Lattice fermions – topic for remainder of class (including chiral symmetry)

Discretizing fermions leads to the infamous doubling problem, several resolutions of which we will discuss.

In numerical simulations, including the fermions & their effects is usually the most "expensive" part of the calculation.

We will start with free fermions, where doubling already raises its head.

We begin immediately in Euclidean space, where the action is (in the ctm)

$$S = -\int_x \overline{\psi}\,(\gamma_\mu \partial_\mu + m)\,\psi$$

Euclidean Dirac matrices $\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu}\cdot 2$ $\quad \gamma_\mu^+ = \gamma_\mu$ $\quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \gamma_5^+$

$\{\gamma_\mu, \gamma_5\} = 0$

What are $\overline{\psi}$ & ψ here? They are Grassman (anticommuting) functions, which are the integration variables of the fermionic path integral.

$$Z = \int [D\psi][D\overline{\psi}]\, e^{-S} = \mathrm{Tr}\, e^{-\hat{H}/T}$$

If the Euclidean time extent is $1/T$ and ψ & $\overline{\psi}$ satisfy APBC in time.

$$H = \int d^3x\, \hat{\chi}^+(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta)\,\hat{\chi}$$

We will not repeat the derivation of this result.

Note that, for this formula to hold, $\bar{\Psi}$ & Ψ must be treated as independent Grassman variables.

Also recall that

$$Z = \int [D\Psi][D\bar{\Psi}]\, e^{\int \bar{\Psi}(\not{\partial}+m)\Psi} = \det(\not{\partial}+m)$$

from the rules of Grassman integration. The precise meaning of the determinant will be clear when we discretize.

Another key result:

$$\frac{1}{Z}\int [D\Psi][D\bar{\Psi}]\, e^{S}\, \Psi(x)\,\bar{\Psi}(y) = -\left[(\not{\partial}+m)^{-1}\right]_{xy}$$

Indices written anticipating matrix notation

Inverse of operator appearing in quadratic form in action.

This is the Wick-rotated form of $\dfrac{1}{i\not{\partial}_M - m}$

$= -i\langle 0|T[\hat{\chi}(x)\ \hat{\chi}^{+}\gamma_0(y)]|0\rangle$

(when temp. $T \to 0$).

In other words, the Euclidean path integral reproduces the Wick rotated time-ordered correlators.

This is true for all correlators & including interactions & we will accept this result w/out derivation.

Now let's discretize:

$$a^{3/2}\psi(x) \to \psi_n \quad ; \quad a^{3/2}\bar{\psi}(x) \to \bar{\psi}_n .$$

fields live on sites, as for scalars.

$a\,\partial_\mu \longrightarrow \Delta_\mu^+$ forward deriv $\Delta_\mu^+\psi_n = \psi_{n+\mu} - \psi_n$

or Δ_μ^- backward deriv $\Delta_\mu^-\psi_n = \psi_n - \psi_{n-\mu}$

or $\dfrac{\Delta_\mu^+ + \Delta_\mu^-}{2}$ symmetric deriv.

gives $\dfrac{\psi_{n+\mu} - \psi_{n-\mu}}{2}$

or...

For the scalar we chose the most local choices, Δ_μ^+ or Δ_μ^-, which led to the same result.

But here the action has only a first derivative, & the choice matters.

- In fact, both $\Delta_\mu^\pm$ are clearly unphysical, since they lead to a propagator

  $\dfrac{1}{\not{\Delta}^\pm + M}$ which only propagates either forward or backward in each dir^n.
  (and violates lattice rot^n invariance)

(Do a large m expansion, giving $\frac{1}{M} - \frac{\not{\Delta}^\pm}{M^2} + \cdots$)

- Alternatively, ∂_μ is an <u>antiHermitian</u> operator (as it is ik_μ when diagonalized),

  while $\Delta_\mu^\pm$ are neither Hermitian or antiHermitian:

$$\underset{\text{matrix}}{(\Delta_\mu^+)^\dagger} = -\Delta_\mu^-$$

Working with nearest neighbors, this forces us to use

$$a\partial_\mu \longrightarrow \frac{\Delta_\mu^+ + \Delta_\mu^-}{2}$$

which is antiHermitian & flips sign under 180° rot $\perp$

$$\text{So} \quad -S = \int \bar{\psi}(\not{\partial}+M)\psi$$

$$\longrightarrow \underbrace{\sum_{n,\mu} \bar{\psi}_n \gamma_\mu \frac{(\psi_{n+\mu}-\psi_{n-\mu})}{2}\psi_n + \sum_n \overbrace{(am)}^{M_0} \bar{\psi}_n \psi_n}_{\text{so-called "naive" lattice fermion action}}$$

