Pert. th^y & the continuum limit of (non-abelian) lattice gauge theories

$$S_{gauge} = -\frac{2}{g^2} \sum_{n,\mu<\nu} \mathrm{Re}\,\mathrm{tr}\, P_{n,\mu\nu}$$

Basic idea: as $g \to 0$, minimizing action forces $\mathrm{tr}\,P_{n,\mu\nu} \to N_c$, ie. $P_{n,\mu\nu} \to \mathbb{1}$.

This does not fix the $U_{n,\mu}$ due to gauge invariance, so gauge fix to bring $U_{n,\mu} \simeq \mathbb{1}$, e.g. with an axial gauge, or Landau gauge, Feynman gauge, ...

Expand $U_{n,\mu} = e^{-ig A_{n,\mu}}$ matrix field (dim'less) in powers of g.

lattice forward derivative

$$S_{gauge} = \sum_{n,\mu,\nu} \frac{\mathrm{tr}}{2} \left\{ (\Delta^+_\mu A_{n,\nu} - \Delta^+_\nu A_{n,\mu})(\Delta^+_\mu A_{n,\nu} - \Delta^+_\nu A_{n,\mu}) \right\} + \text{“}gA^3\text{”} + \text{“}g^2A^4\text{”} + \text{“}g^3A^5\text{”} + \dots$$

get all powers of A!

Implement gauge fixing by Fadeev–Popov procedure as in continuum $\Rightarrow$ ghosts + $S_{g.f.}(A)$

$$\left. S_{gauge} + S_{g.f.} \right|_{\text{quadratic in } A} = -\sum_{n,\mu,\nu} \frac{\mathrm{tr}}{2} A_\nu \underbrace{\Delta^-_\mu \Delta^+_\mu}_{\text{Lattice Laplacian}} A_\nu$$

in Feynman gauge.

For more details see my INT lecture notes.

Before proceeding into the details of lattice PT, let me bring up a fundamental concern.

When we gauge fixed we assumed that we could set $U_{n,\mu} \sim \mathbb{1}$ everywhere (so that $P_{n,\mu\nu} \approx 1$ too)

In other words the links & plaquettes are ordered over long distances.

But if the theory is confining, we know that

$$W(L,T) \propto e^{-\sigma L T} \xrightarrow[L,T\to\infty]{} 0 \quad .$$

This is INCONSISTENT with $U_{n,\mu} \sim \mathbb{1}$ everywhere.

Thus, at best we can gauge fix $U_{n,\mu} \sim 1$ over a region of size $\sim 1/\sqrt{\sigma}$. Thus PT fails for modes with $k \lesssim \sqrt{\sigma}$ – long-wavelength modes.

This holds just as much in the continuum & is the familiar statement that PT holds for high-energy modes.

Just as in the ct$^{\underline{m}}$ we need to check for every calculation that it is NOT sensitive to IR modes.

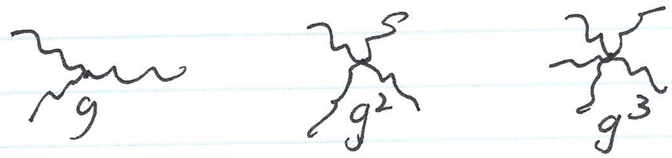
Back to PT.

color indices $a, b = 1, N_c$

Gluon prop. $\overset{k}{\sim\sim}$ $\quad \dfrac{\delta_{\mu\nu}\,\delta_{ab}}{\sum_\mu \hat{k}_\mu^2}$ (Feynman gauge)

$\left\langle \sum_n e^{-ik.n} A_{n,\mu}\; A_{n,\nu} \right\rangle$

Vertices



similar in form to continuum vertices

Non-renormalizable interaction?

OK since, if write

$A_{n,\mu} = a\,\bar{A}_{n,\mu}$,

then vertex is

$\sim a^2 \partial \bar{A}_{n,\mu}^{\,5}$

suppressed by $1/\Lambda$ as in an EFT, except here we know the exact form (we have an UV completion)

Also have ghosts

$- - - - \qquad \dfrac{\delta_{ab}}{\hat{k}^2}$

& ghost-gluon vertices

for more details see, e.g. Montvay & Münster.

Finally, we need that

$$dU_{n,\mu} = \prod_a dA_\mu^a \,(1 + \text{corrections})$$

See Montvay & Münster

We will NOT go into details here — lattice PT is straightforward but messy & has been mechanized to a considerable extent beyond 1-loop.

