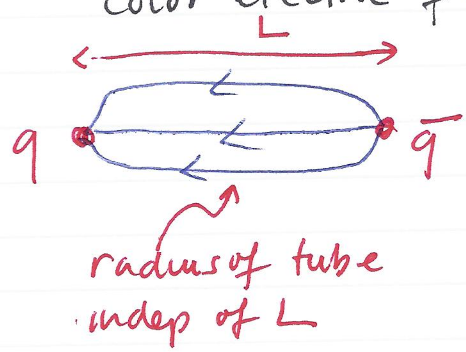
Confinement in pure gauge theory

Confinement is the signature non-perturbative property of QCD — there are no free quarks or gluons, only color-singlet hadrons.

- Weak coupling provides no guidance as to the long distance degrees of freedom (though is v. useful for short distances)

We don't have real quarks yet, but we can discuss the issue w/ scalar quarks just as well — it is color & not spin that matters.

The cartoon of confinement is that, between a q & $\bar{q}$ (or ϕ & $\phi^\dagger$) a "tube" of color electric flux forms as they are pulled apart



Unlike in QED (E&M) where the flux spreads & $V(L) \propto \frac{1}{L}$

Force $\propto 1/L^2$

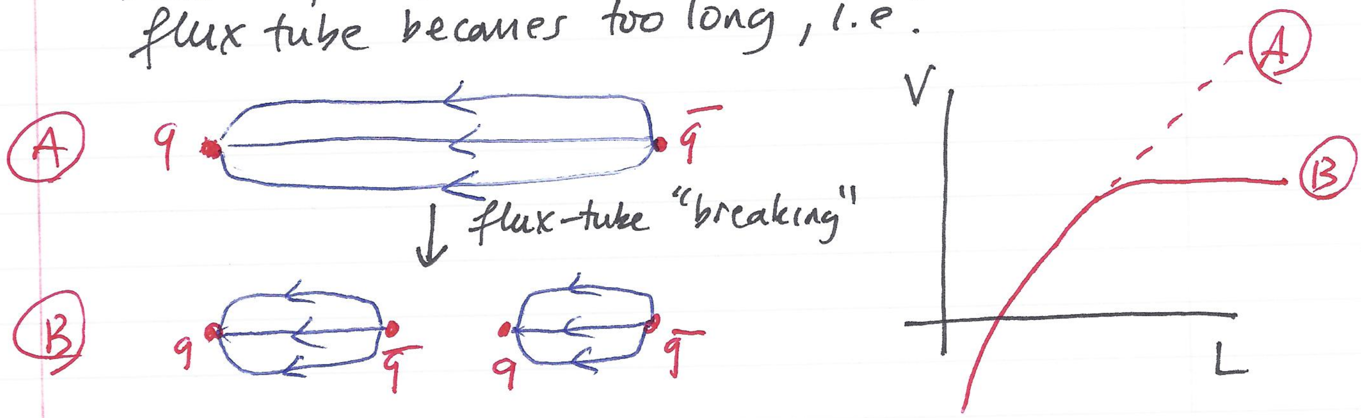
Here $V(L) = \sigma L$ for large $L \Rightarrow$ Force $= \sigma$

"string tension"

- so takes infinite energy to pull q & $\bar{q}$ apart.

Wait, you say! What about vacuum fluctuations?

$q\bar{q}$ pairs appear in the QCD vacuum, &, in the strong chromoelectric field can tunnel to a state of equal or lower energy when the flux tube becomes too long, i.e.



So, despite confinement,
we can pull the initial $q\bar{q}$ pair apart w/ finite energy.

Thus $V(L)$ is not useful as an "order parameter" of confinement in general, except....
if $m_q \to \infty$. Then there are no $q\bar{q}$ pairs and confinement does imply $V(L) = \sigma L$ for large L.

So we need a way of determining $V(L)$ from a Euclidean calculation.

