w/ doubling.

Is fermion doubling avoidable? Yes & No.

There are some famous "no-go" theorems saying that having an equal # of LH & RH fermions is essential.

The most well-known is that of H. B. Nielsen & M. Ninomiya, Nucl. Phys B185 (1981) 20; B 193 (1981) 173
This is in the lattice Hamiltonian formulation; the key conditions are translational invariance, locality & Hermiticity of $\hat{H}$, plus important conditions on the fermion-number charge operators.

I will instead summarize the results of L. Karsten & J. Smit, Nucl. Phys. B183 (1981) 301 who work in the path integral formalism.
They show that LH & RH fermions come in pairs if

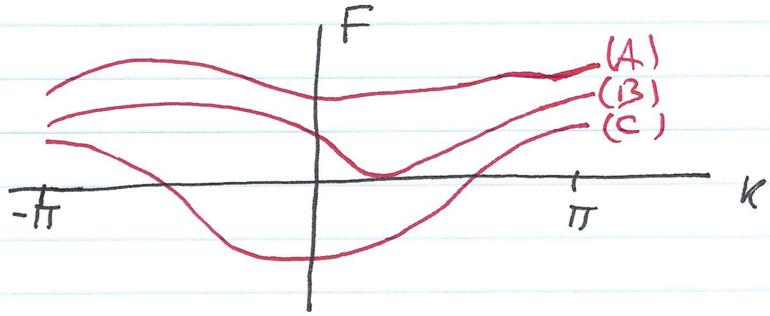
- S is translationally inv. $\Rightarrow$ mom. space is periodic & smooth (differentiable)
- Interactions are local $\Rightarrow$ propagator is cts fn in mom. space
- $\not{\partial}$ is discretized by an antihermitian operator, which thus has imaginary eigenvalues.
- One works in ∞ volume $\Rightarrow$ mom. space is cts.

In 1-d the argument is simple. In the massless theory ($m_0 = 0$):

$$G(k)^{-1} = i\gamma_1 F(k)$$

antihermeticity

real, cts, periodic fcn.



(A) F has no zeros $\Rightarrow$ G has no massless poles.

(B) F has a double-zero $\Rightarrow$ G has a double pole (unphysical)

(C) If F has one zero, it has an odd # of other zeros. Half the crossings have $F' > 0$, other half have $F' < 0$.

$\Rightarrow$ (as in free case) Half are LH, half are RH.

In 4-d the minimal doubling is a single LH & RH pair. Such theories, however, violate hypercubic rotation symmetry & for this & other reasons have not been used practically in large simulations.

- No-go theorems can be avoided by violating their assumptions, & we will return to how this can be done with Ginsparg-Wilson fermions.

- First, though, we will discuss briefly the fermions in use for present studies of QCD – the major focus of lattice research.

- QCD is a vector theory – quarks are massive (though some are light) & have LH & RH components.
  So doubling – in the sense of having LH & RH pairs – is not a problem.
  Having 16 fermions IS however a problem.

Staggered fermions

These allow 16 to be reduced to 4, which, perhaps, one can live with.

Key observation: naive fermions have a hidden $SU(4)$ symmetry.

Write $\psi_n = \Gamma_n \chi_n \quad ; \quad \bar{\psi}_n = \bar{\chi}_n \Gamma_n^{+}$

$\Gamma_n = \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3} \gamma_4^{n_4} \quad ; \quad \Gamma_n^{+} \Gamma_n = 1.$

called "spin diagonalization"

$\Rightarrow \quad \bar{\psi}_n \psi_n = \bar{\chi}_n \chi_n$

$$\bar{\psi}_n \gamma_\mu \psi_{n\pm\mu} = \bar{\chi}_n \underbrace{\Gamma_n^{+} \gamma_\mu \Gamma_{n\pm\mu}}_{\eta_\mu(n) \times \text{ identity matrix}} \chi_{n\pm\mu}$$

n-dependent sign.