I will simply quote the result of a straightforward but tedious 1-loop calculation of the 3-gluon vertex fcn.

obtained from $\Gamma^{(2)}$ by some standard def^n

$$\Gamma^{(3)}_{R,\mu a,\nu b,\rho c}(k_1, k_2) = Z^{3/2}\, \Gamma^{(3)}_{\mu a,\nu b,\rho c}(k_1, k_2) \equiv g_R(\mu)\, V^{(3)}_{\mu a,\nu b,\rho c}$$

where $k_1^2 = k_2^2 = \underbrace{(k_1+k_2)^2}_{(-k_3)^2} = \mu^2$ $\left\{\begin{matrix}\text{symmetric} \\ \text{MOM} \\ \text{regularization} \\ \text{point}\end{matrix}\right\}$

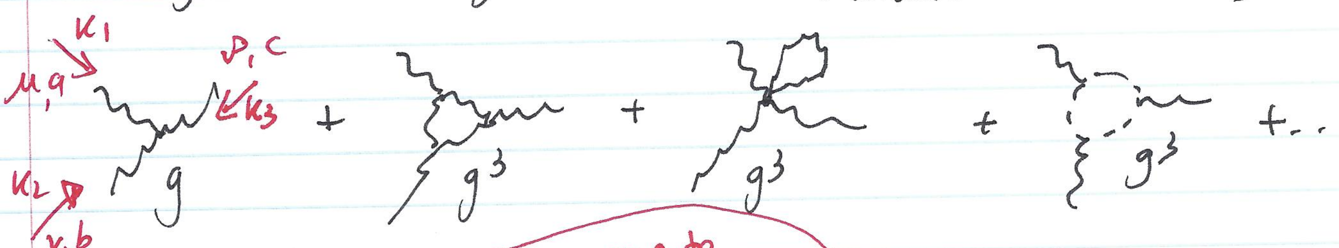
e.g. $k_1 = \frac{\mu}{\sqrt{2}}(1,1,0,0)$

$k_2 = \frac{\mu}{\sqrt{2}}(0,-1,1,0)$

$k_3 = \frac{\mu}{\sqrt{2}}(-1,0,-1,0)$

& $V^{(3)}_{\mu a,\nu b,\rho c}(k_1, k_2) = \underbrace{i f_{abc}\left\{\delta_{\mu\nu}(\hat{k}_1 - \hat{k}_2)_\rho \cos\frac{k_3}{2} + 2\text{ perms.}\right\}}_{\text{tree-level form}}$

Diagrams through 1-loop order include:



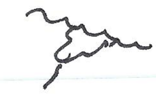
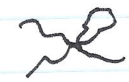
Lead to

opposite sign to ϕ^4 theory

$$g_R(\mu) = g\left\{1 + g^2\left[-\beta_0 \underbrace{\ln(\mu)}_{a\bar{\mu}} + \underset{\text{constant}}{C_L} + \underbrace{O(\mu^2 \ln\mu)}_{\propto a^2 \text{ if } \bar{\mu} \text{ held fixed}}\right] + O(g^4)\right\}$$

universal first term in β-fcn

$$\beta_0 = \frac{11 N_c - 2N_f}{3(16\pi^2)}$$ $N_f = 0$ here

Diagrams are either log divergent (e.g. )
or quadratically divergent ( "tadpoles")

Log divergences lead to $\ln \underbrace{a}_{\frac{1}{\text{UV cut off}}}\underbrace{\bar{\mu}}_{\text{IR cut off}}$

- if $\bar{\mu} >> \sqrt{\sigma}$ then avoid IR region & PT calc. can be reliable

Quad divergences lead to constants — (numerically large) contrib. to C_L

-  gives (up to factors) the same tadpole integral $I(0)$ we encountered in the scalar theory

- In ctm language the vertex is

  $\propto a^2 \bar{\mu}$

  while the loop integral gives $C_1/a^2 \sim \int \frac{d^4k}{k^2}$

  so the combination is $\cancel{a^2}\bar{\mu}\, C_1/\cancel{a^2}$

- Integral is UV dominated so PT is reliable

Get same form in any regularization, except with different C_L (& different cut-off effects).

Note that defining $g_R(\mu)$ in this way is called the MOM scheme. It works also in QCD (i.e. w/ fermions) where it is applied for $m_q=0$.

This is possible because μ provides the IR cut-off.

So it is a mass-indep. scheme.

It is also regularization indep. (like our scheme for ϕ^4 theory) – it can be used w/ any regularization, unlike, say, the $\overline{MS}$ scheme.

Indeed it can be used in a (quasi-) non-perturbative way with lattice regularization.

One takes a numerically generated gauge config., fixes to lattice Landau gauge (say), & calculates $\Gamma^{(3)}_R$ & thus $g_R(\mu)$. This effectively sums up to all orders in the bare coupling g.