Going to momentum space

convenient choice of sign

$$\psi_n = \int_k e^{ik\cdot n}\psi(k) \qquad \bar{\psi}_n = \int_k e^{-ik\cdot n}\bar{\psi}(k)$$

$$-S = \int_k \bar{\psi}(k)\left\{\left[\sum_\mu \underbrace{\frac{\left(e^{ik_\mu}-e^{-ik_\mu}\right)}{2}}_{i\sin k_\mu \equiv i s_\mu}\gamma_\mu\right] + M_0\right\}\psi(k)$$

NOTE: $\sin k_\mu$ not $\sin k_\mu/2$. Needed for periodicity.

$$= \int_k \bar{\psi}(k)\,(i\not{s} + M_0)\,\psi(k)$$

$\sum_\mu s_\mu \gamma_\mu$

$$\Rightarrow G_{np} = -\langle \psi_n \bar{\psi}_p \rangle = -\int_{k,q} e^{ik\cdot n}\underbrace{\langle \psi(k)\bar{\psi}(q)\rangle}_{(2\pi)^4\delta^4_{per}(k-q)\ \frac{-1}{i\not{s}+M_0}} e^{-iq\cdot p}$$

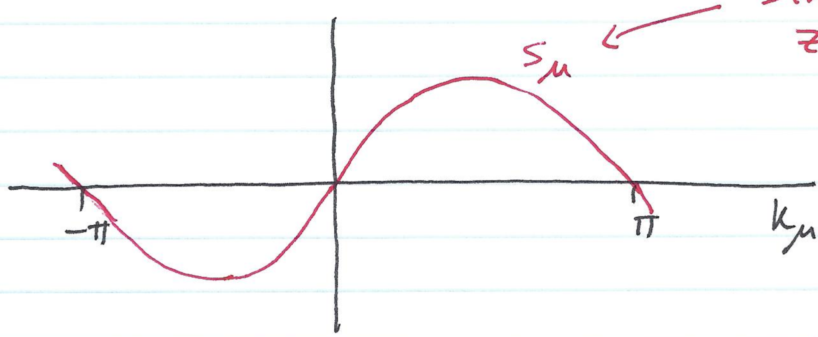
$$= \int_k e^{ik\cdot(n-p)}\frac{1}{i\not{s}+M_0}$$

- All this looks as expected. If $k_\mu = \bar{k}_\mu a$, $m_0 = \bar{m} a$ & send $a \to 0$ w/ $\bar{k}_\mu, \bar{m}$ fixed, momentum space prop. becomes

$$\frac{1}{i\not{k} + m_0} \longrightarrow \frac{1}{a(i\bar{\not{k}} + \bar{m})} = \frac{1}{a} \frac{-i\bar{\not{k}} + \bar{m}}{\bar{k}^2 + m^2}$$

and have, upon inverse Wick rotation, pole at $\bar{k}^2 = -m^2$.

- But here come the doublers.



since it has one zero (at $k_\mu = 0$) & is periodic & continuous $\Rightarrow$ at least one more zero (here at $k_\mu = \pi$)

- When we calculate propagator G_{np} have to integrate over all k_μ, including near π.

- Note that as $a \to 0$, $\bar{k}_\mu \approx 0$ & $\bar{k}_\mu \approx \frac{\pi}{a}$ become infinitely far apart - distinct region of k_μ-space.

- So expand k_μ in vicinity of second zero:

$$k_\mu = \pi + \bar{k}_\mu' a \quad \Rightarrow \quad S_\mu \approx -a \bar{k}_\mu' \qquad \text{one direction}$$

$$k_\nu = \bar{k}_\nu' a \quad \Rightarrow \quad S_\nu \approx a \bar{k}_\nu' \qquad \text{other 3 directions.}$$

- Thus $\not{s} \approx a\left[-\bar{K}_\mu' \gamma_\mu + \sum_{\nu\neq\mu} \bar{K}_\nu' \gamma_\nu\right]$

- To make this look standard, define a new set of γ-matrices:

$$\gamma_\mu' = -\gamma_\mu \quad ; \quad \gamma_\nu' = \gamma_\nu \quad (\nu \neq \mu)$$