But this is straightforward since $V(L)$ can be related to an energy of a state, & those we can read off from our Euclidean 2-point correlators.

Consider $\underbrace{C(\vec{x},t)}_{\text{the cfm Euclidean correlator}} = \left\langle \begin{array}{c} \phi_2(\vec{y},t) \\ \downarrow L(\vec{x},\vec{y};t) \\ \phi_1^{\dagger}(\vec{x},t) \end{array} \quad \begin{array}{c} \phi_2^{\dagger}(\vec{R},0) \\ \uparrow L(\vec{R},0;0) \\ \phi_1(\vec{0},0) \end{array} \right\rangle$

where $\vec{y} = \vec{x} + \vec{R}$.
& L are Wilson lines, and ϕ_1 & ϕ_2 are two flavors of scalar quarks.

Each of the $\phi^{\dagger} L \phi$ is gauge invariant.

When we write the correlator in operator form we get

$$C(\vec{x},t) = \sum_n \langle 0 | \hat{\mathcal{O}}^{\dagger}_{(\vec{x})} | n \rangle \frac{e^{-E_n t}}{2E_n} \langle n | \hat{\mathcal{O}}_{(0)} | 0 \rangle$$

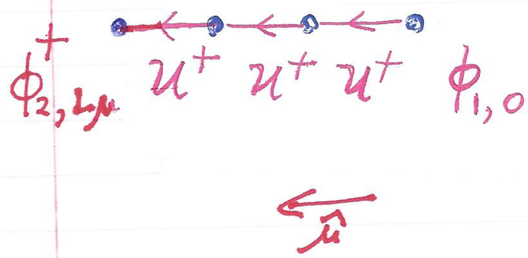
where $\hat{\mathcal{O}}$ is an operator which creates a scalar quark of flavor 2 & a scalar antiquark of flavor 1, separated by distance $\vec{R}$, and a mess of color electric flux in between to make the object gauge invariant.

The intermediate states are those having the requsite quantum numbers.

If $M_\phi \to \infty$, the quark has $\vec{v}=0$ since otherwise the kinetic energy will be infinite, so we expect $C(\vec{x},t) \propto \delta(\vec{x})\, \tilde{C}(t)$, & we then want to know the lowest energy state:

$$E_1 = \underbrace{\text{const}}_{\text{includes } 2M_\phi + \text{some renormalization but indep of } \vec{R}} + V(\vec{R}) \longleftarrow \text{we want large } \vec{R} \text{ dependence of } V$$

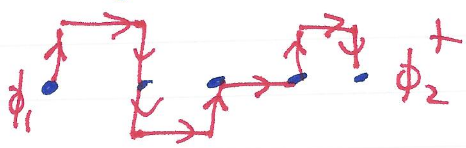
On the lattice the field which "creates" the desired state is



(choosing the Wilson line to lie along the $\hat{\mu}$ axis).

Notes:

- One can choose any lattice path between the endpoints e.g.

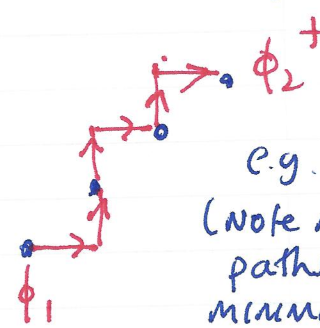


- As long as they have the same symmetry under rotations around the axis, and as long as they live in a single timeslice, these will all couple to the same state.

[If one uses a non-trivial representation of the rot^n group about the axis then one gets an excited flux tube — interesting (⇒ "hybrid" mesons) but not our present focus.]

- One does not need to separate ϕ_1 & ϕ_2^+ along an axis — any direction will do e.g.

Indeed, it is an important check of the restoration of rot. inv. that all directions must give the same $V(R)$ in the continuum limit



e.g. diagonal (note many paths of minimal length.)

