Strong coupling expansion — small β

$$S = -\frac{\beta}{N_c} \sum_{\square} \mathrm{Re}\,\mathrm{tr}\,\square$$

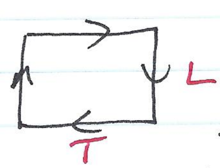
Note that $\dfrac{\mathrm{Re}\,\mathrm{tr}\,\square}{N_c}$ is bounded between -1 & $+1$ since $\square \in SU(N_c)$

- upper bound $\Rightarrow \square = \mathbb{1}$
- for even N_c can attain lower bound w/ $\square = -1$
- for odd N_c, I think the minimum is $-1 + \frac{2}{N_c}$, obtained w/ $\square = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & \ddots \end{pmatrix}$

Thus functional integral is well defined w/ $\beta < 0$

$\Rightarrow$ expect expansion in β to have finite radius of convergence (& can show rigorously)

[For even N_c, I think that can flip sign of β by change of variables — exercise; show this.]

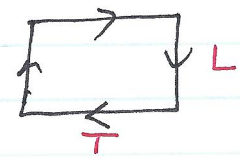
Use to calculate $\langle$  $\rangle$,

glueball masses & $\langle$ Polyakov line $\rangle$.

- Will sketch method + some examples on HW
- High order calculations have been done using technical tricks
- Not useful quantitatively (probably — although recent revival for nuclear physics by O. Philipsen).
- Main use is QUALITATIVE.

Wilson loop in strong coupling expansion

$$W(L,T) = Z^{-1} \int \mathcal{D}U \, e^{\frac{\beta}{N_c} \sum_{\square} \mathrm{Re\,tr}\,\square}$$



$\beta = 0$: $S \to 0 \Rightarrow$ only have $\mathcal{D}U \Rightarrow$ random gauge fields.

Use $\int dU \, 1 = 1$

For all N_c $\longrightarrow$ $\int dU \, U_{ab} = 0$

For $N_c > 2$ $\longrightarrow$ $\int dU \, U_{ab} U_{cd} = 0$

show using $dU = d(UV)$ $\Rightarrow$ only get non-zero result if integrand has a singlet piece

All N_c $\longrightarrow$ $\int dU \, \underbrace{U_{ab} U^{\dagger}_{cd}} = \dfrac{\delta_{bc}\delta_{ad}}{N_c}$

fundamental $\times$ antifundamental $=$ singlet $+$ adjoint

Check: $U = U'V \quad dU = dU'$

$$\Rightarrow \int dU \, U_{ab} U^{\dagger}_{cd} = \int dU' \, U'_{ab'} V_{b'b} V^{\dagger}_{cc'} U'^{\dagger}_{c'd}$$

$$= \frac{\delta_{ad}\delta_{b'c'}}{N_c} V_{b'b} V^{\dagger}_{cc'} = \frac{\delta_{ad}\,(V^{\dagger}V)_{cb}}{N_c}$$

$$= \frac{\delta_{ad}\delta_{cb}}{N_c} \quad \Rightarrow \text{result is self consistent}$$

Normalization: contract with δ_{bc}

$$\int dU \, U_{ab} U^{\dagger}_{cd} \, \delta_{bc} = \delta_{ad} \cancelto{1}{\int dU \, \mathbb{1}} \stackrel{?}{=} \frac{\delta_{bc}\delta_{ad}}{N_c} \delta_{bc} = \delta_{ad} \quad \checkmark$$

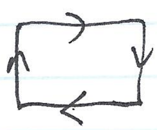
For $N_c = 3$ $\quad \int dU \; U_{ab} \, U_{cd} \, U_{ef} = N \, \varepsilon_{ace} \, \varepsilon_{bdf}$

since $3 \times 3 \times 3$ includes singlet.

How get normalization? Contract with $\varepsilon_{ace} \, \varepsilon_{bdf}$

$$\left.\begin{array}{ll} \text{LHS} \rightarrow & \int dU \, (\det U) \, 3! = 3! \\ \text{RHS} \rightarrow & N \cdot (3!)^2 \end{array}\right\} \Rightarrow N = 1/3!$$

OK – now to the calculation:

$\int DU$  $= 0$

since have $\int dU \, U = 0$ for each link in loop.