If μ is large enough, $\bar{\mu} \gg \sqrt{\sigma}$, non-pert terms ($\propto e^{-1/g^2}$) are small, as are discretization errors if $\bar{\mu} a \ll 1$.

This method – called non-perturbative renormalization or NPR – allows one to determine Z factors & g_R to all orders in PT with statistical (& some systematic) errors.

It is logically very similar to the use of simulations in the Lüscher-Weisz study of ϕ^4 theories – just with different renormalization conditions.

We found $g_R(\mu) = g\left\{1 + g^2\left[-\beta_0 \ln\mu + C_L\right] + O(g^5, a^2)\right\}$

There are several important things we learn from this result.

(1) To take the ctm limit we set $\mu = \bar{\mu} a$, fix $\bar{\mu}$, send $a \to 0$ (so $\mu \to 0$) & think of g_R as $g_R(\bar{\mu})$.

We want to do this in such a way that the physical quantity $g_R(\bar{\mu})$ is fixed.

Since $\mu \to 0$, $-\beta_0 \ln\mu \to \infty \Rightarrow$ we must decrease g as $a \to 0$.

This is asymptotic freedom at work, the opposite phenomenon from that in ϕ^4 theory or from $U(1)$ gauge theory.

It means that PT does control the approach to the ctm limit; once $g(a)$ is small enough, reducing a makes it smaller.

Mathematically

$$\frac{d}{d\ln a} g_R(\bar{\mu}) = 0 = \frac{dg}{d\ln a}(1+O(g^2)) - g^3\beta_0(1+O(g^2))$$

up to a^2 terms which we ignore

$$\Rightarrow \frac{dg}{d\ln a} = \underbrace{\beta_0 g^3 + \beta_1 g^5 + \ldots}$$

Beta-fcn for bare lattice coupling $\beta(g)$

This is one way of looking at renormalization –

- We can vary the bare coupling as a function of the cutoff $1/a$ so that $1/a \to \infty$ & physical quantities have a finite limit.

- Note that $g_R(\bar{\mu})$ for all choices of $\bar{\mu}$ are simultaneously made finite (as are higher-order vertices)

$\Rightarrow$ $\frac{dg}{d\ln a}$ cannot depend on $\bar{\mu}$

$\Rightarrow$ $\frac{dg}{d\ln a}$ cannot depend on a (since the only way it could do so is through $\bar{\mu}a = \mu$)

$\Rightarrow$ $\frac{dg}{d\ln a}$ is a function of g alone

- In fact can show that β_0 & β_1 are universal (more on this below) (e.g. same for any lattice discretization)

Solving the equation leads to

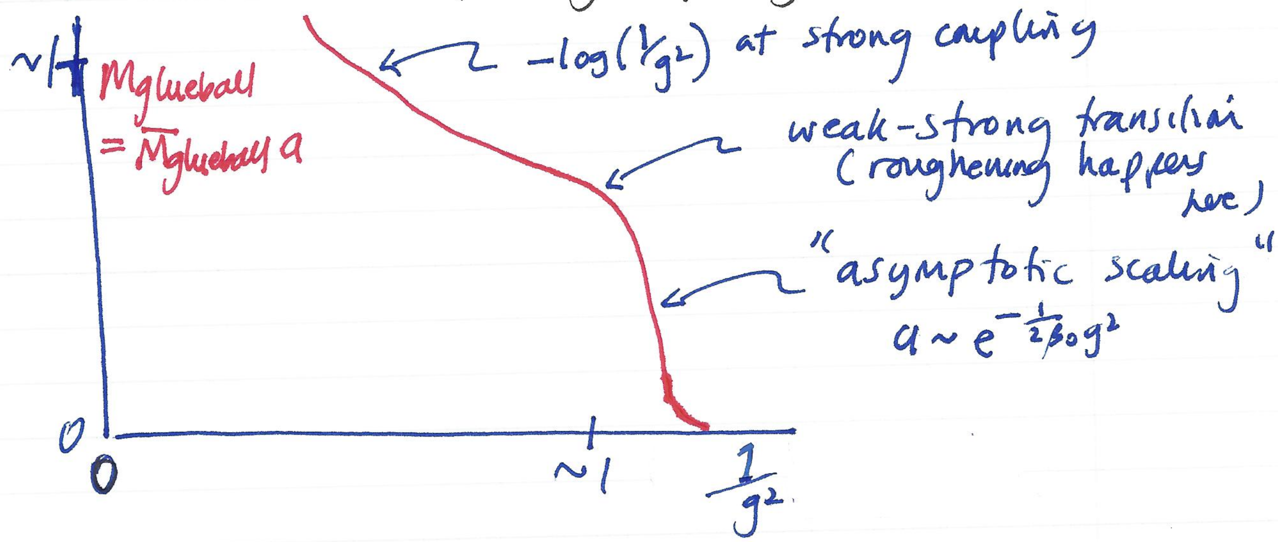
$$a\Lambda_{lat} = e^{-\frac{1}{2\beta_0 g^2(a)}} \left(g^2(a)\beta_0\right)^{-\frac{\beta_1}{2\beta_0^2}} \left[1 + O(g^2)\right]$$

integration constant

convention.