- So let's calculate this correlator on the lattice, in the heavy "quark" limit.

$$G(\vec{n}, n_4) = \frac{1}{Z}\int_{U}\int_{\phi_1,\phi_1^+}\int_{\phi_2,\phi_2^+} e^{-S_{gauge} - S_{\phi_1} - S_{\phi_2}}$$

$$\left(\phi^+_{1,(\vec{n},n_4)} U \dots U\, \phi_{2,(\vec{n}+L\hat{\mu},n_4)}\right)\left(\phi^+_{2,L\hat{\mu}}\, U^+ \dots U^+ \phi_{1,0}\right)$$

$$S_\phi = \sum_n (M_0^2 + 8)\,\phi_n^+\phi_n + \sum_n \frac{g_0}{4!}(\phi_n^+\phi_n)^2 - \sum_{n,p}\phi_n^+ H_{np}\phi_p$$

picking "ϕ^4" for definiteness

- Hopping matrix

$$H_{np} = \delta_{n,p+\hat{\mu}}\, U^+_{n,\mu} + \delta_{n,p-\hat{\mu}}\, U_{n,\mu}$$

(same as before but now with U's)

- Rescale fields: $\phi_n\sqrt{M_0^2+8} = \tilde{\phi}_n$

$$S_\phi = \sum_n \tilde{\phi}_n^+\tilde{\phi}_n - \sum_{np}\tilde{\phi}_n^+\frac{H_{np}}{M_0^2+8}\tilde{\phi}_p + \sum_n \frac{g_0}{4!}\frac{(\tilde{\phi}_n^+\tilde{\phi}_n)^2}{(M_0^2+8)^2}$$

So $G^{(0)}_{np} = \left(\frac{1}{1 - H/(M_0^2+8)}\right)_{np}$ (free propagator)

Set up for expansion in $1/M_0^2$

Now take $M_0^2 \to \infty$

$$G^{(0)} = 1 + \frac{H}{M_0^2+8} + \frac{H^2}{(M_0^2+8)^2} + \cdots$$

vertices suppressed by $\frac{1}{M_0^4}$

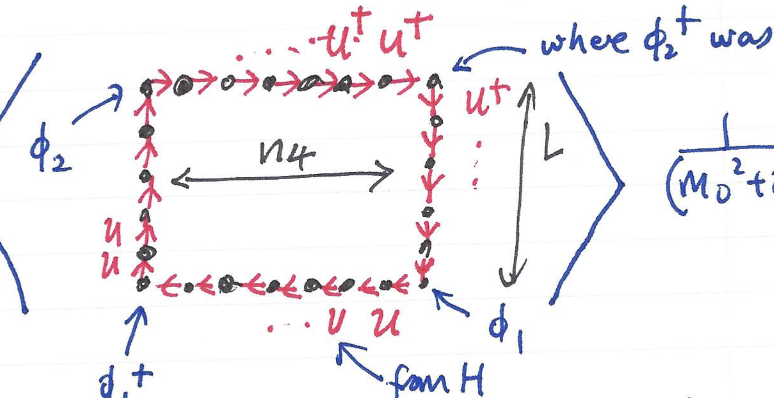
$\Rightarrow$ quarks will propagate, i.e. hop, over minimal distance possible with no interactions

Thus for our correlator, the leading term will be for $\vec{n} = \vec{0}$, i.e. when the two operators are at matching spatial positions, so that the scalar prop. can go along the time direction.

This involves n_4 hops, each with a U or $U^\dagger$.

So we can do the $\int_{\phi_2, \phi_2^\dagger}$ & $\int_{\phi_1, \phi_1^\dagger}$, obtaining

$$\frac{\delta_{\vec{n},\vec{0}}}{Z_{gauge}} \int_U e^{-S_{gauge}} \Big\langle \quad \Big\rangle \frac{1}{(M_0^2+8)^{2n_4}}$$



$$= \frac{\delta_{\vec{n},0}}{(M_0^2+8)^{2n_4}} \Big\langle \begin{matrix}\text{Wilson loop} \\ \text{(rectangular)}\end{matrix} \Big\rangle_U \rightarrow W(L, n_4)$$

$$= C(\vec{n}, n_4)\left(1 + O\left(\tfrac{1}{M_0^2}\right)\right)$$

if use non-straight spatial paths, get different shapes

So the heavy (scalar) quark propagates forward in Euclidean time picking up a $\mathcal{U}^{+}/_{M_0^2+8}$ for each hop

– very similar to HQET – in fact, spin irrelevant in heavy quark limit.