What about Z?

$$Z_{\beta=0} = \int DU = 1$$

Strategy now becomes clear – to get a non zero answer need either

 or  etc. or (for $N_c = 3$)  or 

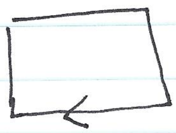
on each link.

Thus leading contribution to numerator occurs when "tile" the Wilson loop

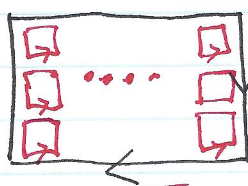
$$e^{\frac{\beta}{N_c}\mathrm{Re\,tr}\,\square} = e^{\sum_{\square}\frac{\beta}{2N_c}(\square + \square)}$$

this includes the trace

$$= \sum_n \frac{1}{n!}\left(\frac{\beta}{2N_c}\right)^n \left[\sum_{\square}(\square + \square)\right]^n$$

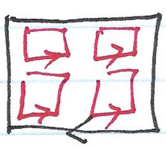
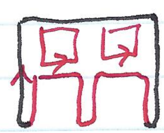
So $\int DU\, e^{-S}$ 

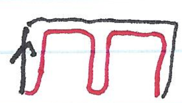
different ways of obtaining LT plaquettes from

$= \frac{1}{\cancel{(LT)!}}\left(\frac{\beta}{2N_c}\right)^{LT}\int DU$  $\cancel{(LT)!}$ + higher order in β.

Doing U integrals; using $\int dU\, U_{ab}U^\dagger_{cd} = \frac{\delta_{ad}\delta_{bc}}{N_c}$

i.e. $\int dU$  $= \frac{1}{N_c}$ 

so e.g.  $\longrightarrow \frac{1}{N_c^2}$  $\overset{uu^\dagger=1}{=\!=} \frac{1}{N_c^2}$ 

$\longrightarrow \frac{1}{N_c^4}$  $= \frac{1}{N_c^4} \cdot \;\; = \frac{1}{N_c^3}$

For an $L\times T$ loop get

$$\left[\left(\frac{1}{N_c}\right)^T\right]^L \cdot N_c = N_c^{1-LT}$$

final trace

unzip each layer

# of layers

Thus obtain (since $Z = 1 +$ higher order in β)

$$W(L,T) = N_c \left(\frac{\beta}{2N_c^2}\right)^{LT} + \text{higher order}$$

$$= e^{\underbrace{LT}_{} \log(\beta/2N_c^2) + \log N_c}$$

(dim'less) Area of loop — get Area law!

Can read off string tension

$$\sigma = -\log(\beta/2N_c^2) \qquad \beta = 2N_c/g^2$$

$$= -\log(1/(N_c g^2))$$

$$= \log(N_c g^2)$$

So we find confinement ($\sigma \neq 0$) at strong coupling — for all N_c!

intuitively related to almost complete decorrelation of links

<u>Notes:</u>

- result as written valid for $N_c \geq 3$
- For $N_c = 2$ $\int dU\, U_{ab} U_{cd} \neq 0$ so get additional contributions
- Holds also for $U(1)$ gauge theory (see next page)

  with $\sigma = -\log(\beta/2)$ $\beta = 1/g^2$.
- $\sigma \to \infty$ as $g \to \infty$

  if $\sigma = \sigma_{phys}\, a^2 \Rightarrow a \to \infty$!

U(1) Lattice gauge theory : $U_{n,\mu} \in U(1)$

- Euclidean continuum action $\quad S = \frac{1}{4}\int_x F_{\mu\nu}F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

- Lattice action $\quad S_{lat} = -\beta \sum_{n,\mu<\nu} \operatorname{Re} P_{n;\mu\nu}$ $\quad$ No trace required

w/ plaquette having same form as for $SU(N_c)$

$$P_{n;\mu\nu} = U_{n,\mu}\, U_{n+\mu,\nu}\, U_{n+\nu,\mu}^{+}\, U_{n,\nu}$$

- Classical ctm limit : $\quad U_{n,\mu} = e^{-iag A_\mu(n+a\hat{\mu}/2)}$

as before (see 10.9) $\quad \operatorname{Re} P_{n;\mu\nu} = 1 - \frac{g^2}{2} a^4 F_{\mu\nu}^2 + O(a^6)$

So $S_{lat} \to \text{const} + \frac{\beta g^2}{4}\int_x \sum_{\mu\nu} F_{\mu\nu}^2 + O(a^2)$

extra factor of 2 from conversion from $\sum_{\mu<\nu}$ to $\sum_{\mu\nu}$

To match ctm action need $\boxed{\beta = \frac{1}{g^2}}$
(so doesn't fit into $SU(N_c)$ result $\beta = 2N_c/g^2$
with $N_c \to 1$; $SU(1)$ = trivial $\neq U(1)$).