Still satisfy $\{\gamma_\mu', \gamma_\nu'\} = 2g_{\mu\nu}$

Unitarily equivalent:

$$\gamma_\rho' = (\gamma_\mu\gamma_5)\,\gamma_\rho\,\underbrace{(\gamma_\mu\gamma_5)^\dagger}_{=(\gamma_\mu\gamma_5)^{-1}} \qquad (\text{no sum on } \mu)$$

- Then $\not{s} \approx a \sum_\rho \bar{K}_\rho' \gamma_\rho' = a\,\not{\bar{K}}'$

so propagator again has nearby pole (different from original pole)

- Can play the same game in all four directions independently:

$$K_\mu \approx 0 \text{ or } \pi \quad \text{each } \mu.$$

$\Rightarrow$ 2^4 different, equally physical, poles

$\Rightarrow$ 16 physical particles – all degenerate fermions

$\Rightarrow$ "DOUBLING" problem

To bring home the point that the doublers are all "equally physical" let's calculate "hybrid" propagator

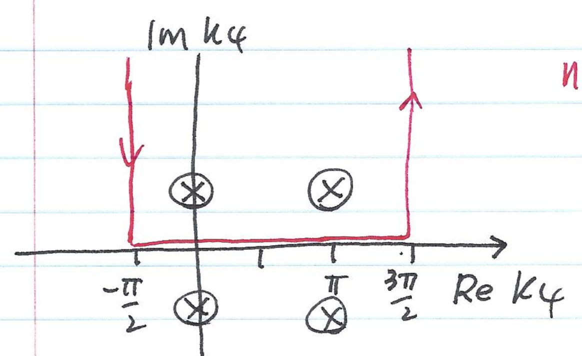
$$G(\vec{k}, n_4) = \sum_{\vec{n}} e^{-i\vec{k}\cdot\vec{n}} G_{(\vec{n};n_4),0}$$

$$= \int_{k_4} e^{ik_4 n_4} \frac{-i\not{s} + M_0}{s^2 + M_0^2}$$

$$\propto \sum_{\substack{\text{states} \\ \text{with mom} = \vec{k}}} C_{\text{state}}\, e^{-E_{\text{state}} n_4} \; ?$$

so can read off energies of states that appear?

Convenient to run k_4 integral from $-\pi/2$ to $3\pi/2$



$n_4 > 0 \Rightarrow$ can close above

vertical parts cancel by periodicity

poles at $k_4 = n\pi \pm iE \quad ; n = 0, 1$

$$\Rightarrow s_4 = (\pm i \sinh E)(-1)^n$$

$$s_4^2 + \vec{s}^2 + M_0^2 = 0 \quad \Rightarrow \quad E = \sinh^{-1}[\vec{s}^2 + M_0^2] > 0$$

Residue is thus: $(-1)^{n n_4} e^{-E n_4} \left(\frac{-i\not{s} + M_0}{2 s_4 c_4} \right) \Big|_{s_4 = i\sinh E}$

Final result: (including $n_4 < 0$ as well)

$$G(\vec{k}, n_4) = \frac{\pm \sinh E\, \gamma_4 - i\vec{\gamma}\cdot\vec{s} + M_0}{\sinh(2E)}\, e^{-E|n_4|}$$

$n_4 \gtrless 0$

(terms cancel when $n_4 > 0$)

$$+ \frac{\mp \sinh E\, \gamma_4 - i\vec{\gamma}\cdot\vec{s} + M_0}{\sinh(2E)}\, (-1)^{n_4} e^{-E|n_4|}$$

This does NOT fulfill our expectations, because of the $(-1)^{n_4}$ factor: this contribution to G alternates in time.

Corresponds to failure of link-reflection positivity (see Montvay & Münster or Smit for details):
$\hat{T}^2$ is physical, but not $\hat{T}$.

Second state has same energy as first, but with different "internal" wavefcn — think of single site translation in time as $(\hat{T}^2)^{\frac{1}{2}} *$ internal symmetry transformation.

Residue of 2nd state is identical to that of 1st state if replace $\gamma_4 \Rightarrow \gamma_4' = -\gamma_4$ (& $\vec{\gamma}' = \vec{\gamma}$).

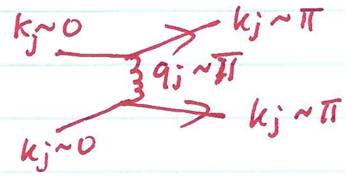
(Residues are outer products of spinor wavefcns, themselves sol^{n}s of (lattice version) of free Dirac eq.)