$\eta_1 = 1 \ ; \ \eta_2 = (-1)^{n_1} \ ; \ \eta_3 = (-1)^{n_1+n_2} \ ; \ \eta_4 = (-1)^{n_1+n_2+n_3}$

$$S_0 - S = \sum_n \left\{ \sum_\mu \eta_\mu(n)\, \bar{\chi}_n \left(U_{n,\mu}\chi_{n+\mu} - U^{+}_{n-\mu,\mu}\chi_{n-\mu}\right) + i m_0 \bar{\chi}_n \chi_n \right\}$$

Including ints. w/ gauge fields.

Recalling that χ_n has 4 (Dirac) components, we see that $\chi_n \to V\chi_n \quad \bar{\chi}_n \to \bar{\chi}_n V^+$

with $V \in SU(4)$ is a global $SU(4)$ invariance.

In other words, in this writing, the different Dirac components are uncoupled.

To get "staggered" (or "Kogut-Susskind" or "Susskind") fermions one simply deletes three of the four components:

$\chi_n \rightarrow$ single component Grassman field.

$\Rightarrow$ 4-fold reduction in d.o.f.

$\Rightarrow$ 4 continuum Dirac (LH+RH) fermions

$\Rightarrow$ Role of Dirac matrices played by phases.

Still have 16 poles, but these are organized into 4 4-component ctm fermions.

We do have 4 "light" quarks – u, d, s & c – and, by introducing mass terms, can make staggered fermions give 4 non-degenerate quarks in ctm. limit.
However, need to break most lattice symmetries, leading to challenging operator-mixing & tuning problems.

$\Rightarrow$ not used in practice.

[See my web site for slides of a recent talk revisiting this issue.]

What is done in practice is to have one staggered field FOR EACH PHYSICAL QUARK,

e.g. χ_u, χ_d, χ_s & χ_c in present simulations.

$\Rightarrow$ in ctm limit have

extra d of f called "taste"

4 u, 4 d, 4 s & 4 c quarks.

So fermion determinant is not what we want

e.g. $\int [D\chi_u][D\bar{\chi}_u]\, e^{\bar{\chi}(D_{stag}+M_u)\chi} = \det(D_{stag}+M_u)$

$$\xrightarrow[\text{cont. limit}]{} \det{}^4(\not{D}+M_u)$$

one ctm. flavor.

To avoid, take 4th root of determinant — possible (& straightforward) in simulation algorithms. ("Rooting").

Valid formally, but away from ctm. limit 4 tastes are NOT degenerate (since remaining lattice symmetry is a discrete subgroup of $SU(4)_{taste}$)

so $\det^{1/4}(D_{stag}+M_u) \Rightarrow$ non-local lattice action.

There are many arguments (but no proof) that non-locality irrelevant in ctm limit, and lots of numerical tests.

[For reviews see, e.g. S. Sharpe, hep-lat/0610094]

Used in practice mainly in the US; numerically efficient + some other technical advantages.

[though analysis is quite complicated...]

Wilson fermions - reducing down to one LH + RH pair

Basic idea: make remaining 15 pairs have physical masses of $O(1/a)$ so that they are irrelevant for the ctm limit theory

Since atleast one component of k_μ is close to π for a doubler, its position space wave fcn alternates in sign in atleast one direction

This suggests adding a term to the lattice action which has no impact for $k \sim 0$ & acts like an $O(1)$ mass term [in lattice units] when one or more $k_\mu \sim \pi$.

Such a term is the "Wilson term"

$$S_W = r \sum_{n,\mu} \frac{1}{2} \bar{\psi}_n \Delta_\mu^+ \Delta_\mu^- \psi_n$$

this is just the bosonic action which we know has no doublers

parameter allowing one to vary strength of Wilson term

or $$S_W = r \sum_{n,\mu} \frac{1}{2} \bar{\psi}_n \left(\psi_{n+\mu} - 2\psi_n + \psi_{n-\mu} \right)$$

can make gauge invariant by adding appropriate $\mathcal{U}$'s

$$= \frac{r}{2} \int_k \bar{\psi}(k) \left(-\hat{k}^2 \right) \psi(k)$$

recall $\hat{k}^2 \equiv \sum_\mu \hat{k}_\mu^2 = \sum_\mu 4 \sin^2 \frac{k_\mu}{2}$