- Haar measure :

$$U = e^{i\theta} \qquad \int dU = \frac{1}{2\pi}\int_0^{2\pi} d\theta$$

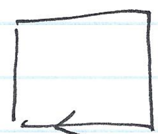
$$\left.\begin{aligned} &\int dU\, U^n = \delta_{n0} = \int dU\, U^{+n} \\ &\int dU\, U U^{+} = 1 \end{aligned}\right\} \text{ Or more generally } \int dU\, U^n U^{+m} = \delta_{nm} .$$

- Strong coupling expansion for Wilson loop for $U(1)$

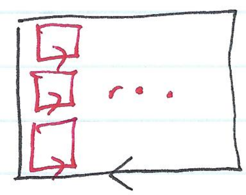
  Same steps as on 12.4

$$-S = \frac{\beta}{2}\left(\square + \square\right)$$

$$\int DU\, e^{-S}$$



$$= \frac{(LT)!}{(LT)!}\left(\frac{\beta}{2}\right)^{LT} \underbrace{\int DU}_{1}$$



$+$ higher order

← integrals are easier here

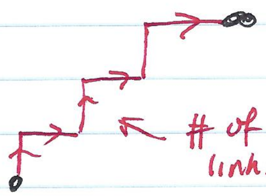
$$= \left(\frac{\beta}{2}\right)^{LT} + \dots$$

$$= \exp\left\{-LT \log(2/\beta)\right\}$$

$$\Rightarrow \sigma = +\log(2/\beta) = +\log(2g^2) \quad \text{as } \beta \to 0.$$

Further notes on string tension in strong coupling

- Can choose heavy quark-antiquark pair to lie on a diagonal. Then Wilson-loop has "taxi-driver" zig-zag shape in space & straight lines in time.



# of spatial links is L_{TD} ← taxi-driver distant

Minimal area tiling maintains zig-zag shape

$$\Rightarrow \langle W \rangle = N_c \left(\frac{\beta}{2N_c^2} \right)^{T L_{TD}}$$

$$= e^{-T L_{TD} \log(2N_c^2/\beta) + \log N_c}$$

compare to $e^{-T V(R)} + \text{subleading}$

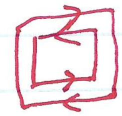
$$\Rightarrow V(R) = \sigma L_{TD}.$$

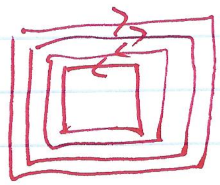
at leading order in β expansion

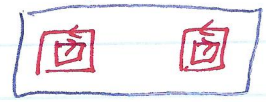
clearly violates rot^n invariance

- Going to higher order.

– When does denominator, Z, get a correction to 1?

* For general N_c get β^2 contrib ,

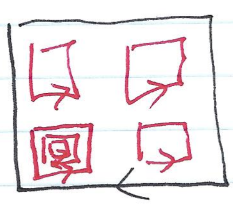
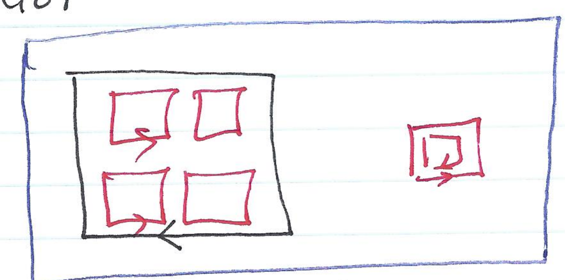
β^4 contrib  (all on same plaquette) etc.,

as well as 

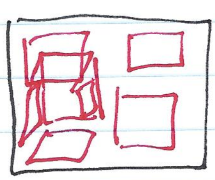
and "cube" contrib  at β^6.

* For $N_c = 3$, also get  at β^3.

– Corrections to numerator

 &  etc.

extra β^2 extra β^2

&  cubic "bump" at β^4.

– All terms involving multiple U's on single links can be treated at once using character expansion
See Montvay & Münster for extensive discussion

– Can show that area law is maintained to all orders in β.