Note that automatically get the trace of the product of $\mathcal{U}$'s & $\mathcal{U}^{+}$'s – a gauge invariant object

Now, from continuum analysis on (11.3) or from corresponding transfer matrix analysis

$$C(\vec{0}, n_4) = \sum_n |\langle 0 | \mathcal{O} | n \rangle|^2 \frac{e^{-E_n n_4}}{2E_n}$$

using continuum form

$$\xrightarrow[n_4 \text{ large}]{} |\langle 0 | \mathcal{O} | 1 \rangle|^2 \frac{e^{-E_1 n_4}}{2E_1}$$

$$= e^{-2\ln(M_0^2+8)\, n_4}\ W(L, n_4)$$

$$\Rightarrow -\ln W(L, n_4) + \text{const}.n_4 + \text{const}' \underset{n_4 \text{ large}}{=} E_1 n_4$$

$$\text{So}\quad E_1 \underset{n_4 \text{ large}}{\sim} -\frac{\ln W(L, n_4)}{n_4} + \text{const}$$

For confinement want $E_1 = \sigma L$ for L large.

$$\Rightarrow \quad \ln W(L, n_4) \underset{n_4, L \text{ large}}{\sim} -\sigma L n_4 = -\sigma(\text{Area})$$

$$\text{or } W(L, n_4) \underset{n_4, L \text{ large}}{\sim} e^{-\sigma(\text{Area})}$$

"Area law" test of confinement (Wilson)

If no confinement then

$$W(L, n_4) \underset{n_4, L \text{ large}}{\sim} e^{-c(L+n_4)}$$

"perimeter term" (same for L & n_4 by Euclid. inv.)

Summary: if $W(L,T)$ falls very rapidly, as $e^{-\text{Area}\cdot\sigma}$, then theory (pure gauge theory) is confining.

$\Rightarrow$ very rapid decorrelation of $\mathcal{U}$'s as move apart

More generally can use $W(L,T)$ to obtain heavy-quark potential.

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Alternative condition at $T > 0$. Recall that if have finite # of lattice sites in t direction, N_t, then

$$Z = \mathrm{tr}\, \hat{T}^{N_t} = \mathrm{tr}\, e^{-\hat{H} a N_t}$$

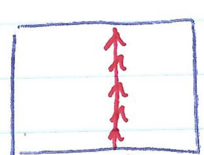
$$= \mathrm{tr}\, e^{-\hat{H}/T} \quad \text{with } T = 1/aN_t.$$

Boltzmann statistical ensemble at temp T

Add a heavy quark to the ensemble $(M_0 \gg aT)$

- we know how it propagates - forward in time, not moving, picking up $\mathcal{U}^+$ factors.
- for N_t finite we can make a gauge invariant object by closing $\mathcal{U}^+$'s into a loop

$$\frac{Z_\phi}{Z} = \frac{\int [D\mathcal{U}]\, e^{-S_{gauge}}}{\int [D\mathcal{U}]\, e^{-S_{gauge}}}$$



$\mathrm{tr}(\mathcal{U}^+\mathcal{U}^+ \cdots \mathcal{U}^+) = L$ "Polyakov line"

$$= e^{-\frac{F_\phi}{T}}$$

free energy of static quark (up to $\frac{1}{(M_0^2+8)}$ factors).

Spin is irrelevant as large M_0, so all the matters is that there is a color-triplet object in the ensemble at the fixed position.