$k_\mu = 0 \Rightarrow \hat{k}_\mu^2 = 0$ while $k_\mu = \pi \Rightarrow \hat{k}_\mu^2 = 4$

So full action becomes (in free theory)

$$-S_{naive} - S_W = \int_k \bar{\psi}(k)\left(i\not{s} + M_0 + \frac{r}{2}\hat{k}^2\right)\psi(k)$$

$$\Rightarrow G_{np} = \int_k e^{ik\cdot(n-p)} \underbrace{\frac{1}{i\not{s} + M_0 + \frac{r}{2}\hat{k}^2}}_{\frac{-i\not{s} + M_0 + \frac{r}{2}\hat{k}^2}{s^2 + (M_0 + \frac{r}{2}\hat{k}^2)^2} \equiv G(k)}$$

If set $M_0 = 0$, $G(k)$ only has a pole at $k=0$.

Near $k_\mu = \pi$, $\forall \mu$, for example,

$$G(k) \propto \frac{1}{0 + (\frac{r}{2}\cdot 16)^2}$$

More generally, would-be doublers pick up "masses" of

$$\frac{r}{2}\cdot\{4, 8, 12, 16\}$$

If r finite as $a \to 0$, these are dimensionfull masses $m \sim 1/a$, so doublers decouple from ctm theory.

Typically set $r=1$, for then can show reflection positivity, & some other simplifications occur.

[Can also see lack of doublers by calculating the position-space propagator. (HW).]

This all looks very good for QCD – a single Dirac fermion with a mass $\overline{m} = M_0/a$.

What's the downside? Loss of chiral symmetry.

Also: since Wilson term has Laplacian, it becomes $a \int_x \underbrace{\overline{\psi}(x)}_{\text{const}} \frac{D^2}{2} \psi(x)$ in naive ctm. limit

$\Rightarrow$ discretization errors $\propto a$

Lattice Scalar field th'y & staggered/naive fermions have errors $\propto a^2$

Can "improve" Wilson fermions in several ways to remove $\mathcal{O}(a)$ errors, e.g. "twisted mass" fermions.

So the main issue is the lack of chiral symmetry. Recall that this plays a major role in ctm model-building, because it allows light fermion masses to be "natural" (not ruined by radiative corrections).

Let's recall the story; first in Minkowski space

Consider a theory w/ N_f massless Dirac flavors

Action is invariant under global trans:

N_f component vector $\to \psi_L \to L\psi_L$; $\in SU(N_f)$; $\psi_L^\dagger \Rightarrow \psi_L^\dagger L^\dagger \Rightarrow \overline{\psi}_L = \psi_L^\dagger \gamma_0 \to \overline{\psi}_L L^\dagger$

why? $\mathcal{L} = \overline{\psi} i\not{\partial} \psi = \overline{\psi}_L i\not{\partial} \psi_L + \overline{\psi}_R i\not{\partial} \psi_R$

so can rotate in flavor space independently on LH & RH fields.

Thus also have $\psi_R \to R\psi_R$ & $\overline{\psi}_R \to \overline{\psi}_R R^\dagger$ symmetry.

- Here I have used $P_L \gamma_\mu = \gamma_\mu P_R$ w/ $P_{L \atop R} = \frac{1 \mp \gamma_5}{2}$
- Now the mass term

$$m \bar{\psi} \psi = m\left(\overline{\psi_L}\, \psi_R + \overline{\psi_R}\, \psi_L\right)$$

is NOT invariant under $SU(N_f)_L \times SU(N_f)_R$.

$\Rightarrow$ if $m_{bare} = 0$ then $m_{phys} = 0$ also,

and if $m_{bare} \neq 0$ then $m_{phys} \propto m_{bare}$.

In particular do not get $m_{phys} \propto \Lambda$.

- Aside: chiral group is $SU(N_f)$ & not $U(N_f)$ because for a single flavor transformations in which $L \neq R$ are anomalous.
  Only vector transformations ($L = R$) are good symmetries & these do not forbid a mass term
- Returning to our lattice action, we first have to deal w/ a peculiarity of the Euclidean path integral for fermions

$\bar{\psi}_n$ & ψ_n are not related: $\bar{\psi}_n \neq \psi_n^+ \gamma_0$

They are independent.