Glueball masses in strong coupling

Recall that we can determine energies of states from Euclidean two-point correlator

$$C(n_4) = \langle \sum_{\vec{n}} \mathcal{O}^*_{\vec{n},n_4} \, \mathcal{O}_0 \rangle$$

function of U which creates glueball

$$= \sum_m c_M e^{-M_m n_4}$$

read off masses from fall-off of $\vec{p}=0$ correlator

Simplest choice is $\mathcal{O}_n = \sum_{\mu<\nu} c_{\mu\nu} P_{n,\mu\nu}$

– sum of plaquettes.

Choosing different coefficients one can project onto states lying in different irreps of the lattice timeslice symmetry group (cubic group + parity & charge conjugation)

– fun finite group theory but no time for that here

For scalar glueball use

$$\mathcal{O}_S = \mathrm{Re}(P_{12} + P_{13} + P_{23})$$

– in continuum becomes $J^{PC} = 0^{++}$ state

For tensor glueball use

$$\mathcal{O}_T = \mathrm{Re}(P_{12} - P_{13}) \text{ or } \mathrm{Re}(P_{21} - P_{23})$$

– in continuum becomes part of (2/5) $J^{PC} = 2^{++}$ states.

– rest of tensors need different operators.

Let's calculate the scalar glueball correlator.

To avoid coupling to the vacuum we use

$$\overline{O_s} = O_s - \langle O_s \rangle$$

in operator language this is $\langle 0 | \hat{O}_s | 0 \rangle$ = VEV

Then $\langle \overline{O_{s,n}} \; \overline{O_{s,0}} \rangle$

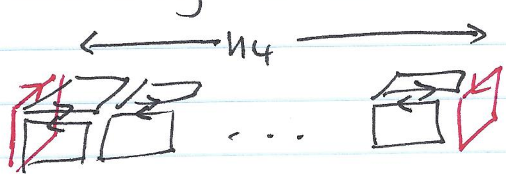
$$= \langle O_{s,n} \, O_{s,0} \rangle - \underbrace{\langle O_{s,0} \rangle^2}_{\text{using translation invariance}}$$

i.e. the connected correlator.

So we want

$$\langle (\underset{\vec{n},n_4}{\Box} + \underset{\vec{n},n_4}{\Box}) \; (\underset{0}{\Box} + \underset{0}{\Box}) \rangle_c$$

The contribution with the smallest power of β has $\vec{n} = 0$ & has the plaquettes connected by a "tube"



$O = \frac{1}{2}(P_{12} + P_{12}^* + \ldots)$

$(\frac{1}{2})^2 \times 6$

$$\text{So } C(n_4) = \left(\frac{\beta}{2N_c}\right)^{4n_4} \cdot \frac{3}{2} \cdot \underbrace{\left(\frac{1}{N_c^4}\right)^{n_4} \cdot N_c}_{\text{from } \int dU}$$

$$= \frac{3N_c}{2} \left(\frac{\beta}{2N_c^2}\right)^{4n_4}$$

$$= \frac{3N_c}{2} \, e^{-\log(2N_c^2/\beta)\, 4n_4}$$

β^2 for SU(2)

So $M(\text{scalar}) = 4\log(2N_c^2/\beta) + O(\beta)$

$= 4\sigma$ at leading order.

The same diagrams contribute to the tensor glueball at leading order. In fact

$M(\text{tensor}) = M(\text{scalar}) + O(\beta^4)$

first order a difference occurs.

These expansions have been extended to β^8 for SU(2) & SU(3) theories.

The key point here is that for $0 \le \beta < \beta_c$ all lattice gauge theories are confining, have a mass gap, & a complicated spectrum of glueballs.
This is true also for U(1) & Z_N gauge theories — the group changes the numerical factors but not the qualitative features.

However, this is confinement in the regime where a is large & there are huge discretization errors. For example, we found

$M_{scalar} = 4\sigma + O(\beta)$

But in physical units this is

$\underbrace{M_{scalar}^{phys}\, a = 4\,\sigma^{phys}\, a^2}_{\text{so two quantities give inconsistent determinations of } a\text{!}}$ since they have different dimensions!

What happens as β increases? Depends on gauge group.