- Recalling that can take continuum limit & obtain a free ct^M propagator when $\epsilon \to 0$ (& $M_0 \to 0$), we see that there are 8 such places (each $k_j \to 0$ or π).

- In fact, if make suitable changes in $\vec{\gamma}$ (as described above), $G(\vec{k}, n_4)$ is invariant under $k_j \to k_j + \pi$.

- For a free theory, each choice of $\vec{k}$ leads to a non-interacting pair of states (& the members of the pair are decoupled too).

But when include interactions (e.g. in QCD) then diagrams like



lead to interactions between them.

So we have to deal w/ the doublers.

- N.B. Coupling quarks to gluons is easy.

$$\psi_n \to \psi_{n,j} \qquad \bar{\psi}_n \to \bar{\psi}_{n,j} \qquad j = 1, N_c$$

gauge transf. $\psi_n \to V_n \psi_n \qquad \bar{\psi}_n \to \bar{\psi}_n V_n^+$

gauge-invariant kinetic term: (mass term unchanged)

$$\sum_{n,\mu} \bar{\psi}_n \gamma_\mu \left(U_{n,\mu} \psi_{n+\mu} - U^+_{n-\mu,\mu} \psi_{n-\mu} \right)$$

In PT, get $U_{n,\mu} \sim 1$ by gauge transform, and then analysis of doublers goes through.

However, due to interactions, do not have exact degeneracy of 16 quarks except when $a \to 0$. (No time to go into this.)

Is doubling avoidable? Is it a technical or a fundamental problem?

We will spend much of the rest of the quarter discussing this.

Doubling is intimately related to chiral symmetry & to anomalies in gauge theories.

To start, let's go back to naive free fermions, set $m_0 = 0$ & insert a LH projector

$$-S \to \sum_{n,\mu} \bar{\psi}_n \gamma_\mu P_L \left(\psi_{n+\mu} - \psi_{n-\mu} \right)$$

$\left(\frac{1-\gamma_5}{2}\right)$ picks out $\psi_{nL} = P_L \psi_L$

Note that this picks out $\bar{\psi}_n P_R \equiv \bar{\psi}_{nL}$ ← not a typo just a definition

since $\gamma_\mu P_L = P_R \gamma_\mu$, $P_R = \frac{1+\gamma_5}{2}$

This reduces the d.o.f. by two, and naively leads to a ctm limit with 16 LH fields.

However, recall that in order to obtain the standard ctm propagator (or standard residue of the exp. fall off) we need to use γ_μ' & not γ_μ.

$$\gamma_\mu' = (\gamma_{45})^{\sigma_4}(\gamma_{35})^{\sigma_3}(\gamma_{25})^{\sigma_2}(\gamma_{15})^{\sigma_1} \cdot \gamma_\mu (\gamma_{51})^{\sigma_1}(\gamma_{52})^{\sigma_2}(\gamma_{53})^{\sigma_3}(\gamma_{54})^{\sigma_4}$$

with $\sigma_\mu = 0$ or 1 & $\gamma_{15} = \gamma_1 \gamma_5$ etc.

Then $\gamma_5' = \gamma_1'\gamma_2'\gamma_3'\gamma_5' = (-1)^{\sigma_1+\sigma_2+\sigma_3+\sigma_4}\gamma_5$

$\Rightarrow$ for half the doublers $\gamma_5' = \gamma_5$ & for rest $\gamma_5' = -\gamma_5$

- This means that $\frac{1-\gamma_5}{2} = \frac{1+\gamma_5'}{2}$ for half of the doublers

  – they are really RHanded!

  So get 8 LH & 8 RH cfm fermions – a vector combination!

- This holds also in the presence of gauge fields,

  and this is a good thing. If not, one could

  have, say, an SU(3) gauge theory w/ 16 LH fermions.

  But such a gauge theory is known to be anomalous – the fermion determinant $\det(\not{D}+M)$

  includes covariant derivative

  is not invariant under gauge transformations in general.
  (One way of seeing this is that the measure is not invariant.)

  Since the lattice theory is manifestly gauge invariant & regulated – it does not have anomalies.

  So it had to be that the fermion complement was non-anomalous – and the specific remedy the theory found was to become a vector gauge theory.

- This leaves many questions: How many doublers do there have to be in 4-d – what is the minimal number? Ans. 2
  Can one make a lattice regularization of a chiral gauge theory (LH & RH fermions in different reps)?
  Ans: Yes for abelian case; uncertain in general.