But since in Minkowski operator treatment

$$\overline{\psi_L} = \overline{\psi_L}\, P_R \quad \& \quad \overline{\psi_R} = \overline{\psi_R}\, P_L$$

We define the L & R $\bar{\psi}$'s in the path integral to have these same projections, which are opposite to the naive choices.

So, on the lattice, $\psi_n = \psi_{nL} + \psi_{nR}$

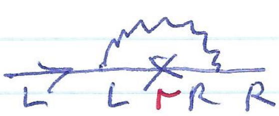
$$\overline{\psi_n} = \overline{\psi_{nL}} + \overline{\psi_{nR}}$$

$$\mathcal{L} - S = \sum_{n\mu}\left(\overline{\psi_{nL}}\, \gamma_\mu \Delta_\mu \psi_{nL} + \overline{\psi_{nR}}\, \gamma_\mu \Delta_\mu \psi_{nR}\right)$$
$$+ m_0 \sum_n \left(\overline{\psi_{nL}}\, \psi_{nR} + \overline{\psi_{nR}}\, \psi_{nL}\right)$$
$$+ \frac{r}{2} \sum_{n,\mu} \left(\overline{\psi_{nL}}\, \Delta_\mu^+ \Delta_\mu^- \psi_{nR} + \overline{\psi_{nR}}\, \Delta_\mu^+ \Delta_\mu^- \psi_{nL}\right)$$

Neither the m_0 term nor the r term are invariant under chiral transformations: chiral symmetry is lost by introducing the Wilson term.

Once one introduces gauge interactions, this means that setting $m_0 \to 0$ does NOT lead to a massless quark

Diagrammatically, this arises as follows:



$$\delta m \sim g^2 r \int_{k < 1/a} \frac{1}{k^2} \frac{1}{\not k}\, a k^2 \frac{1}{\not k} \sim g^2 \frac{a}{a^2} \sim g^2 r / a$$

$$\Rightarrow \delta m_0 \sim g^2 r$$

This is similar to the ϕ^4 case, except here it is δm_0 rather than δm_0^2.

It means that m_0 must be tuned (by, it turns out, a negative amount) in order to find the massless point — again, fine tuning.

This can be done, numerically, by looking for the value of m_0 for which the pion mass vanishes, since $M_\pi^2 \propto m_{phys}$ in QCD w/ SSB of chiral symmetry.

[In fact, even if chiral symm. was not spont. broken, one can find the massless point using the "PCAC quark mass" — no time to discuss.]

- Tuning one parameter is not that big of a deal. The real problem w/ Wilson fermions is that, when one wants to calculate matrix elements of the electroweak Hamiltonian, H_W, there is additional operator mixing on the lattice, which leads to lots of practical challenges

- Standard notation for Wilson fermions — same idea as for scalars.

$$-S = \sum_n \bar\psi_n \psi_n (m_0 + 4) - \sum_{n\mu} \Big\{ \bar\psi_n \frac{1-\gamma_\mu}{2} U_{n,\mu} \psi_{n+\mu} + \bar\psi_n \frac{1+\gamma_\mu}{2} U^+_{n-\mu,\mu} \psi_{n-\mu} \Big\}$$

$$= \sum_n \bar\psi_n' \psi_n' + 2K \sum_{n\mu} \Big\{ \bar\psi_n' \frac{1-\gamma_\mu}{2} U_{n,\mu} \psi'_{n+\mu} + \bar\psi_n' \frac{1+\gamma_\mu}{2} U^+_{n-\mu,\mu} \psi'_{n-\mu} \Big\}$$

with $\bar\psi_n' = \sqrt{m_0+4}\, \bar\psi_n$ & $\psi_n' = \sqrt{m_0+4}\, \psi_n$

and the hopping parameter

$$K = \frac{1}{2(m_0+4)}$$

. For large m_0, can do a hopping parameter expansion, as for the scalar theory.

At tree level, the critical K at which $m_0 = 0$ is $K_c = 1/8$.
Including interactions $K_c > 1/8$.