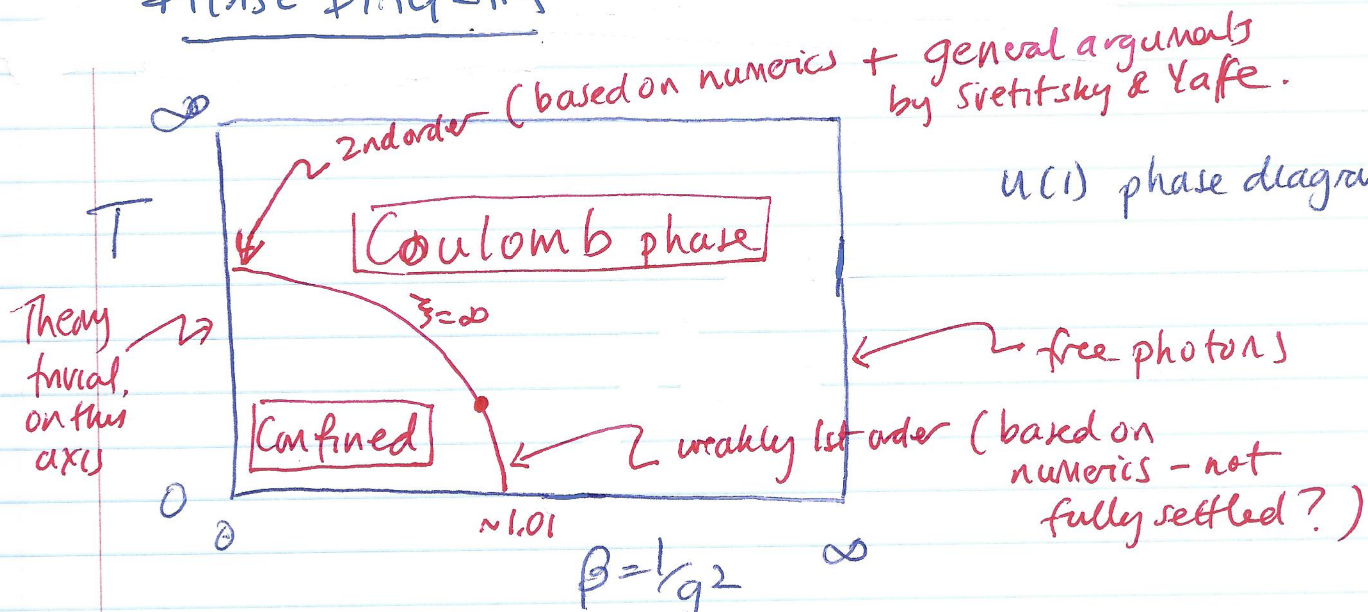
For $U(1)$, at $\beta_c \approx 1.01$ there is a phase transition (weakly first order?) to a weak-coupling phase ("Coulomb phase") in which

$$\underbrace{V(r)}_{\text{heavy quark potential}} \propto 1/r + \text{const.} \quad \& \quad W(L,T) \propto e^{-c\overbrace{(L+T)}^{\text{perimeter dependence}}} + \ldots$$

The existence of this transition was first proved analytically by Alan Guth (of inflation fame), although for a different form of the lattice action.

This is expected, since P.T. in the weak coupling limit gives free photons.

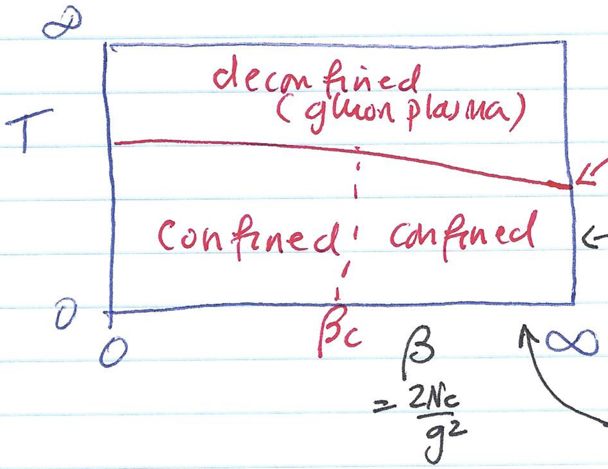
PHASE DIAGRAM



U(1) phase diagram

N.B. Hamiltonian formulation very interesting – related to "x-y" or planar Heisenberg model

For $SU(2)$ & $SU(3)$ numerical simulations show that there is no Coulomb phase (at $T=0$)
$\Rightarrow$ confinement lasts until $g \to 0$ ($\beta \to \infty$)