or $$g^2(a) = \frac{1}{-2\beta_0 \ln(a\Lambda_{lat})} + \text{higher order.}$$

This means that if we calculate any physical quantity in lattice units, e.g. $\sqrt{\sigma}$, $M_{glueball}$, then it will drop very rapidly as $g \to 0$.



All quantities fall in the same way, implying that their ratios become constant

$$\frac{M_{glueball}}{\sqrt{\sigma}} \xrightarrow[a \to 0]{} \text{const.}$$ ← continuum limit result.

Need $M_{glueball} \ll 1$ to avoid discretization errors, & $M_{glueball} N_s \gg 1$ to avoid finite volume errors

$\Rightarrow \quad 1 \gg \overline{M}_{glueball}\, a \gg 1/N_s$

$\Rightarrow \quad N_s \gg\gg 1$ ($\sim$50–100 is good)

& narrow range of allowable a

$\Rightarrow$ very narrow range of allowable/useful values of g for simulations ($g \sim 1$)

* Note that $\xi = \frac{1}{M} \propto e^{1/2\beta_0 g^2}$ as $g \to 0$

Compare this to $\xi \propto (K-K_c)^{-1/2}$ in mean-field approximation for ϕ^4 theory

Power-law in parameter has been replaced by an essential singularity.

(2) Returning to

$$g_R(\bar{\mu}) = g\left\{1 + g^2\left[-\beta_0 \ln \bar{\mu} a + C_L\right] + \dots\right\}$$

we can also derive a more conventional RG eq.

Recall that renormalization theory implies that $\Gamma_R^{(n)}$ are finite functions of .... g_R & the physical momenta, here $\bar{\mu}$. No reference to the bare coupling or a can remain.

$$\Rightarrow \left.\frac{d\, g_R(\bar{\mu})}{d \ln \bar{\mu}}\right|_{g,a} = -\tilde{\beta}(g_R)$$

cannot depend explicitly on $\bar{\mu}$ by dimensional analysis.

By explicit calculation

$$\text{LHS} = -\beta_0 g^3 + O(g^5) = -\beta_0 g_R^3 + O(g^5)$$

Thus the MOM coupling depends on $\bar{\mu}$ in the usual way.

(3) Relating $g_{\overline{MS}}$ to g (bare lattice coupling).

Can calculate $g_R(\bar{\mu})$ in the $\overline{MS}$ scheme, finding

$$g_R(\bar{\mu}) = g_{\overline{MS}} \left\{ 1 + g_{\overline{MS}}^2 \left[\beta_0 \ln\left(\frac{\bar{\mu}_{\overline{MS}}}{\bar{\mu}}\right) + C_{\overline{MS}} \right] + O(g^4) \right\}$$

$\bar{\mu}_{\overline{MS}}$ is the $\overline{MS}$ renormalization scale, at which $g_{\overline{MS}}$ is evaluated.

Equating this with our lattice result leads to

$$g = g_{\overline{MS}} \left\{ 1 + g^2 \left[\overbrace{(C_{\overline{MS}} - C_L)}^{-0.234} + \ln(\bar{\mu}_{\overline{MS}}\, a) \right] + O(g^4) \right\}$$

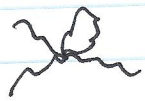
or $g_{\overline{MS}}^2$ - equivalent at this order

evaluated at scale $1/a$

evaluated at scale $\bar{\mu}_{\overline{MS}}$.

$\Rightarrow$ If $g_{\overline{MS}} \sim 1$ (so $\alpha_{\overline{MS}} = g_{\overline{MS}}^2/4\pi \sim 0.08$ - small)

then $g^2(\frac{1}{a} = \bar{\mu}_{\overline{MS}}) \approx 0.5$

Large difference can be primarily traced to tadpole diagrams e.g. .

If $\alpha_{\overline{MS}}(\bar{p})$ is a good expansion parameter for a process involving momentum $\sim \bar{p}$, then

$\alpha_{LAT}(\bar{p}) = g^2/4\pi$ will be a poor expansion parameter.

This appears to be the case. $\Rightarrow$ Always rexpress lattice PT results in terms of $\alpha_{\overline{MS}}$ or $\alpha_{MOM} = g_R(\bar{\mu})^2/4\pi$