If we have confinement, then we expect
$F_\phi \to \infty$ & $\langle L \rangle = 0$.
Polyakov line

What one finds, e.g. from numerical simulations, is that $\langle L \rangle = 0$ at low temperatures,

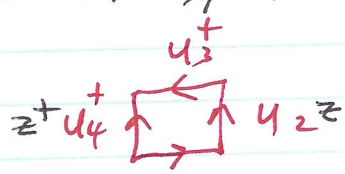
while for $T > T_c \sim \Lambda_{QCD}$ $\langle L \rangle \neq 0$.

Deconfinement transition into (quark-) gluon plasma phase. Color flux can be screened (even by gluons alone). For $N_c = 3$, transition is first order. (pure gauge theory).

This actually is an example of SSB: there is a global Z_{N_c} symmetry under which all time-directed links on a single timeslice (any one) are multiplied by $z \in Z_{N_c}$

$$\mathcal{U}_{\vec{n}, n_4; \mu=4} \to z\, \mathcal{U}_{\vec{n}, n_4; \mu=4} \qquad \forall \vec{n}, \text{ fixed } n_4$$

- leaves gauge action invariant



$z z^\dagger = 1$
& $[z, u_j] = 0$
so cancels from $\mathrm{tr}\, u_1 u_2 u_3^\dagger u_4^\dagger$

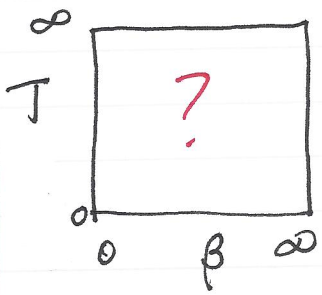
- is global (indep. of $\vec{n}$)
- NOT a gauge transform
- $L \to z^\dagger L$

* If Z_{N_c} unbroken, $\langle L \rangle = 0$; $T < T_c$

* If Z_{N_c} broken, $\langle L \rangle \neq 0$ is order param $T > T_c$

Summary for pure gauge theory

Can study as function of $\beta \propto 1/g^2$ & T = temp.



What is phase diagram?

Does confinement hold & where?

What is the spectrum?

Where can we take the continuum limit?

Details depend on N_c & we'll focus mostly on $N_c = 3$.

Useful analytic expansions?

$\beta \to \infty$ ($\Rightarrow g^2 \to 0$) – (lattice regularized) perturbation theory

degrees of freedom are gluons & find universal β-fcn with asymptotic freedom.

To hold $g_R(p_{phys})$ fixed (defined, e.g. in MOM scheme) must take bare coupling $g \to 0$ as $a \to 0$ (logarithmically).

[opposite from ϕ^4 where need $g_0 \to \infty$ & can't reach $a = 0$].

$\Rightarrow$ can have a non-trivial continuum limit as $\beta \to \infty$

"Dimensional-transmutation" gives a natural scale from PT:

$$g^2(a) \sim \frac{\#}{-\ln(\Lambda a)}$$

bare coupling

Lattice version of Λ parameter of pure gauge theory

- What are the excitations at this scale?

  - glueballs!

  Particles "created" by gauge invariant "operators" such as $\sum_{\vec{n}} \mathrm{tr}\, P_{n,\mu\nu}$ or other Wilson-loops.

  Simplest expectation is that there is a mass gap from the vacuum to the lightest glueball.

  $$M_{glue} = c\,\Lambda \qquad c > 0$$

  But also possible that $c = 0$ – gapless.

  And we expect at high T that we revert to a weak-coupling regime where gluons, not glueballs, are the relevant d.o.f.

- So, we need to study the theory in more details.

  For qualitative, analytic insight, very useful to use the strong coupling expansion – in powers of β.

  This is also unfamiliar to continuum QFT practitioners, so worth learning.

  Few other strong-couplings tools – best is AdS/CFT – which can study QCD-like theories in the ctm for g large & N_c large.