$\boxed{\zeta = \infty \text{ on this line.}}$

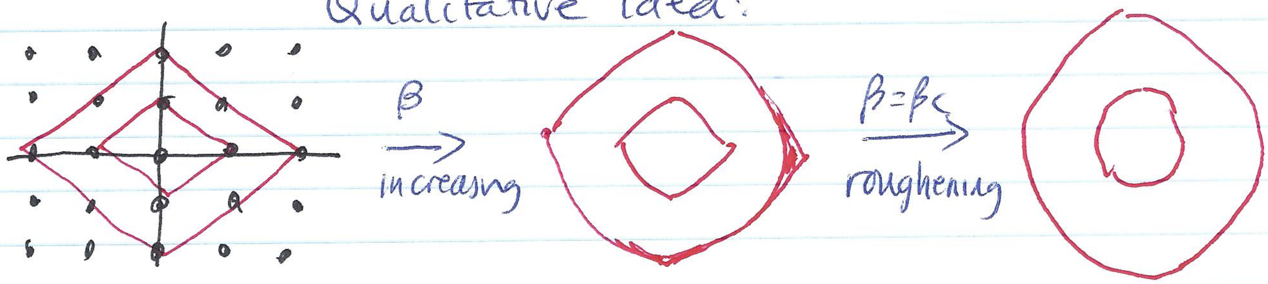
$g^2 \sim \frac{\#}{-\ln a\Lambda}$ or $a \sim \frac{1}{\Lambda} e^{-\#/g^2}$

Much more could be said about nature of finite T transition – in QCD this is a huge research area, with direct connections to heavy-ion collisions.

What is β_c? Place where strong coupling expansion fails (but confinement survives)

Called the "roughening transition" / thought to be "infinite" order.

Qualitative idea:



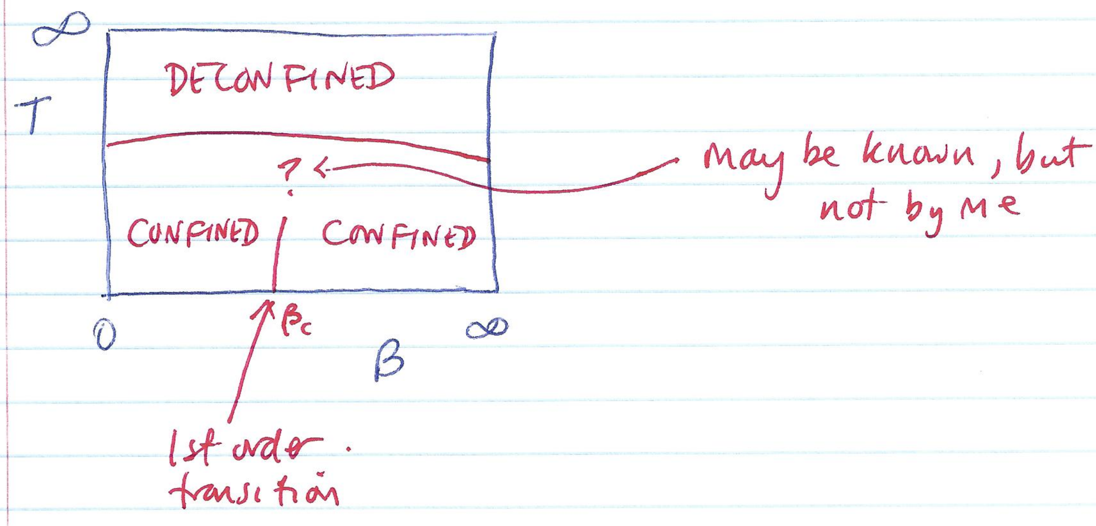
$\beta \to 0$
equipotentials have sharp corners

curved edges but sharp corners

smooth equipotentials

Alternatively: transverse fluctuations of on-axis flux tube diverge as $\ln(L)$ L = lattice size in space.

For $SU(N_c)$, $N_c \geq 4$ the phase diagram is different:



For $T=0$ there IS a transition at $\beta_c < \infty$.
However, the theory is confining on both sides!

The transition can be avoided by changing the action, e.g. including a term with adjoint form $\sum_{\square} |\mathrm{tr}\,\square|^2$.

The transition is a "bulk transition" – it happens at large a & is a lattice artifact.

[In stat. mech., phase diagrams are littered with such transitions.]

Moral: a phase transition is not necessarily an enemy of confinement.